Advanced Topics in Stochastic Analysis
- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 7

Let B be a standard 1D Brownian motion. Let $\delta \in \mathbb{R}$ and $x > 0$. A *Bessel process* of dimension δ started from x is the solution X to the SDE

$$\mathrm{d}Z_t^x = \frac{\delta - 1}{2Z_t^x}\,\mathrm{d}t + \mathrm{d}B_t, \qquad\qquad Z_0^x = x.$$

The solution exists and is unique up to the stopping time ("lifetime") $T_x := \sup\big\{t > 0 \;\big|\; \inf_{s\in[0,t]} Z_s^x > 0\big\}$.

1. Let $0 < x < y$ and let $(Z_t^x)_{t\in[0,T_x)}$ and $(Z_t^y)_{t\in[0,T_y)}$ be Bessel processes of dimension $\delta \in \mathbb{R}$ started from $x = Z_0^x$ and $y = Z_0^y$, respectively.

   (a) Let $\lambda > 0$. Show that $t \mapsto \lambda Z^x_{\lambda^{-2}t}$ is a Bessel process of dimension δ started from λx.

   (b) Show that $T_x \le T_y$ almost surely, and $Z_t^x < Z_t^y$ almost surely for all $t \in [0, T_x)$.

   (c) Show that, almost surely for all $t \in [0, T_x)$, we have

   $$\begin{aligned} Z_t^x &\ge B_t + x, \quad \text{if } \delta \ge 1 \\ Z_t^x &\le B_t + x, \quad \text{if } \delta \le 1. \end{aligned}$$

   (d) Show that $\mathbb{P}\big[\inf_{0\le t<T_x} Z_t^x > 0\big] = 1$ if and only if $\delta > 2$.

2. Let $0 < x < y$ and let $(Z_t^x)_{t\in[0,T_x)}$ and $(Z_t^y)_{t\in[0,T_y)}$ be Bessel processes of dimension

   $$\delta = 1 + \frac{4}{\kappa} \in (1, \infty), \qquad \text{with } \kappa > 0,$$

   started from $x = Z_0^x$ and $y = Z_0^y$, respectively. Define

   $$R_t := Z_t^y - Z_t^x, \qquad U_t := \frac{R_t}{Z_t^y}, \qquad N_t := F(U_t),$$

   where

   $$F(u) := \int_u^1 \frac{\mathrm{d}s}{s^{2-8/\kappa}(1-s)^{4/\kappa}}, \qquad u \in [0,1].$$

   Calculate the Itô differentials of these processes. In particular, show that N is a local martingale. You may use the fact that

   $$F''(u) + \left(\frac{2(1-4/\kappa)}{u} - \frac{4/\kappa}{1-u}\right)F'(u) = 0, \qquad u \in (0,1).$$

3. Let $\kappa \in (4,8)$ and let $(K_t)_{t\geq 0}$ be the SLE(κ) hulls. Prove that almost surely,

$$\bigcup_{t>0} \overline{K_t} = \overline{\mathbb{H}}.$$

4. Let U be a simply connected domain conformally equivalent to $\mathbb{D}$. Fix $z_0 \in U$. Let $\mathrm{crad}_U(z_0) := |f'(0)|$ be the *conformal radius* of U seen from z_0, where $f\colon \mathbb{D} \to U$ is a conformal bijection such that $f(0) = z_0$.

   (a) Prove that $\mathrm{crad}_U(z_0)$ is well-defined.

   (b) Prove that $\mathrm{crad}_U(z_0)$ is comparable to $\mathrm{dist}(z_0, \partial U)$, that is,

   $$\frac{1}{4}\,\mathrm{crad}_U(z_0) \leq \mathrm{dist}(z_0, \partial U) \leq \mathrm{crad}_U(z_0).$$

   [Hint: Koebe]

   (c) Let $(g_t)_{t\geq 0}$ be a Loewner chain and $(K_t)_{t\geq 0}$ the corresponding hulls. Show that

   $$\mathrm{crad}_{H_t}(z_0) = \frac{2\,\mathrm{Im}(g_t(z_0))}{|g_t'(z_0)|}, \qquad \text{for } z_0 \in H_t := \mathbb{H} \setminus K_t \text{ and } t \in [0, \tau_{z_0}).$$

5. Let $(g_t)_{t\geq 0}$ be a Loewner chain, $(K_t)_{t\geq 0}$ the corresponding hulls, and $(W_t)_{t\geq 0}$ the driving function.

   (a) Prove that

   $$\partial_t \log |g_t'(z)| = 2\,\frac{\big(\mathrm{Im}(g_t(z))\big)^2 - \big(\mathrm{Re}(g_t(z) - W_t)\big)^2}{\Big(\big(\mathrm{Im}(g_t(z))\big)^2 + \big(\mathrm{Re}(g_t(z) - W_t)\big)^2\Big)^2}, \qquad z \in H_t := \mathbb{H} \setminus K_t \text{ and } t \in [0, \tau_z).$$

   (b) Let $z \in \mathbb{H}$. Prove that

   $$t \mapsto \log \frac{|g_t'(z)|}{\mathrm{Im}(g_t(z))} := d_t(z)$$

   is increasing for $t \in [0, \tau_z)$. Conclude using Problem 4 that

   $$\frac{1}{2} \inf_{0\leq t<\tau_z} \mathrm{dist}(z, \partial H_t) \leq \lim_{t\nearrow\tau_z} e^{-d_t(z)} \leq 2 \inf_{0\leq t<\tau_z} \mathrm{dist}(z, \partial H_t).$$