Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 2

1. Let $f \in Hol(U)$ where U is a bounded domain (open, connected subset of $\mathbb{C}$). Prove that if there exists $z_0 \in U$ such that
$$|f(z_0)| \geq |f(w)| \quad \text{for all } w \in U,$$
then f is constant.

2. Let $f \in Hol(U)$, where U is a bounded domain. Suppose that f extends continuously to the boundary ∂U. Show that
$$|f(z)| \leq \sup_{w \in \partial U} |f(w)| = \max_{w \in \overline{U}} |f(w)| \quad \text{for all } z \in U.$$

3. Let $f \in Hol(\mathbb{D})$, $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Prove that if either $|f'(0)| = 1$, or there exists $w \in \mathbb{D} \setminus \{0\}$ such that $|f(w)| = |w|$, then $f(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$ (i.e., f is a rotation).

4. Recall that $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, a complex manifold with two charts. The continuity and holomorphicity of maps f on $\hat{\mathbb{C}}$ are defined using the coordinate charts as follows.
   - If $f(\infty) = \infty$, f is continuous/holomorphic at ∞ iff $z \mapsto \frac{1}{f(1/z)}$ is continuous/holomorphic at 0,
   - If $f(\infty) \neq \infty$, f is continuous/holomorphic at ∞ iff $z \mapsto f(1/z)$ is continuous/holomorphic at 0,
   - If $z_0 \in \mathbb{C}$ and $f(z_0) = \infty$, then f is continuous/holomorphic at z_0 iff $z \mapsto 1/f(z)$ is continuous/holomorphic at z_0.
   - If $z_0 \in \mathbb{C}$ and $f(z_0) \neq \infty$, the definitions are the usual ones.

   Now, let $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ and define
$$f(z) = \frac{az + b}{cz + d}.$$
   (a) Prove that $f \in Hol(\hat{\mathbb{C}})$. Prove also that f is invertible, and hence conformal.
   (b) Check that if $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$, then f is bijective $\mathbb{H} \to \mathbb{H}$.
   (c) Check that if $a = 1 = d$ and $b = \overline{c} \in \mathbb{D}$, then f is bijective $\mathbb{D} \to \mathbb{D}$.

5. Prove that there is no conformal (i.e., holomorphic and injective) map $\hat{\mathbb{C}} \to \mathbb{C}$, nor $\hat{\mathbb{C}} \to \mathbb{D}$, nor $\mathbb{C} \to \mathbb{D}$.

6. Let $U \subset \hat{\mathbb{C}}$ be a simply connected domain such that $\hat{\mathbb{C}} \setminus U$ contains at least two points. Fix $z_0 \in U$. (The following items are part of the proof of the Riemann mapping theorem.)
   (a) Show that there exists a conformal map $f \colon U \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.
   [Hint: WLOG, let $a, \infty \in \hat{\mathbb{C}} \setminus U$ and $a \neq \infty$. Consider the function $z \mapsto \sqrt{z - a}$.]
   (b) Suppose $f, g \colon U \to \mathbb{D}$ are two conformal bijections such that $f(z_0) = 0 = g(z_0)$ and $f'(z_0) > 0$ and $g'(z_0) > 0$. Prove that $f = g$.