Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 2

Hand in Friday, 8th of May, before the lecture

Exercise 1 *[3 Pkt]*

Let $V = \{v : \mathbb{N} \to \mathbb{R} \mid \|v\| < \infty\}$, where $\|v\| = \sup_{n\in\mathbb{N}} |v_n|$. Obviously, $(V, \|\cdot\|)$ is a metric space. Show that the closed unit sphere $B = \{v \in V \mid \|v\| = 1\}$ is not compact.

Exercise 2 *[3 Pkt]*

Show that each σ-finite measure μ on some measurable space $(\Omega, \mathcal{F})$ has a representation of the form $\mu = \sum_{n=0}^{\infty} a_n \mu_n$, where for all n, $a_n \geq 0$ and μ_n is a probability measure on $(\Omega, \mathcal{F})$.

Exercise 3 *[1+1+2+2+2 Pkt]*

Let $\mathcal{C}$ and $\mathcal{D}$ be classes of random variables.

1. Show that $\mathcal{C}$ is uniformly integrable, if and only if $\sup_{X\in\mathcal{C}} \mathbb{E}[|X| 1_{\{|X|>K\}}] \to 0$ as $K \to \infty$.
2. Show that $\mathcal{C} + \mathcal{D} := \{X + Y, X \in \mathcal{C}, Y \in \mathcal{D}\}$ is uniformly integrable, if $\mathcal{C}$ and $\mathcal{D}$ are uniformly integrable.
3. Let $g : [0, \infty) \to [0, \infty)$ be such that $g(x)/x \to \infty$ as $x \to \infty$. If $\sup_{X\in\mathcal{C}} \mathbb{E}[g(|X|)] < \infty$, show that $\mathcal{C}$ is uniformly integrable.
4. If there exists an integer $p > 1$ such that $\sup_{X\in\mathcal{C}} \mathbb{E}[|X|^p] < \infty$, show that $\mathcal{C}$ is uniformly integrable.
5. If $\mathbb{E}[\sup_{X\in\mathcal{C}} |X|] < \infty$, show that $\mathcal{C}$ is uniformly integrable.

Exercise 4 *[2+2+2 Pkt]*

Let $Y, X, X_1, X_2, \ldots$ be random variables in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and $X_n \to X$ in $\mathcal{L}^2$. Show that

1. $\lim_{n\to\infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2]$.
2. $\lim_{n\to\infty} \mathbb{E}[X_n Y] = \mathbb{E}[XY]$.
3. $\mathcal{L}^2 \subset \mathcal{L}^1$.