Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 1

Hand in Friday, 1st of May 2020 before the lecture

Exercise 1 *[5 Pkt]*

Find the open, closed and compact subsets of the metric space $(\mathbb{Z}^d, \rho)$, where $\mathbb{Z}$ is the set of integer numbers and ρ is the euclidean metric in $\mathbb{R}^d$. Define the corresponding Borel-σ-algebra $\mathcal{B}(\mathbb{Z}^d)$.

Exercise 2 *[5 Pkt]*

Let $\{(X_n, \mathcal{A}_n, \mu_n)\}_{n\in\mathbb{N}}$ be a family of measure spaces, where the sets X_n are pairwise disjoints. We define the measure space $(X, \mathcal{A}, \mu)$, where $X = \bigcup_n X_n$,

$$\mathcal{A} = \{B : B \cap X_n \in \mathcal{A}_n \text{ for all } n\}$$

and $\mu(B) = \sum_n \mu_n(B \cap X_n)$. Show that

1. $\mathcal{A}$ is a σ-algebra;
2. μ is a measure;
3. μ is σ-finite if and only if all μ_n are σ-finite.

Exercise 3 *[5 Pkt]*

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $A_1 \triangle A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$ for $A_1, A_2 \in \mathcal{A}$. Show that:

1. If $A_1, A_2 \in \mathcal{A}$ and $\mu(A_1 \triangle A_2) = 0$, then $\mu(A_1) = \mu(A_2)$.
2. If the measure space is complete, $A_1, A_1 \triangle A_2 \in \mathcal{A}$ and $\mu(A_1 \triangle A_2) = 0$, then we have that $A_2 \in \mathcal{A}$.

Remark: A measure space $(X, \mathcal{A}, \mu)$ is complete if the σ-algebra $\mathcal{A}$ contains all the subsets of μ-null sets, i.e., if $B \in \mathcal{A}$, $\mu(B) = 0$ and $A \subset B$ then $A \in \mathcal{A}$.