Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 4

NB: In this sheet, the problems are long because there is a lot of explanation. So don't be afraid!
(If you didn't learn Stochastic calculus yet, you can skip this Exercise Set and return to it in a few weeks. The next one won't assume knowledge of this one, and we'll return to this only a little later.)

1. Let X be a semimartingale with

$$\mathrm{d}X_t = F(t)\,\mathrm{d}t + \sum_{j=1}^{m} G_j(t)\,\mathrm{d}B_t^{(j)},$$

   where $(B_t^{(1)}, \ldots, B_t^{(m)})$ is a m-dimensional (standard) Brownian motion. Prove that X is a local martingale if and only if $\mathbb{P}[F(t) = 0 \text{ for almost every } t] = 1$.

2. Let

$$X_t = \int_0^t G(s)\,\mathrm{d}B_s, \quad t \in [0, \infty)$$

   be a continuous local martingale with respect to the natural filtration $\big(\mathcal{F}_t\big)_{t\in[0,\infty)}$ of the Brownian motion B. Define the stopping times

$$\sigma(r) := \inf\{t \geq 0 \mid \langle X \rangle_t \geq r\}, \quad r \in [0, \infty).$$

   Suppose that $\lim_{t\to\infty} \langle X \rangle_t = \infty$ almost surely. Finally, define

$$Y_t := X_{\sigma(t)}, \quad t \in [0, \infty).$$

   In this problem, we will prove that Y is a standard 1-dimensional Brownian motion with respect to the filtration $\big(\mathcal{F}_{\sigma(t)}\big)_{t\in[0,\infty)}$, where $\mathcal{F}_{\sigma(t)} := \{A \in \mathcal{F} \mid A \cap \{\sigma(t) \leq s\} \in \mathcal{F}_s \text{ for all } s \geq 0\}$.

   (a) Fix $a \in \mathbb{R}$. Show that the following process is a continuous local martingale :

$$M_t = \exp\left(\mathrm{i}aX_t + \frac{a^2}{2}\langle X \rangle_t\right), \quad t \in [0, \infty).$$

   (b) Fix $r \in [0, \infty)$. Show that $M_{t\wedge\sigma(r)}$ is a continuous bounded martingale for $t \in [0, \infty)$.

   (c) Show that for any $0 \leq s \leq r$ and for any $a \in \mathbb{R}$, we have

$$\mathbb{E}\left[\exp\left(\mathrm{i}a(Y_r - Y_s)\right) \mid \mathcal{F}_{\sigma(s)}\right] = \exp\left(-\frac{a^2}{2}(r-s)\right).$$

(d) Deduce that $Y_r - Y_s \sim N(0, r-s)$ is independent of $\mathcal{F}_{\sigma(s)}$.

3. Let $f \in Hol(U)$, assume that f is not a constant function, and fix $z \in U$. Define

$$\sigma(t) := \inf\left\{ s \geq 0 \;\middle|\; \int_0^s |f'(B_r)|^2 \,\mathrm{d}r = t \right\}, \quad 0 \leq t < \int_0^{\tau_U} |f'(B_r)|^2 \,\mathrm{d}r,$$

where B is the complex (2D) Brownian motion started from z and $\tau_U = \inf\{t \geq 0 \mid B_t \notin U\}$.

(a) Show that the following map is strictly increasing:

$$t \mapsto \int_0^t |f'(B_r)|^2 \,\mathrm{d}r.$$

(b) Show that if f is conformal, then

$$\int_0^{\tau_U} |f'(B_r)|^2 \,\mathrm{d}r = \inf\{t \geq 0 \mid \widetilde{B}_t \notin f(U)\},$$

in distribution, where $\widetilde{B}$ is the complex (2D) Brownian motion started from $f(z)$.

4. Let $U \subset \mathbb{C}$ be a domain. Let $h \colon \overline{U} \to [0, \infty)$ be continuous, and suppose that h is harmonic inside U. Prove that $h(z) \geq \mathbb{E}_z[h(B_{\tau_U})]$, for all $z \in U$, where $B \sim \mathbb{P}_z$ and τ_U are as in the above exercise.

5. Using the Beurling estimate, prove that there exist constants $C, \alpha \in (0, \infty)$ such that the following holds. Let $0 < r < R < \infty$ and let $\gamma \colon [a, b] \to \mathbb{C}$ be a curve s.t. $|\gamma(a)| = r$ and $|\gamma(b)| = R$. Then for all $z \in \mathbb{C}$ with $|z| \leq r$, we have

$$\mathbb{P}_z[B[0, \tau_{R\mathbb{D}}] \cap \gamma[a, b] = \emptyset] \leq C \left(\frac{r}{R}\right)^{\alpha},$$

where $B \sim \mathbb{P}_z$ and $\tau_{R\mathbb{D}} = \inf\{t \geq 0 \mid B_t \notin R\mathbb{D}\}$, and $R\mathbb{D} = \{Rz \mid z \in \mathbb{D}\}$ is the R-scaled disc.

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