Advanced Topics in Applied Probability

- Introduction to Lattice Models

Exercises denoted by ($\star$) are harder or use additional theory.

Exercises – Set 2

Throughout, we consider a finite or countably infinite connected graph $G = (V, E)$ (assuming for ease that there are no loops nor multiple edges and that $\deg(v) < \infty$ for all $v \in V$), with edge weights (conductances) $w \colon E \to (0, \infty)$ denoted as w_e for $e \in E$ or $w_{u,v}$ for $\langle u, v\rangle \in E$, where we set $w_{u,v} := 0$ if $\langle u, v\rangle \notin E$.

Let $f \colon V \to \mathbb{R}$ be a function on the vertices. Recall that we call f harmonic in $U \subset V$ if

$$f(u) = \sum_{v \in V} p_{u,v} f(v) \quad \text{for all } u \in U, \qquad \text{where} \qquad p_{u,v} = \frac{w_{u,v}}{W_u} \qquad \text{and} \qquad W_u := \sum_{v \in V} w_{u,v}.$$

For $U \subset V$, we define its boundary as $\partial U := \{v \in V \mid \langle v, u\rangle \in E \text{ and } v \notin U,\, u \in U\}$.

1. (a) (**Maximum principle**) Assume that f is harmonic in a connected set $U \subset V$. Show that if

   $$f(u_0) = \sup_{v \in V} f(v) \quad \text{for some } u_0 \in U,$$

   then $f(u) = f(u_0)$ for all $u \in \overline{U} := U \cup \partial U$.

   (b) Assume that f is harmonic in V and G is finite. Show that f is constant.

2. (**Hitting probabilities**) For each $u \in V$, let $\mathbb{P}_u$ be the probability measure of the random walk $X = (X_n)_{n \in \mathbb{Z}_{\geq 0}}$ on G started at $X_0 = u$, and for a subset $A \subset V$, let $\tau_A := \inf\{n \in \mathbb{Z}_{\geq 0} \mid X_n \in A\}$.

   (a) Let $U \subset V$ and fix $s \in U$. Show that the function

   $$u \mapsto \mathbb{P}_u[X_n = s \text{ for some } 0 \leq n < \tau_{V \setminus U}]$$

   is harmonic on $U \setminus \{s\}$. What are its boundary values?

   (b) Assume that G is finite. Let $A, B \subset V$ such that $A \cap B = \emptyset$. Show that the function

   $$u \mapsto \mathbb{P}_u[\tau_B < \tau_A]$$

   is harmonic on $V \setminus (A \cup B)$. What are its boundary values?

3. (**Existence and Uniqueness for Dirichlet problem**) Assume that G is finite. Let $A \subset V$ be non-empty and let $g \colon A \to \mathbb{R}$ be given. Show that there exists a unique function $f \colon V \to \mathbb{R}$ such that

   - f is harmonic on $V \setminus A$,
   - $f(u) = g(u)$ for all $u \in A$.

   [Hint: for uniqueness, use maximum principle; for existence, use random walk]

4. ($\star$) (**Martingale argument**) Let $f\colon V \to \mathbb{R}$ be harmonic on $U \subset V$. Show that

$$(f(X_{n\wedge\tau_{V\setminus U}}))_{n\in\mathbb{Z}_{\geq 0}}$$

is a martingale for the random walk $X = (X_n)_{n\in\mathbb{Z}_{\geq 0}}$ on G.
Using this, can you find an alternative proof for the Uniqueness in Exercise 3?

5. (**Green's function**) Let $A \subset V$ be a non-empty subset of vertices of G. We define the *Green's function*

$$\mathcal{G}_A\colon V\times V \to [0,\infty), \qquad \mathcal{G}_A(u,v) := \mathbb{E}_u\Big[\sum_{n=0}^{\tau_A-1} \mathbf{1}\{X_n = v\}\Big].$$

(Note that for $A=\emptyset$, the Green's function $\mathcal{G}(u,v) = \mathcal{G}_\emptyset(u,v) = \sum_{n=0}^{\infty} p_{u,v}$ may be infinite.)

(a) Show that $W_u\,\mathcal{G}_A(u,v) = W_v\,\mathcal{G}_A(v,u)$, where $W_u := \sum_{v\in V} w_{u,v}$.

(b) Show that, for fixed $v\in V$ and $u\notin A$, we have

$$\Delta_u \mathcal{G}_A(u,v) = \begin{cases} -\mathbf{1}\{u=v\}, & u\notin A,\\ 0, & u\in A,\end{cases}$$

where Δ_u denotes the discrete *Laplacian operator* acting on the variable u, defined for functions $f\colon V\to\mathbb{R}$ as

$$\Delta f(u) := \sum_{\langle v,u\rangle\in E} p_{u,v}(f(v)-f(u)) = \sum_{v\in V} p_{u,v}(f(v)-f(u)),$$

where $p_{u,v}$ is the transition probability of X, related to $w_{u,v}$ as $p_{u,v} = w_{u,v}/W_u$.

(c) Show that $f\colon V\to\mathbb{R}$ is harmonic in $U\subset V$ if and only if $\Delta f(u) = 0$ for all $u\in U$.

6. (**Potential function and effective resistance**) Let $s,t\in V$ be the source and sink in (G,w). Let $(i_{u,v})$ be a current satisfying the Kirchoff's laws, let $(\phi_{u,v})$ be the associated potential difference obtained from Ohm's law, and let $\phi\colon V\to\mathbb{R}$ be an associated potential function.

(a) Show that ϕ is harmonic on $V\setminus\{s,t\}$.

(b) Let's normalize ϕ via $\phi(s)=0$. Show that then we have

$$\phi(t) = \frac{\mathcal{E}(i)}{|i|}, \qquad \text{where} \qquad \mathcal{E}(i) = \frac{1}{2}\sum_{u\sim v} w_{u,v}(\phi(v)-\phi(u))^2 \qquad \text{and} \qquad |i| := I_s := \sum_{v\in V} i_{s,v}.$$

(c) ($\star$) Show that the effective resistance $R_{\text{eff}} := \frac{1}{|i|}(\phi(t)-\phi(s))$ of (G,w) can be expressed as

$$R_{\text{eff}}(s,t) = \frac{\mathcal{G}_{\{t\}}(s,s)}{W_s}, \qquad \text{where} \qquad W_u := \sum_{v\in V} w_{u,v},$$

where $\mathcal{G}$ is the Green's function from Exercise 5.

Upon finding mistakes and/or typos, please contact me!