Advanced Topics in Applied Probability
- Introduction to Lattice Models

Exercises denoted by $(\star)$ are harder or use additional theory.

Exercises – Set 4

1. (**Self-avoiding walks (SAW)**) A self-avoiding walk (SAW) is a lattice path that visits no vertex more than once. Let $\sigma_n := \#\{\text{SAW of length } n \text{ on } \mathbb{Z}^d \text{ starting from the origin}\}$. Show that $\sigma_{n+m} \leq \sigma_n \sigma_m$ for all $n, m \in \mathbb{N}$, and using this, prove that the following limit exists:
$$d \leq \lim_{n\to\infty} \sigma_n^{1/n} \leq 2d - 1.$$
Can you find better bounds for this limit?

2. (**Kolmogorov's zero-one law and ergodicity**) Consider a countably infinite graph $G = (V, E)$. Let $\Omega = \{0,1\}^E$ be endowed with the cylinder sigma-algebra $\mathcal{F}$. Define the tail sigma-algebra as
$$\mathcal{T} := \bigcap_{\substack{E' \subset E \\ \text{finite subset}}} \sigma\{\omega(e) \,|\, e \notin E'\}.$$
Events $A \in \mathcal{T}$ are called *tail events.*
   (a) Consider percolation on Ω. Show that $\{\exists \text{ an infinite cluster}\}$ is a tail event.
   (b) Show that for any $A \in \mathcal{F}$, there exists a sequence $A_n \in \mathcal{F}$ defined on finite sets $E_n \subset E$ such that $\mathbb{P}[A \Delta A_n] \to 0$ as $n \to \infty$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.
   (c) Let $(X(e))_{e\in E}$ be i.i.d. random variables on Ω. Using (b), show that $A \in \mathcal{T}$ implies $\mathbb{P}[A] \in \{0,1\}$.
   (d) Suppose now that $G = \mathbb{Z}^d$. We say that $A \in \mathcal{F}$ is *translation invariant* if
$$A = \pi_x A := \{\pi_x(\omega) \,|\, \omega \in A\} \qquad \text{for all translations} \quad \pi_x(y) := y + x, \quad x, y \in V,$$
   where $\pi_x(\omega) := (\omega(\pi_{-x}(e)))_{e\in E}$. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is called *ergodic* if we have $\mathbb{P}[A] \in \{0,1\}$ for all translation invariant events $A \in \mathcal{F}$. Prove that percolation on Ω is ergodic.

3. (**Influence and Russo's formula**) Consider percolation $\mathbb{P}_p$ on edges of a finite graph $G = (V, E)$ with $p \in (0,1)$. The *(conditional) influence* of an edge $e \in E$ for an event A is
$$I_A(e) := \mathbb{P}_p[A \,|\, \omega(e) = 1] - \mathbb{P}_p[A \,|\, \omega(e) = 0].$$
   (a) Show that if A is an increasing event, then
$$I_A(e) = \mathbb{P}_p[\omega^e \in A] - \mathbb{P}_p[\omega_e \in A] = \mathbb{P}_p[\mathbf{1}_A(\omega^e) \neq \mathbf{1}_A(\omega_e)],$$
$$\text{where} \qquad \omega^e(e') = \begin{cases} 1 & \text{if } e' = e, \\ \omega(e') & \text{if } e' \neq e, \end{cases} \qquad \omega_e(e') = \begin{cases} 0 & \text{if } e' = e, \\ \omega(e') & \text{if } e' \neq e. \end{cases}$$

(b) An edge is called *pivotal* for A if $\mathbf{1}_A(\omega^e) \neq \mathbf{1}_A(\omega_e)$. Show that if A is increasing, then this is equivalent to $\omega^e \in A$ and $\omega_e \notin A$. Then show that for any increasing event A, we have

$$\liminf_{\delta \to 0} \frac{\mathbb{P}_{p+\delta}[A] - \mathbb{P}_p[A]}{\delta} \geq \mathbb{E}[\#\{e \in E \text{ pivotal for } A\}].$$

[Hint: Perturb $\mathbb{P}[\omega(e) = 1]$ first only for edges in a finite box and use Russo's formula there. Then take a limit.]

4. ($\star$) (**Kirchoff's Matrix-Tree Theorem**) Consider Wilson's algorithm on a finite connected graph $G = (V, E)$ for generating a sample of a spanning tree T on G, with some enumeration of the vertices $V = \{v_1, v_2, \ldots, v_{n-1}, v_n, t\}$, with t the last (root) vertex, and with transition probabilities $p_{u,v}$ for the random walk. For $A \subset V$, let $\mathcal{G}_A$ be the Green's function (cf. Exercise 5, Set 2).

(a) Suppose $(\eta_1, \eta_2, \ldots, \eta_m)$ are the branches of T produced by the algorithm, generated in this order. Concatenate all of them to obtain the ordered set $\{u_1, u_2, \ldots, u_{\ell_m}\}$ of vertices visited by the algorithm; so as vertex-paths, $\eta_1 = (u_1, u_2, \ldots, u_{\ell_1})$, $\eta_2 = (u_{\ell_1+1}, u_{\ell_1+2}, \ldots, u_{\ell_2})$, …, so that $u_{\ell_1} = t$ is the root (by Wilson's algorithm) and the last step of each of the other walks $\eta_2, \eta_3, \ldots, \eta_m$ belongs to the already generated collection (in particular, the vertices in the ordered set $\{u_1, u_2, \ldots, u_{\ell_m}\}$ are not distinct). Show that the probability to obtain this sample is

$$\prod_{k=1}^{m} \prod_{j=\ell_{k-1}+1}^{\ell_k - 1} p_{u_j, u_{j+1}} \mathcal{G}_{A_j}(u_j, u_j),$$

where $A_1 = \{u_0\}$, $A_2 = \{u_0, u_1\}$, $A_3 = \{u_0, u_1, u_2\}$, …, $A_n = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$, writing the root as $u_0 = t$ and using the convention that $\ell_0 = 0$.

(b) Show that for any $A \subset V$ and $u \in V \setminus A$, we have

$$\mathcal{G}_A(u, u) = \frac{\det(-\Delta^{(A \cup \{u\})})}{\det(-\Delta^{(A)})}$$

where Δ is the Laplacian operator (cf. Exercise 5, Set 2) regarded as an $(|V| \times |V|)$-matrix, and $\Delta^{(B)}$ is the minor of Δ obtained by removing the rows and columns associated to vertices in B.

(c) Let $\lambda_1, \lambda_2, \ldots, \lambda_{|V|-1}$ be the non-zero eigenvalues of Δ (all are negative). Show that

$$\#\{T \subset G \mid T \text{ is a spanning tree of } G\} = \frac{1}{|V|} \prod_{j=1}^{|V|-1} (-\lambda_j).$$

5. ($\star$) (**Markov Chain - Tree Theorem**) Let X be an irreducible Markov chain in a finite state space S, with associated oriented graph $\vec{H} = (S, \vec{E})$ having vertex set S, edge set $\vec{E} = \{\overrightarrow{\langle e_-, e_+ \rangle} \mid p_{e_-, e_+} > 0\}$, and p_{e_-, e_+} being the transition probabilities of X. The *Markov Chain - Tree Theorem* asserts that the stationary measure of X on S is proportional to

$$\pi_x \propto \sum_{A \in \mathcal{A}_x} \alpha(A) \qquad \text{where} \qquad \alpha(A) = \prod_{\overrightarrow{\langle e_-, e_+ \rangle} \in A} p_{e_-, e_+}, \tag{1}$$

where $\mathcal{A}_x$ is the set of spanning arborescences of $\vec{H}$ with root x. This can be proven using an auxiliary Markov chain on the space $\mathcal{A} = \prod_{x \in S} \mathcal{A}_x$ as follows. For each $\overrightarrow{\langle r, x \rangle} \in \vec{E}$ and $A \in \mathcal{A}_r$, denote

$$F(A, \overrightarrow{\langle r, x \rangle}) := (A \cup \{\overrightarrow{\langle r, x \rangle}\}) \setminus \{\overrightarrow{\langle x, y \rangle}\} \in \mathcal{A}_x,$$

where y is the unique neighbor vertex of x such that $\overrightarrow{\langle x, y \rangle} \in A$. Define a Markov chain Y on the space $\mathcal{A} = \prod_{x \in S} \mathcal{A}_x$ via the transition probabilities

$$\mathbb{P}\big[Y_n = F(A, \overrightarrow{\langle r, x \rangle}) \,|\, Y_{n-1} = A\big] := p_{r,x}, \quad n \in \mathbb{N}, \ \overrightarrow{\langle r, x \rangle} \in \vec{E}, \ A \in \mathcal{A}_r.$$

(a) Show that the Markov chain Y on $\mathcal{A}$ is irreducible.

(b) Show that the weight $\alpha(A)$ gives the stationary measure for Y.

(c) Conclude that (1) gives the stationary measure for the original Markov chain X.

Upon finding mistakes and/or typos, please contact me!