Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 10

1. Let γ be the SLE(κ) curve for $\kappa \in (0,4]$, and let $(g_t)_{t\geq 0}$ be the corresponding conformal maps normalized at ∞ and $(W_t)_{t\geq 0}$ the driving process. Let $A \subset \overline{\mathbb{H}}$ be a hull such that $0 \notin A$ and ∂A is a Jordan curve. Assume that $T := \inf\{t \geq 0 \mid \mathrm{dist}(\gamma(t), A) = 0\} < \infty$, denote $z = \gamma(T) \in \partial A$, and choose $\delta > 0$ small enough such that $\ell := [z, \delta\hat{n}] \subset A$, where $\hat{n}$ is the inward unit normal vector of ∂A.

   (a) By considering a complex Brownian motion started from ℓ, show that there exists a constant $r > 0$ such that
   $$g_T(\ell) - W_T \subset \{w \in \mathbb{H} \mid \mathrm{Im}(w) \geq r|\mathrm{Re}(w)|\}.$$

   (b) Show that for every $r > 0$, there exist $C, \alpha \in (0,\infty)$ such that if $\gamma\colon [0,1] \to \mathbb{H}$ is a curve with $0 < |\gamma(0)| = \varepsilon < 1 = |\gamma(1)|$ and
   $$\gamma[0,1] \subset \{w \in \mathbb{H} \mid \mathrm{Im}(w) \geq r|\mathrm{Re}(w)|\},$$
   then for the Brownian excursion E in $\mathbb{H}$, we have
   $$\mathbb{P}_0[E[0,\infty) \cap \gamma[0,1] = \emptyset] \leq C\varepsilon^{\alpha}.$$
   Why is the assumption that $\gamma[0,1]$ lies inside a cone needed?

   (c) For $m \in \mathbb{N}$, define the stopping times $\sigma_m := \inf\{t \geq 0 \mid |\gamma(t) - z| = \frac{1}{m}\}$, so $\sigma_m \nearrow T$ as $m \to \infty$. Deduce that for the Brownian excursion E in $\mathbb{H}$, we have
   $$\lim_{m\to\infty} \mathbb{P}_{\gamma(\sigma_m)}[E[0,\infty) \cap A = \emptyset\} = 0.$$

2. Let $\kappa > 0$ and $\rho > -2$. Consider the SLE(κ,ρ) process, i.e., the Loewner chain with driving process $(W_t)_{t\geq 0}$ and force point $(X_t)_{t\geq 0}$ satisfying the SDE system
   $$\begin{aligned} \mathrm{d}W_t &= \frac{\rho}{W_t - X_t}\,\mathrm{d}t + \sqrt{\kappa}\,\mathrm{d}B_t, \qquad W_0 = 0,\\ \mathrm{d}X_t &= \frac{2}{X_t - W_t}\,\mathrm{d}t, \qquad X_0 = x \in \mathbb{R}. \end{aligned}$$
   (parameterized by half-plane capacity), up to a blow-up time. Let $(Z_t)_{t\geq 0}$ be the solution to
   $$\mathrm{d}Z_t = \left(\frac{\rho+2}{\kappa}\right)\frac{1}{Z_t}\,\mathrm{d}t + \mathrm{d}B_t, \qquad Z_0 = 0,$$
   again up to a blow-up time. Check that
   $$\int_0^t \frac{\mathrm{d}s}{Z_s} < \infty$$
   and using this, deduce that the SLE(κ,ρ) process is well-defined also when $x \nearrow 0$ or $x \searrow 0$.