Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 3

1. Let U, V be simply connected domains whose boundaries are Jordan curves (i.e., homeomorphic images of the circle $\partial\mathbb{D}$). Let $z_1, z_2, z_3 \in \partial U$ and $w_1, w_2, w_3 \in \partial V$ appear in counterclockwise order along the boundary. Show that there exists a unique conformal bijection $f\colon U \to V$ such that $f(z_i) = w_i$ for $i = 1, 2, 3$.

2. Define the function classes

$$S := \{f \text{ conformal on } \mathbb{D} \mid f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \text{ in } \mathbb{D}\}$$
$$\Sigma := \{g \text{ conformal on } \mathbb{D}^* \mid g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \text{ in } \mathbb{D}^*\}, \qquad \mathbb{D}^* := \hat{\mathbb{C}} \setminus \mathbb{D}.$$

   (a) Let $f \in S$. Check that $g(z) := \frac{1}{f(1/z)} \in \Sigma$ and $0 \notin g(\mathbb{D}^*)$, and

$$g(z) = z - a_2 + \frac{a_2^2 - a_3}{z} + \cdots .$$

   (b) Let $g \in \Sigma$ and $0 \notin g(\mathbb{D}^*)$. Check that $f(z) := \frac{1}{g(1/z)} \in S$ and

$$f(z) = z - b_0 z^2 + (b_0^2 - b_1) z^3 + \cdots .$$

   (c) Let $f \in S$. Show that there exists $h \in S$ such that

$$\begin{aligned} h(-z) &= -h(z), \\ (h(z))^2 &= f(z^2), \\ h(z) &= z + \frac{a_2}{2} z^3 + \cdots \quad \text{for all } z \in \mathbb{D}. \end{aligned}$$

   [Hint: Consider $g\colon z \mapsto \sqrt{\frac{f(z)}{z}}$, and then $z \mapsto z g(z^2)$.]

3. (Schwarz reflection principle) Let $f \in Hol(B_+)$, where $B_+ = B(0, r) \cap \mathbb{H}$. Write $f(z) = u(z) + \mathrm{i} v(z)$ for $u, v\colon B_+ \to \mathbb{R}$ harmonic. Suppose that

$$\lim_{z \to x} v(z) = 0 \qquad \text{for all } x \in (-r, r).$$

   (a) Show that v has a unique harmonic extension to $B(0, r)$, and $v(\bar{z}) = -v(z)$ for all $z \in B(0, r)$.
   (b) Show that f has a unique holomorphic extension to $B(0, r)$, and $f(\bar{z}) = \overline{f(z)}$ for all $z \in B(0, r)$.

4. Let $K, K_1, K_2 \subset \overline{\mathbb{H}}$ be hulls and $r > 0$ and $x \in \mathbb{R}$.

   (a) Show that $\mathrm{hcap}(rK) = r^2\, \mathrm{hcap}(K)$.
   (b) Show that $\mathrm{hcap}(K + x) = \mathrm{hcap}(K)$.
   (c) Show that $\mathrm{hcap}(K_1 \cup K_2) = \mathrm{hcap}(K_1) + \mathrm{hcap}(g_{K_1}(K_2))$, where $g_{K_1} : \mathbb{H} \setminus K_1 \to \mathbb{H}$ is the conformal bijection normalized at ∞.