Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 3

Hand in Friday, 15th of May, 2020

Exercise 1 *[5 Pkt]*

Use Jensen's inequality to prove the Hölder inequalties for $p, q \in (0, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Exercise 2 *[5 Pkt]*

Let μ be a finite measure on a measurable space $(\Omega, \mathcal{A})$. Let $f : \Omega \to \mathbb{R}$ be measurable, nonnegative and integrable. Define

$$\mu_f(A) = \int_A f \, d\mu \qquad \forall A \in \mathcal{A}.$$

Show that μ_f is a measure on $(\Omega, \mathcal{A})$.

Exercise 3 *[5 Pkt]*

Let $\mathcal{M}$ be the set of all measures on the measurable space $(\Omega, \mathcal{A})$.

1. Let $\mu \sim \nu$, if and only if $\mu \ll \nu$ and $\nu \ll \mu$. Show that $\mu \sim \nu$ is an equivalence relation.
2. Show that for finite measures μ and ν, $\mu \sim \nu$ is equivalent to $0 < \frac{\mathrm{d}\nu}{\mathrm{d}\mu} < \infty$ μ-a.e.

Hint: You may use the Radon-Nikodým theorem

Exercise 4 *[5 Pkt]*

Let $(\Omega, \mathcal{A})$ be a measurable space, where the σ-algebra $\mathcal{A}$ contains all points, i.e. $\{\omega\} \in \mathcal{A}$ for all $\omega \in \Omega$. Let μ and ν be finite and discrete measures on $\mathcal{A}$.

1. Give a necessary and a sufficient condition for $\nu \ll \mu$.
2. Assume $\nu \ll \mu$. Calculate all densities of ν with respect to μ.

Hint: The measure μ is called *finite and discrete* if there are at most countably many $\omega_i \in \Omega$ and $p_i > 0$ with $\sum_i p_i < \infty$, such that

$$\mu = \sum_i p_i \delta_{\omega_i}.$$