Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 9

Hand in Friday, June 26, 2020

Exercise 1 *[6 Pkt]*

1. Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale such that $\mathbb{E}[e^{tX_n}] < \infty$ for all $n \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Prove that
$$\mathbb{P}\left(\max_{1\le k\le n} X_k \ge x\right) \le e^{-tx}\mathbb{E}[e^{tX_n}],$$
for all $t > 0$ and $x \in \mathbb{R}$.

2. Let $(\xi_n)_{n\in\mathbb{N}}$ be i.i.d random variables with $\mathbb{P}(\xi_1 = 1) = 1/2 = \mathbb{P}(\xi_1 = -1)$. Let $X_n = \sum_{k=1}^n \xi_k$. Prove that
$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\sqrt{2n\log n}} \ge 1+\epsilon\right) = 0$$
for all $\epsilon > 0$.
Hint: Use $X_n \le \max_{1\le k\le n} X_k$*, part 1., the fact that* $\log\cosh x \le x^2/2$ *and the Borel Cantelli lemmas.*

Exercise 2 *[4 Pkt]*

Let M be a martingale in L^2 with $M_0 = 0$. As in Definition 4.26, let $[M] = ([M]_n)_n$ be given by
$$[M]_n \equiv \sum_{k=1}^n (M_k - M_{k-1})^2 \quad \text{for all } n \in \mathbb{N}.$$
Show that

1. $M^2 - [M] = (C \bullet M)$, where $C = (C_n)_n$ is given by $C_n = 2M_{n-1}$ for all $n \in \mathbb{N}$,
2. If M is bounded in L^2, then $(C \bullet M)$ is bounded in L^1,
3. $(C \bullet M)$ is a martingale.

Exercise 3 *[10 Pkt]*

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_n)_n$ is a filtration. Let τ, τ_1 and τ_2 be stopping times with respect to $(\mathcal{F}_n)_n$. Show that

1. In the definition of stopping times, one could require that $\{\tau_i \leq n\} \in \mathcal{F}_n$ for all n instead of $\{\tau_i = n\} \in \mathcal{F}_n$ for all n,
2. $\tau_1 + \tau_2$, $\tau_1 \vee \tau_2 := \max\{\tau_1, \tau_2\}$, $\tau_1 \wedge \tau_2 := \min\{\tau_1, \tau_2\}$ are stopping times,
3. $\mathcal{F}_\tau$ is a σ-algebra,
4. If $\tau_1 \leq \tau_2$, then $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$,
5. $\mathcal{F}_{\tau_1 \wedge \tau_2} = \mathcal{F}_{\tau_1} \cap \mathcal{F}_{\tau_2}$,
6. If $F \in \mathcal{F}_{\tau_1 \vee \tau_2}$, then $F \cap \{\tau_2 \leq \tau_1\} \in \mathcal{F}_{\tau_1}$,
7. $\mathcal{F}_{\tau_2 \vee \tau_1} = \sigma(\mathcal{F}_{\tau_1}, \mathcal{F}_{\tau_2})$,
8. If X is an adapted process, then X_τ is $\mathcal{F}_\tau$-measurable.