Advanced Topics in Applied Probability

- Introduction to Lattice Models

Exercises denoted by ($\star$) are harder or use additional theory.

Exercises – Set 1

Throughout, we consider a finite connected graph $G = (V, E)$ (assuming for ease that there are no loops nor multiple edges), with edge weights (conductances) $w\colon E \to (0, \infty)$ denoted as w_e for $e \in E$.

Considering (G, w) as an electrical network, we also let $s, t \in V$ be the source and sink in (G, w), giving rise to the current i as the unique solution to Kirchoff's laws with size $|i| = 1$. We let ϕ be an associated potential function (from Ohm's law, determined up to an additive constant).

1. (**Trees**) Let T be a tree with n vertices. Show that T has $n - 1$ edges.

2. (**Dual tree and Euler's formula**) Suppose that G is planar with face set F. For each spanning tree $T \subset G$, we associate a subgraph $T^* \subset G^*$ of the dual graph of G so that the vertices of T^* are all vertices of G^* and the edges of T^* are all those edges of G^* that don't cross an edge in T. Show that T^* is a spanning tree on G^*. Then note that this gives a proof for Euler's formula
$$|V| - |E| + |F| = 2.$$

3. ($\star$) (**Gossip**) There are a number $n \geq 4$ of elderly professors, each of which knows some item of gossip that the others don't know. They communicate by telephone and in each conversation they part with all the gossip they know. Show that $2n - 4$ calls are needed before each of them knows everything.

4. (**Bushes and trees**) Consider the electrical network (G, w). For simplicity, let's assume that $w_e = 1$ for all edges $e \in E$. Show that the effective resistance $R_{\text{eff}} = R_{\text{eff}}(s,t) = \phi(t) - \phi(s)$ of (G, w) can be expressed as
$$R_{\text{eff}}(s,t) = \frac{\#\{\text{bushes } B = T_s \cup T_t \text{ on } G\}}{\#\{\text{spanning trees } T \subset G\}},$$
where a *bush* is defined as a spanning forest $B \subset G$ such that B consists of exactly two trees T_s and T_t, such that $s \in T_s$ and $t \in T_t$. Recall that if we fix an edge $\langle a, b\rangle \in E$, then by erasing or adding this edge, we find a bijection
$$\begin{aligned} & \{\text{bushes } B = T_s \cup T_t \text{ on } G \text{ such that } a \in T_s, b \in T_t \text{ or } b \in T_s, a \in T_t\} \\ \longleftrightarrow \quad & \{\text{spanning trees } T \subset G \text{ such that } \langle a, b\rangle \in \gamma_{s,t}(T)\}, \end{aligned}$$
where $\gamma_{s,t}(T)$ is the path in T from s to t.

5. (**Average number of transitions for RW**) Let $X = (X_n)_{n \in \mathbb{Z}_{\geq 0}}$ be the random walk on G started at $X_0 = s$. Recall that the transition probabilities $p_{u,v}$ of X are related to $w_{u,v}$ as as $p_{u,v} = w_{u,v}/W_u$, where $W_u := \sum_{v \in V} w_{u,v}$. For an edge $\langle a, b\rangle \in E$, set
$$u_{a,b} := \mathbb{E}_s[\#\{0 \leq n < \tau_t \,|\, X_n = a \text{ and } X_{n+1} = b\}] - \mathbb{E}_s[\#\{0 \leq n < \tau_t \,|\, X_n = b \text{ and } X_{n+1} = a\}],$$
where $\tau_t := \inf\{n \in \mathbb{Z}_{\geq 0} \,|\, X_n = t\}$. Show that $u_{a,b} = i_{a,b}$ for all $\langle a, b\rangle \in E$.

6. (**Network simplifications**)

(a) Prove the *series law*: in an electrical network with distinct vertices $\{s, v, t\}$ and two edges $\{\langle s, v\rangle, \langle v, t\rangle\}$ with respective resistances $r_1 := 1/w_{s,v}$ and $r_2 := 1/w_{v,t}$, we have $R_{\text{eff}} = r_1 + r_2$.

(b) Prove the *parallel law*: in an electrical network with distinct vertices $\{s, t\}$ and two parallel edges $\{\langle s, t\rangle, \langle s, t\rangle\}$ with respective resistances $r_1 := 1/w_{s,t}$ and $r_2 := 1/w_{s,t}$ (abusing notation), we have $\frac{1}{R_{\text{eff}}} = \frac{1}{r_1} + \frac{1}{r_2}$.

(c) Find a condition for the resistances in the *star-triangle transformation* (see the figure below) so that the transformation is meaningful as a local modification of an electrical network.

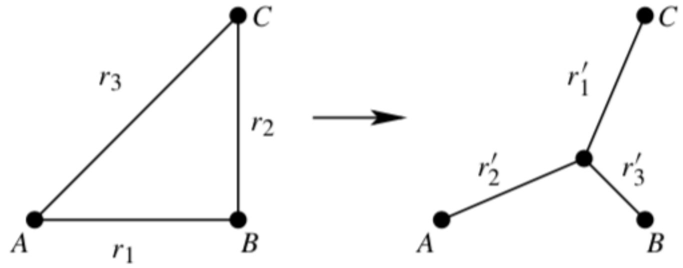

Figure 1.6 Edge-resistances in the star–triangle transformation. The triangle T on the left is replaced by the star S on the right, and the corresponding resistances are denoted as marked.

Figure from: G. Grimmett: Probability on Graphs,
`http://www.statslab.cam.ac.uk/~grg/books/pgs2e-draft.pdf`

Upon finding mistakes and/or typos, please contact me!