Advanced Topics in Stochastic Analysis

# - Introduction to Schramm-Loewner evolution

**Mondays 12–14** and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

## Exercises – Set 11

1. We define the (discrete) *Green's function* for a finite, connected graph $\mathcal{G} = (V, E) \subset \mathbb{Z}^d$ as
$$G_{\mathcal{G}}(x, y) := \mathbb{E}_x[\sum_{k=1}^{\tau_{\mathcal{G}}-1} 1\{X_k = y\}], \quad x, y \in V,$$
where $(X_n)_{n\in\mathbb{N}}$ is a simple random walk on $V$ started from $x$ and $\tau_{\mathcal{G}} := \inf\{n \in \mathbb{N} \mid X_n \notin V\}$.
   (a) Show that $G_{\mathcal{G}}(x, y) = G_{\mathcal{G}}(y, x)$, for all $x, y \in V$.
   (b) Show that, for fixed $y \in V$, we have $\Delta_x G_{\mathcal{G}}(x, y) = -1\{x = y\}$, where $\Delta_x$ denotes the (discrete) Laplacian acting on the variable $x$, defined as
$$\Delta f(x) := \frac{1}{2d} \sum_{(z,x)\in E} (f(z) - f(x)) \quad f\colon V \to \mathbb{R}.$$

2. Let $U \subset \mathbb{C}$ be a bounded domain whose boundary is regular for Brownian motion. Prove that $H_0^1(U) \subset L^2(U)$, where $H_0^1(U)$ is the Hilbert space completion of the space
$$C_0^\infty(U) := \{f\colon \mathbb{C} \to \mathbb{R} \mid f \text{ is smooth and } \operatorname{supp}(f) \subset U \text{ is compact}\}$$
with respect to the *Dirichlet inner product*
$$(f, g)_\nabla := \frac{1}{2\pi} \int_U \nabla f(z) \nabla g(z) \mathrm{d}z.$$

3. Let $\mathcal{G} = (V, E) \subset \mathbb{Z}^d$ be a finite, connected graph. Let $h$ be the (discrete) GFF on $\mathcal{G}$ (with zero b.c.) and let $A \subset V$ be a random subset of vertices, so that we have a coupling $(A, h)$ in the same probability space. We say that the coupling is *local* if the following holds:
For any fixed $B \subset V$, write $h = h_B + h^B$ using the Markov property (for $V \setminus B$), where
   - $h_B$ is (discrete) harmonic on $V \setminus B$ and $h_B$ equals $h$ on $B$,
   - $h^B$ is the GFF on $V \setminus B$ (with zero b.c.), and $h^B$ equals zero on $B$, and
   - $h_B$ and $h^B$ are independent.

   Then the process $h^B$ is independent of the sigma-algebra $\sigma(h_B, \{A = B\})$.
   Check that in the following situations, the couplings are local:
   (a) For all fixed $B \subset V$, the event $\{A = B\}$ is $\sigma(h_B)$-measurable.
   (b) Choose $x \in V$ uniformly at random and independently of $h$, and let $A_0$ be the largest connected component of $V$ such that $x \in A_0$ and $h(x)h(y) > 0$ for all $y \in A_0$ (cluster of the same sign). Take $A := (A_0 \cup \partial A_0) \cap V$ to be $A_0$ with one layer added.

4. Let $(A_1, h)$ and $(A_2, h)$ be two local couplings. Assume that conditionally on $h$, the sets $A_1$ and $A_2$ are independent. Show that $(A_1 \cup A_2, h)$ is a local coupling.

5. Consider the (discrete) GFF $h$ on $\mathcal{G} = (V, E) \subset \mathbb{Z}$ with $V = \{-1, 0, 1\}$ and $E = \{(-1, 0), (0, 1)\}$. Let $\xi$ be a random variable independent of $h$ such that $\mathbb{P}[\xi = 1] = \frac{1}{2} = \mathbb{P}[\xi = -1]$. Let $A_1 := \{\xi\}$ and $A_2 := \{\xi \times \operatorname{sign}(h(0))\}$. Show that $(A_1, h)$ and $(A_2, h)$ are local couplings, but $(A_1 \cup A_2, h)$ is not.