Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 5

1. Prove that for a complex Brownian motion

   (a) starting from $z \in \mathbb{D}$, the hitting density on the circle $S^1 \ni e^{\mathrm{i}\theta}$ is
   $$\frac{1}{2\pi} \frac{1-|z|^2}{|e^{\mathrm{i}\theta} - z|^2},$$

   (b) starting from $z = x + \mathrm{i}y \in \mathbb{H}$, the hitting density on the real line $\mathbb{R} \ni u$ is
   $$\frac{1}{\pi} \frac{y}{(x-u)^2 + y^2}.$$

   (c) starting from $z \in \mathbb{H} \setminus \overline{\mathbb{D}}$, the first hitting density $p_z(e^{\mathrm{i}\theta})$ on the half-circle $\mathbb{H} \cap S^1 \ni e^{\mathrm{i}\theta}$ satisfies
   $$p_z(e^{\mathrm{i}\theta}) = \frac{2}{\pi} \frac{\mathrm{Im}(z)}{|z|^2} \sin(\theta) \left(1 + \mathcal{O}(|z|^{-1})\right), \quad \text{as } z \to \infty.$$

2. Let $K \subset \overline{\mathbb{H} \cap \mathbb{D}}$ be a hull. Prove that
   $$\mathrm{hcap}(K) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{\exp(\mathrm{i}\theta)}[\mathrm{Im}(B_\tau)] \sin(\theta)\, \mathrm{d}\theta,$$
   where $\tau = \tau_{\mathbb{H}\setminus K} = \inf\{t \geq 0 \mid B_t \notin \mathbb{H} \setminus K\}$ and $B \sim \mathbb{P}_{\exp(\mathrm{i}\theta)}$ is the complex (2D) Brownian motion started from $\exp(\mathrm{i}\theta)$.

3. Let B be a standard 1D Brownian motion. For $t \geq 0$, denote $\Delta_{m,n} = B_{tm2^{-n}} - B_{t(m-1)2^{-n}}$. Calculate
   $$\mathbb{E}\Big[\Big(\sum_{m \leq 2^n} \Delta_{m,n}^2 - t\Big)^2\Big]$$
   and use Borel-Cantelli lemma to show that, almost surely,
   $$\sum_{m \leq 2^n} \Delta_{m,n}^2 \to t \qquad \text{as } n \to \infty.$$
   (Thus, sample paths of B behave locally like $\sqrt{t}$.)

4. Let B be a standard 1D Brownian motion. Let $\alpha > 1/2$. Show that almost surely, at every point $t \geq 0$, B fails to be locally α-Hölder continuous.