Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 8

In this exercise sheet, we will prove the following result in several steps. This result describes the probability of the SLE curve to come close to a given point z, in terms of the conformal radius $\mathrm{crad}_{H(z)}(z)$.

Theorem. *Let $\kappa \in (0,8)$. Consider the SLE(κ) curve γ in $(\mathbb{H}; 0, \infty)$. Fix $z \in \mathbb{H}$ and $\varepsilon \in (0, 1/2]$. Then there exists a constant $\alpha = \alpha(\kappa) > 0$ independent of z and ε such that*

$$\mathbb{P}[\mathrm{crad}_{H(z)}(z) \leq \varepsilon \, \mathrm{crad}_{\mathbb{H}}(z)] = c_* \, \varepsilon^{2-d} \left(\sin(\arg z)\right)^{4a-1} (1 + O(\varepsilon^\alpha)) \, , \tag{1}$$

where

$$a = \frac{2}{\kappa}, \qquad d = 1 + \frac{\kappa}{8}, \qquad c_* = 2 \left(\int_0^\pi (\sin u)^{4a} \, \mathrm{d}u \right)^{-1}$$

are κ-dependent constants and $H(z)$ denotes the connected component of the complement $\mathbb{H} \setminus \gamma[0,\infty)$ of the whole curve that contains the point z.

Notation:

- $(g_t)_{t \geq 0}$ is the Loewner chain associated to the SLE with the following parameterization:

  $$\partial_t g_t(z) = \frac{a}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad z \in H_t,$$

  where K_t are the hulls and $H_t := \mathbb{H} \setminus K_t$ (so H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$), and the driving function is $W_t = -B_t$ (here, B is a standard 1D BM and the minus sign is for convenience). The swallowing time of the point z is $\tau_z := \inf\big\{ s > 0 \,\big|\, |g_s(z) - W_s| = 0 \big\}$.

- The time evolution of the point z is governed by the (complex-valued Bessel process) solving the SDE

  $$Z_0 = z, \qquad \mathrm{d}Z_t = \frac{a}{Z_t} \, \mathrm{d}t + \mathrm{d}B_t, \qquad t < \tau_z.$$

- The argument $\Theta_t := \arg(Z_t)$ of this time evolution is a useful quantity. It satisfies the SDE

  $$\Theta_0 = \arg(z), \qquad \mathrm{d}\Theta_t = (1-2a) \frac{(\mathrm{Re}\, Z_t)(\mathrm{Im}\, Z_t)}{|Z_t|^4} \, \mathrm{d}t - \frac{\mathrm{Im}\, Z_t}{|Z_t|^2} \, \mathrm{d}B_t, \qquad t < \tau_z.$$

Exercises:

1. Recall that the time evolution of the conformal radius is encoded in the process satisfying

$$\Upsilon_t = \frac{1}{2}\mathrm{crad}_{H_t}(z), \qquad \frac{\mathrm{d}\Upsilon_t}{\Upsilon_t} = -\frac{2a\,(\mathrm{Im}\,Z_t)^2}{|Z_t|^4}\,\mathrm{d}t, \qquad t < \tau_z,$$

and if $\tau_z < \infty$, we define

$$\Upsilon_t := \Upsilon_{\tau_z} = \frac{1}{2}\mathrm{crad}_{H(z)}(z) \quad \text{for all } t \geq \tau_z,$$

so that

$$\Upsilon_\infty := \lim_{t\to\infty} \Upsilon_t = \frac{1}{2}\mathrm{crad}_{H(z)}(z).$$

Prove that in the *radial time-paramaterization*

$$\sigma(t) := \inf\{s > 0 \mid \log\left(\frac{\Upsilon_0}{\Upsilon_s}\right) = 2at\}$$

(with $\hat{B}$ standard 1D BM) the processes $\hat{\Upsilon}_t := \Upsilon_{\sigma(t)}$ and $\hat{\Theta}_t := \Theta_{\sigma(t)}$ satisfy

$$\hat{\Upsilon}_t = e^{-2at}\Upsilon_0, \qquad \mathrm{d}\hat{\Theta}_t = (1-2a)\cot\hat{\Theta}_t\,\mathrm{d}t + \mathrm{d}\hat{B}_t, \qquad t < T := \inf\left\{s > 0 \,\middle|\, \hat{\Theta}_t \in \{0,\pi\}\right\}.$$

2. Check that when writing $\varepsilon = e^{-2as}$ for certain $s = s_\varepsilon > 0$, the left-hand side of Equation (1) reads

$$\mathbb{P}[\Upsilon_\infty \leq \epsilon\Upsilon_0] = \mathbb{P}[T \geq s_\varepsilon].$$

3. By finding a suitable function $f \in C^2(0,\pi)$, find a local martingale of the form

$$M_t = e^{-\lambda t} f(\hat{\Theta}_t),$$

which we expect to describe the conditional probability $\mathbb{P}[T > s \mid \hat{\Theta}_s = \theta]$.
[Hint: use Itô to find an ODE for f. It will look like something on the right-hand side of Equation (1).]

4. Given this *local* martingale, check that we can define a new probability measure $\mathbb{P}^*$ by tilting $\mathbb{P}$ by the martingale $M_{t\wedge T}$. You will find that under the new measure,

$$\mathrm{d}\hat{B}_t = (4a-1)\cot\hat{\Theta}_t\,\mathrm{d}t + \mathrm{d}B^*_t$$

where B^* is a standard BM under $\mathbb{P}^*$. What is the worry with M not being a true martingale?

5. Prove that

$$\mathbb{P}[T > s] = e^{s(\frac{1}{2}-2a)}(\sin\hat{\Theta}_0)^{4a-1}\mathbb{E}^*[(\sin\hat{\Theta}_s)^{1-4a}].$$

6. Finally, estimate the quantity $\mathbb{E}^*[(\sin\hat{\Theta}_s)^{1-4a}]$ using knowledge about the Bessel process $\hat{\Theta}$ under the measure $\mathbb{P}^*$. Combining this with Exercise 5 you will find the right-hand side of Equation (1).