Advanced Topics in Applied Probability
- Introduction to Lattice Models

Exercises denoted by ($\star$) are harder or use additional theory.

Exercises – Set 6

For subsets $Q_n \subset \mathbb{Z}^d$, we write $Q_n \nearrow \mathbb{Z}^d$ as $n \nearrow \infty$ if the following hold:

- the sequence $(Q_n)_{n\in\mathbb{N}}$ is *increasing*, i.e., $Q_n \subset Q_{n+1}$ for all $n \in \mathbb{N}$, and
- the sequence $(Q_n)_{n\in\mathbb{N}}$ *invades* $\mathbb{Z}^d$, i.e., $\bigcup_{n\in\mathbb{N}} Q_n = \mathbb{Z}^d$.

1. (**Uniqueness of thermodynamic limit**) Consider the weak limit of random-cluster measures on $Q_n := [-n, n]^d \cap \mathbb{Z}^d$, with any fixed boundary conditions ξ (recall Exercise 1(c) in Set 5). Let $\tilde{Q}_n \subset \mathbb{Z}^d$ be another sequence $\tilde{Q}_n \nearrow \mathbb{Z}^d$ as $n \nearrow \infty$. Show that the weak limit is the same:
$$\tilde{\phi}^{\xi}_{p,q} := \lim_{n\to\infty} \phi^{\xi}_{p,q;\tilde{Q}_n} = \lim_{n\to\infty} \phi^{\xi}_{p,q;Q_n} =: \phi^{\xi}_{p,q}.$$

2. (**Equivalence of 2-Potts and Ising models**) Show that the 2-Potts model
$$\pi_{\beta,2}[\sigma] := \frac{1}{Z^{\mathrm{P}}_{\beta,q}} \exp\Big(\beta \sum_{\langle u,v\rangle\in E} \delta_{\sigma_u,\sigma_v}\Big), \qquad \sigma \in \Sigma = \{1,2\}^V, \qquad Z^{\mathrm{P}}_{\beta,2} := \sum_{\sigma\in\Sigma} \exp\Big(\beta \sum_{\langle u,v\rangle\in E} \delta_{\sigma_u,\sigma_v}\Big),$$
is equivalent to the Ising model
$$\lambda_{\beta/2}[\sigma] := \frac{1}{Z^{\mathrm{I}}_{\beta/2}} \exp\Big(\frac{\beta}{2} \sum_{\langle u,v\rangle\in E} \sigma_u\sigma_v\Big), \qquad \sigma \in \Sigma = \{-1,+1\}^V, \qquad Z^{\mathrm{I}}_{\beta/2} := \sum_{\sigma\in\Sigma} \exp\Big(\frac{\beta}{2} \sum_{\langle u,v\rangle\in E} \sigma_u\sigma_v\Big).$$

3. (**Duality for random-cluster models**) Let $G = (V, E)$ be a finite planar graph and let $G^* = (V^*, E^*)$ be its dual graph. For a random-cluster configuration $\omega \sim \phi_{p,q;G}$, we associate the dual random-cluster configuration ω^* by setting $\omega^*(e^*) := 1 - \omega(e)$ for all $e^* \in E^*$, where $e \in E$ is the unique edge crossed by e^*. Show that
$$\phi_{p,q;G}[\omega] = \phi_{p^*,q^*;G^*}[\omega^*], \qquad \text{where} \qquad \frac{p^*}{1-p^*} = \frac{q(1-p)}{p} \quad \text{and} \quad q^* = q.$$
[Hint: Euler's formula]

4. ($\star$) (**Critical probability for random-cluster models**) Fix $q \geq 1$. Let
$$p_c(q) = \sup\{p \geq 0 \,|\, \theta^1(p,q) = 0\} = \sup\{p \geq 0 \,|\, \theta^0(p,q) = 0\}$$
be the critical value for the random-cluster model, where $\theta^b(p,q) := \phi^b_{p,q;\mathbb{Z}^2}[0 \leftrightarrow \infty]$ for $b \in \{0,1\}$. (Recall that the equality $p_c(q)$ for both $b \in \{0,1\}$ follows from convexity of the free energy.)
For example, by using analogous arguments as for percolation ($q = 1$ case), prove that $p_c(q) \geq p_{\mathrm{sd}}(q)$, where $p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ is the solution to the *self-duality equation* $p^* = p$.
[Hint: e.g. prove that $\theta^0(p_{\mathrm{sd}}, q) = 0$ using planar duality and the fact that the number of infinite clusters is a.s. 0 or 1.]

5. (**Kramers-Wannier duality for Ising model**) Let $G = (V, E)$ be a finite planar graph, embedded in the plane, and let $\partial^o V := \{v \in V \mid v \text{ belongs to the boundary of the unbounded face of } G\}$. Let g be an additional "ghost" vertex (geometrically, a new vertex added on the unbounded face of G) and set $\partial E := \{\langle v, g\rangle \mid v \in \partial^o V\}$. We define $\overline{G} := (\overline{V}, \overline{E})$ with vertices $\overline{V} = V \cup \{g\}$ and edges $\overline{E} = E \cup \partial E$. Now, fix $\beta \in (0, \infty)$ and consider the Ising model $\lambda^{\oplus}_{\beta;G}$ on G with plus b.c., with partition function

$$Z^{\oplus}_{\beta;G} := \sum_{\substack{\sigma \in \{-1,+1\}^{\overline{V}}, \\ \sigma_g = +1 \text{ at } g}} \exp\Big(\beta \sum_{\langle u,v\rangle \in \overline{E}} \sigma_u \sigma_v\Big).$$

Let $G^* = (V^*, E^*)$ be the dual graph of G, and define $\beta^* \in (0, \infty)$ so that $\tanh \beta^* = e^{-2\beta}$. Show that

$$2^{\#V^*} (\cosh \beta)^{\#E^*} Z^{\oplus}_{\beta;G} = e^{\beta \# E} Z^{\text{free}}_{\beta^*;G^*},$$

where

$$Z^{\text{free}}_{\beta^*;G^*} := \sum_{\sigma \in \{-1,+1\}^{V^*}} \exp\Big(\beta^* \sum_{\langle u,v\rangle \in E^*} \sigma_u \sigma_v\Big),$$

is the partition function of the Ising model $\lambda^{\text{free}}_{\beta^*;G^*}$ on G^* with free b.c.

6. (**Average magnetization in the Ising model**) Let $G = (V, E)$ be a finite graph and $\lambda_{\beta,h}$ the Ising model on G (with free b.c.) in an *external magnetic field* of constant magnitude $h > 0$,

$$\lambda_{\beta,h}[\sigma] := \frac{e^{-\beta H_h(\sigma)}}{Z_{\beta,h}}, \qquad H_h(\sigma) := -\sum_{\langle u,v\rangle \in E} \sigma_u \sigma_v - h \sum_{v \in V} \sigma_v, \qquad \sigma \in \{-1,+1\}^V,$$

with partition function $(\beta, h) \mapsto Z_{\beta,h}$,

$$Z_{\beta,h} := \sum_{\sigma \in \{-1,+1\}^V} e^{-\beta H_h(\sigma)}.$$

   (a) Show that the *average magnetization* $M_{\beta,h}$ in this model (noting that $\lambda_{\beta,h}[\sigma_v]$ is just the expected value of the spin at the vertex v) equals

$$M_{\beta,h} := \frac{1}{\#V} \sum_{v \in V} \lambda_{\beta,h}[\sigma_v] = \frac{1}{\beta \# V} \frac{\partial}{\partial h} \log Z_{\beta,h}.$$

   (b) Show that the *cumulant generating function* of the total magnetization

$$m := \sum_{v \in V} \sigma_v$$

   can be expressed as

$$\log \lambda_{\beta,h}\big[e^{tm}\big] = \log Z_{\beta,h+t/\beta} - \log Z_{\beta,h},$$

   and the r:th cumulant of m is

$$\kappa_r(m) = \frac{1}{\beta^r} \Big(\frac{\partial}{\partial h}\Big)^r \log Z_{\beta,h}.$$

Upon finding mistakes and/or typos, please contact me!