Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 9

In this exercise sheet, we will discuss the ingredients to prove that SLE(κ) is almost surely generated by a (continuous transient) curve, for any $\kappa \in (0,\infty) \setminus \{8\}$. Unfortunately, the proof fails for $\kappa = 8$, as we'll see.

Theorem. *Let $\kappa \in (0,\infty) \setminus \{8\}$. The SLE($\kappa$) is almost surely generated by a curve γ.*

Notation:

- $(g_t)_{t\geq 0}$ is the Loewner chain associated to the SLE with the following parameterization:
$$\partial_t g_t(z) = \frac{a}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad z \in H_t,$$
where $a = 2/\kappa$, the driving function is $W_t = -B_t$, and K_t are the hulls and $H_t := \mathbb{H} \setminus K_t$.
- $(h_s)_{s\geq 0}$ is the solution to the *reverse* LE (this is almost the same as "backward LE")
$$\partial_t h_t(z) = \frac{-a}{h_t(z) - W_t}, \qquad h_0(z) = z, \qquad z \in \mathbb{H}.$$
- We denote $f_t(z) := g_t^{-1}(z)$ and $\hat{f}_t(z) := g_t^{-1}(z + W_t)$. Note that LE for g_t gives an ODE for $(f_t)_{t\geq 0}$:
$$\partial_t f_t(w) = \frac{-a\, f_t'(w)}{w - W_t}, \qquad f_0(w) = w, \qquad w \in \mathbb{H}. \tag{1}$$
- For all $(y,t) \in [0,\infty) \times [0,1]$, we denote by $V(y,t) := \hat{f}_t(\mathrm{i}y)$.
- We make a dyadic partitioning of $t \in [0,1]$:
$$\mathcal{D}_{2n} := \{k2^{-2n} \mid k = 0,1,\ldots,2^{2n}\}, \quad n \in \mathbb{N}.$$
We are going to control the values of V when $y = 2^{-n} > 0$ is small and the time scale is as in $\mathcal{D}_{2n}$.

Our goal: *By* [2, Proposition 4.28]*, the Theorem follows if we show that V is well-defined and continuous as $y \searrow 0$, so that the curve*
$$\gamma(t) := \lim_{y\searrow 0} V(y,t) = \lim_{y\searrow 0} g_t^{-1}(\mathrm{i}y + W_t)$$
generating the hulls $(K_t)_{t\in[0,1]}$ is well-defined and f_t extends continuously to $\overline{\mathbb{H}}$.

To establish the goal, it suffices to find a bound function $\delta\colon [0,\infty) \to [0,\infty)$ such that $\lim_{\epsilon\searrow 0} \delta(\epsilon) = 0$ and
$$|V(y,t) - V(x,s)| \leq \delta(x + y + |t-s|), \qquad t,s, \in [0,1], \quad x,y > 0. \tag{2}$$
By [2, Lemma 4.32], it turns out that to get this estimate, the following ingredients are sufficient:

(a): There exists a sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_n > 0$, and $\lim_{n\to\infty} r_n = 0$, and $\lim_{n\to\infty} \frac{\sqrt{n}}{\log r_n} = 0$, and

(b): $|\hat{f}_t'(\mathrm{i}\,2^{-n})| \leq 2^n r_n$, for all $t \in \mathcal{D}_{2n}$, and

(c): there exists $c \in (0,\infty)$ such that $|W_{t+s} - W_t| \leq c\sqrt{n}\,2^{-n}$, for all $t \in [0,1]$ and $s \in [0, 2^{-2n}]$.

We'll see why in Exercises 6–9 below.

Exercises, Part 1: *We establish properties* **(a)**, **(b)**, **(c)** *for the SLE.*

0. Check that for *fixed* time $t \geq 0$, the function $z \mapsto \hat{f}_t'(z)$ and the function $z \mapsto h_t'(z)$ have the same law (but it is not true that the joint law of $(\hat{f}_t'(z))_{t\geq 0}$ and the joint law of $(h_t'(z))_{t\geq 0}$ would be the same!). Therefore, instead of estimating $|\hat{f}_t'(z)|$, it suffices to estimate $|h_t'(z)|$.

1. **Set-up:** For fixed $z \in \mathbb{H}$, we consider the process $Z_t = h_t(z) - W_t$ solving the SDE

$$Z_0 = z, \qquad \mathrm{d}Z_t = -\frac{a}{Z_t}\,\mathrm{d}t + \mathrm{d}B_t, \qquad t \geq 0.$$

(Because $t \mapsto \operatorname{Im} Z_t$ is increasing, this is OK for all times.) This is more useful after the time-change $\iota(t) := \inf\{s \geq 0 \mid \frac{\operatorname{Im} Z_s}{\operatorname{Im}(z)} = e^{at}\}$. Then the imaginary part of $\tilde{Z}_t := Z_{\iota(t)}$ is exponentially increasing:

$$\operatorname{Im} \tilde{Z}_t = \operatorname{Im}(z) e^{at}, \qquad \mathrm{d}(\operatorname{Re} \tilde{Z}_t) = -a(\operatorname{Re} \tilde{Z}_t)\,\mathrm{d}t + |\tilde{Z}_t|\,\mathrm{d}\tilde{B}_t,$$

where $\tilde{B}$ is standard 1D BM. It is useful to consider

$$\tilde{K}_t := \frac{\operatorname{Re} \tilde{Z}_t}{\operatorname{Im} \tilde{Z}_t} = \frac{e^{-at}\operatorname{Re} \tilde{Z}_t}{\operatorname{Im}(z)}, \qquad \tilde{L}_t := \sqrt{\tilde{K}_t^2 + 1}$$

which satisfy the SDEs

$$\mathrm{d}\tilde{K}_t = -2a\tilde{K}_t\,\mathrm{d}t + \tilde{L}_t\,\mathrm{d}\tilde{B}_t, \qquad \mathrm{d}\tilde{L}_t = \left(\frac{1}{2}\tilde{L}_t - \left(\frac{1}{2} + 2a\right)\frac{\tilde{K}_t^2}{\tilde{L}_t}\right)\mathrm{d}t + \tilde{K}_t\,\mathrm{d}\tilde{B}_t.$$

To simplify this, we can write

$$\tilde{J}_t := \sinh^{-1} \tilde{K}_t \qquad \Longrightarrow \qquad \begin{cases} \tilde{K}_t = \sinh \tilde{J}_t \\ \tilde{L}_t = \cosh \tilde{J}_t, \end{cases} \qquad \text{and} \qquad \mathrm{d}\tilde{J}_t = -\left(\frac{1}{2} + 2a\right)\tanh \tilde{J}_t\,\mathrm{d}t + \mathrm{d}\tilde{B}_t.$$

Finally, the process $\tilde{h}_t := h_{\iota(t)}$ satisfies

$$\partial_t \log|\tilde{h}_t'(z)| = a\frac{(\operatorname{Re} \tilde{Z}_t)^2 - (\operatorname{Im} \tilde{Z}_t)^2}{|\tilde{Z}_t|^2} = a\left(1 - \frac{2}{\tilde{L}_t^2}\right) = a\left(1 - \frac{2}{(\cosh \tilde{J}_t)^2}\right) = a\left(2(\tanh \tilde{J}_t)^2 - 1\right)$$

Task: Prove that the following process is a *martingale*:

$$\tilde{M}_t = |\tilde{h}_t'(z)|^p\,(\operatorname{Im} \tilde{Z}_t)^{p-\frac{r}{a}}\,(\sin \tilde{\Theta}_t)^{-2r}, \qquad \text{where} \qquad \tilde{\Theta}_t := \arg(\tilde{Z}_t)$$

and $(p, r) \in \mathbb{R}^2$ satisfy $r^2 - (1+2a)r + ap = 0$. [Hint: Identify $\sin \tilde{\Theta}_t$ with an expression involving $\tilde{J}_t$.]

2. **Task:** Prove that

$$\mathbb{E}\left[|\tilde{h}_t'(z)|^p\,(\sin \tilde{\Theta}_t)^{-2r}\right] = \left(\frac{\operatorname{Im}(z)}{|z|}\right)^{-2r} \exp\left(-at\left(p - \frac{r}{a}\right)\right)$$

and if $p, r \geq 0$, then we have

$$\mathbb{P}\left[|\tilde{h}_t'(z)| \geq \lambda\right] \leq \lambda^{-p}\left(\frac{\operatorname{Im}(z)}{|z|}\right)^{-2r} \exp\left(-at\left(p - \frac{r}{a}\right)\right), \qquad \lambda > 0. \tag{3}$$

3. Using the estimate (3), one can obtain the following estimate for the derivative h_t' in the original time parameterization (see [2, Corollary 7.3] and [1, Corollary 5.1]): For every $r \in [0, 1+2a]$, there exists a constant $c(\kappa, r) \in (0, \infty)$ such that for all $t \in [0,1]$, $x \in \mathbb{R}$, and $y \in (0,1]$ and $\lambda \in [e^6, \frac{1}{y}]$, we have

$$\mathbb{P}\left[|h_t'(x + \mathrm{i}y)| \geq \lambda\right] \leq c\lambda^{-p}\left(\frac{y}{|x + \mathrm{i}y|}\right)^{-2r} \delta(y, \lambda), \tag{4}$$

where $p = p(r) = \frac{1}{a}\left((1+2a)\,r - r^2\right) \geq 0$ and

$$\delta(y,\lambda) = \begin{cases} \lambda^{-p+\frac{r}{a}}, & p - \frac{r}{a} > 0, \\ 1 + \log\frac{1}{\lambda y}, & p - \frac{r}{a} = 0, \\ y^{p-\frac{r}{a}}, & p - \frac{r}{a} < 0. \end{cases}$$

Recall that $a = 2/\kappa$. We still have freedom to choose the parameter $r \geq 0$. Note that by choosing $r = r_0 = \frac{1+4a}{4} = \frac{1}{4} + \frac{2}{\kappa}$, which maximises the quantity $2p - \frac{r}{a}$, we have

$$2p(r_0) - \frac{r_0}{a} = \kappa r_0 \left(\left(\frac{1}{2} + \frac{4}{\kappa} \right) - r_0 \right) = \kappa r_0^2 \geq 2$$

and $\kappa r_0^2 = 2$ if and only if $\kappa = 8$.

Task: Verify that if $\kappa \in (0,\infty) \setminus \{8\}$, then choosing these $(r_0, p(r_0))$, the estimate (4) gives for $x = 0$, $y = 2^{-n}$, and $\lambda = 2^{n(1-\alpha)}$, with $n \in \mathbb{N}$ is large enough and $\alpha \in \left(0, 1 - \frac{2}{2p(r_0) - r_0/a}\right)$ small enough,

$$\mathbb{P}\left[|h_t'(\mathrm{i}2^{-n})| \geq 2^{n(1-\alpha)} \right] \leq c\, 2^{-n(2+\varepsilon)}, \tag{5}$$

for some $\varepsilon > 0$. [NB: There are two different cases: $\kappa < 8$ and $\kappa > 8$.]

4. **Task:** Using the dyadic partitioning $\mathcal{D}_{2n}$ for $t \in [0,1]$, show that (5) implies that for any α small enough, there exists a random variable C such that almost surely, $C < \infty$ and

$$|h_t'(\mathrm{i}2^{-n})| \leq C\, 2^{n(1-\alpha)}, \qquad t \in \mathcal{D}_{2n}, \quad n \in \mathbb{N}.$$

5. **Task:** Conclude that all properties **(a)**, **(b)**, **(c)** indeed hold.

Exercises, Part 2: *Why do properties* **(a)**, **(b)**, **(c)** *imply our goal?*

Let's begin by arguing backwards: Let $t \in [0,1]$, $s \in [0, 2^{-2n}]$ and $0 < x, y \leq 2^{-n}$ and write

$$|\hat{f}_t(\mathrm{i}y) - \hat{f}_{t+s}(\mathrm{i}x)| \leq |\hat{f}_t(\mathrm{i}y) - \hat{f}_t(\mathrm{i}2^{-n})| + |\hat{f}_t(\mathrm{i}2^{-n}) - \hat{f}_{t+s}(\mathrm{i}2^{-n})| + |\hat{f}_{t+s}(\mathrm{i}2^{-n}) - \hat{f}_{t+s}(\mathrm{i}x)|. \tag{6}$$

6. **Task:** Estimate the middle term in (6) in terms of $\sup\limits_{u \in [t,t+s]} |\hat{f}_t'(\mathrm{i}2^{-n})|$, by using the ODE (1) for f_t.

7. **Task:** Estimate the first term in (6) in terms of $\sup\limits_{v \in [2^{-j}, 2^{-j+1}]} |\hat{f}_t'(\mathrm{i}v)|$, with a sum over $j = n, n+1, \ldots$.

   (The third term can be estimated similarly.)

8. **Tools:** Using property (b), the ODE (1) for f_t, and Gronwall's Area theorem, one can show that

$$|f_t'(\mathrm{i}2^{-n} + W_{k\,2^{-2n}})| \leq e^6 2^n r_n, \qquad t \in [k\,2^{-2n}, (k+1)\,2^{-2n}], \quad k = 0, 1, \ldots, 2^{-2n} - 1, \quad n \in \mathbb{N}.$$

   Using Koebe distortion theorem, one can show that for any conformal map φ on $\mathbb{H}$, we have

$$|\varphi'(w)| \leq 144^{\frac{|z-w|}{y}+1} |\varphi'(z)|, \qquad \mathrm{Im}(z), \mathrm{Im}(w) \geq y > 0.$$

   Task: Using these facts and property (c), prove that there exists $\beta > 0$ such that

$$|\hat{f}_t'(\mathrm{i}2^{-n})| \leq c e^{\beta\sqrt{n}} 2^n r_n, \qquad t \in [0,1], \quad n \in \mathbb{N},$$

   and furthermore,

$$|\hat{f}_t'(\mathrm{i}y)| \leq c e^{\beta\sqrt{n}} 2^n r_n, \qquad t \in [0,1], \quad y \in [2^{-n}, 2^{-n+1}], \quad n \in \mathbb{N}. \tag{7}$$

9. **Task:** Conclude using (7) that all terms in the expression (6) have the desired bound, so (2) holds.

References

[1] Antti Kemppainen. Schramm-Loewner evolution. SpringerBriefs in Mathematical Physics, 2017. `http://wiki.helsinki.fi/display/mathphys/sle-book`

[2] Gregory Lawler. Conformally Invariant Processes in the Plane. American Mathematical Society, 2005. `http://pi.math.cornell.edu/~lawler/book.ps`