Polyharmonic Fields and Liouville Quantum Gravity Measures on Tori of Arbitrary Dimension: from Discrete to Continuous

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Abstract

For an arbitrary dimension n, we study:

- the Polyharmonic Gaussian Field h_L on the discrete torus $\mathbb{T}^n_L = \frac{1}{L}\mathbb{Z}^n/\mathbb{Z}^n$, that is the random field whose law on $\mathbb{R}^{\mathbb{T}^n_L}$ given by

$$c_n\, e^{-b_n \left\|(-\Delta_L)^{n/4} h\right\|^2} dh,$$

  where dh is the Lebesgue measure and Δ_L is the discrete Laplacian;

- the associated discrete Liouville Quantum Gravity measure associated with it, that is the random measure on $\mathbb{T}^n_L$

$$\mu_L(dz) = \exp\left(\gamma h_L(z) - \frac{\gamma^2}{2}\mathbf{E} h_L(z)\right) dz,$$

  where γ is a regularity parameter.

As $L \to \infty$, we prove convergence of the fields h_L to the Polyharmonic Gaussian Field h on the continuous torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, as well as convergence of the random measures μ_L to the LQG measure μ on $\mathbb{T}^n$, for all $|\gamma| < \sqrt{2n}$.

Contents

Introduction

We study Gaussian random fields and the associated LQG measures on continuous and discrete tori of arbitrary dimension. The random field h on the continuous torus is a particular case of the co-polyharmonic field introduced and analyzed in detail in [5] in great generality on all 'admissible' manifolds of even dimension. One of the main goals now is to study the approximation of these fields and the associated LQG measures by their discrete counterparts.

The *polyharmonic fields* h on $\mathbb{T}^n \cong [0,1)^n$ and h_L on $\mathbb{T}^n_L \cong \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^n$ for $L \in \mathbb{N}$ are centered Gaussian random fields with covariance functions

$$\mathbf{E}\Big[h(x)\, h(y)\Big] = k(x,y) := \frac{1}{a_n}\, \mathring{G}^{n/2}(x,y)\ ,$$
$$\mathbf{E}\Big[h_L(x)\, h_L(y)\Big] = k_L(x,y) := \frac{1}{a_n}\, \mathring{G}^{n/2}_L(x,y)\ .$$

given in terms of the integral kernel for the 'grounded' inverse of the (continuous and discrete, resp.) poly-Laplacian $(-\Delta)^{n/2}$ and $(-\Delta_L)^{n/2}$, and a normalization constant $a_n := \frac{2}{\Gamma(n/2)\,(4\pi)^{n/2}}$. With this choice of the normalization constant

Lemma 0.1 (cf. Lemma 1.4)**.**

$$\left| k(x,y) - \log\frac{1}{d(x,y)} \right| \le C\ .$$

Characterization of the Discrete Polyharmonic Field. Let $n, L \in \mathbb{N}$ be given and assume for convenience that L is odd, let $\mathbb{Z}^n_L = \{-\frac{L-1}{2}, -\frac{L-1}{2}+1, \ldots, \frac{L-1}{2}\}^n$, and set $N := L^n$ and

$$c_n := \left(\frac{a_n}{2\pi N}\right)^{\frac{N-1}{2}} \cdot \prod_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left(4L^2 \sum_{k=1}^n \sin^2\left(\pi z_k/L\right)\right)^{n/4}\ .$$

Define the measure $\widehat{\boldsymbol{\nu}}(h)$ on $\mathbb{R}^N \cong \mathbb{R}^{\mathbb{T}^n_L}$ by

$$d\widehat{\boldsymbol{\nu}}(h) := c_n\, e^{-\frac{a_n}{2N}\|(-\Delta_L)^{n/4} h\|^2}\, d\mathcal{L}^N(h)\ ,$$

and denote by $\boldsymbol{\nu}$ its push forwards under the map

$$h \mapsto \mathring{h}, \quad \mathring{h}_v := h_v - \frac{1}{N}\sum_{v=1}^N h_v\ .$$

In other words, $\boldsymbol{\nu} = \widehat{\boldsymbol{\nu}}\left(\,\cdot\,\Big|\sum_{v=1}^N h_v = 0\right)$.

Furthermore,

$$\mathring{T}_*\boldsymbol{\nu} = \mathring{\mathbf{P}} \qquad \mathring{T}_*^{-1}\mathring{\mathbf{P}} = \boldsymbol{\nu}$$

where $\mathring{\mathbf{P}}$ denotes the distribution of the 'grounded white noise' on $\mathbb{T}^n_L$, explicitly given as

$$d\mathring{\mathbf{P}}(\Xi) = \frac{1}{(2\pi)^{\frac{N-1}{2}}}\, e^{-\frac{1}{2N}\|\Xi\|^2}\, d\mathcal{L}_H^{N-1}(\Xi)$$

on the hyperplane $H = \{\Xi \in \mathbb{R}^N : \sum_{v=1}^N \Xi_v = 0\}$, and where

$$\mathring{T} : h \mapsto \Xi = \sqrt{a_n}\,(-\Delta_L)^{n/4} h \qquad \mathring{T}^{-1} : \Xi \mapsto h = \frac{1}{\sqrt{a_n}}\,\mathring{G}_L^{n/4}\Xi\,.$$

Theorem 0.2 (cf. Thm. 2.4)**.** *The distribution of the discrete polyharmonic field on $\mathbb{T}^n_L$ is given by the probability measure $\boldsymbol{\nu}$ on $\mathbb{R}^{\mathbb{T}^n_L} \cong \mathbb{R}^N$.*

Convergence of the Random Fields. As $L \to \infty$, the polyharmonic fields h_L on the discrete tori converge to the polyharmonic field h on the continuous torus. This convergence of the fields, indeed, holds in great generality.

For a precise formulation, one either has to specify classes of test functions on $\mathbb{T}^n$ which admit traces on $\mathbb{T}^n_L$, or unique ways of extending functions on $\mathbb{T}^n_L$ onto $\mathbb{T}^n$.

Theorem 0.3 (cf. Thm. 3.7)**.** *For all $f \in \bigcup_{s>n/2} \mathring{H}^s(\mathbb{T}^n)$,*

$$\langle h_L, f\rangle_{\mathbb{T}^n_L} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \textit{in } L^2(\mathbf{P}) \textit{ as } L \to \infty.$$

Let $\mathcal{D}_L \subset \mathcal{C}^\infty(\mathbb{T}^n)$ denote the linear span of the eigenfunctions φ_z for the negative Laplacian with associated eigenvalues $0 < \lambda_z < (L\pi)^2$, or more explicitly,

$$\mathcal{D}_L := \left\{ f:\ f(x) = \sum_{z\in\mathbb{Z}^n_L} \Big[\alpha_z \cos(2\pi x\cdot z) + \beta_z \sin(2\pi x \cdot z)\Big],\ \alpha_z, \beta_z \in \mathbb{R} \right\}.$$

Theorem 0.4 (cf. Thm. 3.9)**.** *For all $f \in \mathring{H}^{-n/2}(\mathbb{T}^n)$,*

$$\langle h_{L,\sharp}, f\rangle_{\mathbb{T}^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \textit{in } L^2(\mathbf{P}) \textit{ as } L \to \infty$$

where $h^\omega_{L,\sharp}$ for every ω denotes the unique function in $\mathcal{D}_L$ which coincides with h^ω_L on $\mathbb{T}^n_L$.

The same convergence assertion also holds for the so-called *spectrally reduced polyharmonic field* $h_L^{-\circ}$ on $\mathbb{T}^n_L$ given in terms of the eigenbasis $\{\varphi_z\}_{z\in\mathbb{Z}^n_L}$ of the discrete Laplacian Δ_L as

$$h_L^{-\circ}(x) := \sum_{z\in\mathbb{Z}^n_L} \left(\frac{L^2}{\pi^2|z|^2}\sum_{k=1}^n \sin^2(\pi z_k/L)\right)^{-n/4} \langle h_L|\varphi_z\rangle_{\mathbb{T}^n_L} \cdot \varphi_z(x)\,.$$

Our convergence results apply to the case of arbitrary dimension n. In dimension $n \le 4$, several results are available in the literature for the convergence of other discrete Fractional Gaussian Fields of integer order to the corresponding counterpart in the continuum, including e.g., the *odometer* for the *sandpile model*, or the *membrane model*. For a comparison of these results with those in the present work, see §3.2.5 below.

Convergence of the Random Measures. The convergence questions for the associated random measures are more subtle. Again, of course, one expects that the Liouville measure μ_L on the discrete tori converge as $L \to \infty$ to the Liouville measure μ on the continuous torus. This convergence of the random measure, however, only holds for small parameters γ.

Theorem 0.5 (cf. Thm. 4.7)**.** *Assume $|\gamma| < \sqrt{\frac{n}{e}}$, and let a be an odd integer ≥ 2. Then in $\mathbf{P}$-probability and in $L^1(\mathbf{P})$,*

$$\mu_{a^\ell} \to \mu \qquad \textit{as } \ell \to \infty.$$

Analogous convergence results hold for the random measures associated with the Fourier extensions of the discrete polyharmonic fields and the reduced discrete polyharmonic field, in the latter case even in the whole range of subcriticality $\gamma \in (-\sqrt{2n}, \sqrt{2n})$.

Theorem 0.6 (cf. Thm. 4.9, Thm. 4.10)**.** *For* $|\gamma| < \sqrt{\frac{n}{e}}$,

$$\mu_{L,\sharp} \to \mu \qquad \text{as } L \to \infty,$$

and for $|\gamma| < \sqrt{2n}$,

$$\mu^{-\circ}_{L,\sharp} \to \mu \qquad \text{as } L \to \infty.$$

Uniform Integrability of the Random Measures. As an auxiliary result of independent interest, we provide a direct proof of the uniform integrability of (discrete, semi-discrete, and continuous) random measures on the multidimensional torus.

Theorem 0.7 (cf. Thm. 4.11)**.** *Assume that* $|\gamma| < \sqrt{\frac{n}{e}}$. *Then*

$$\sup_L \mathbf{E}\Big[\big|\mu_L(\mathbb{T}^n_L)\big|^2\Big] < \infty$$

and

$$\sup_L \mathbf{E}\Big[\big|\mu_{L,\sharp}(\mathbb{T}^n)\big|^2\Big] < \infty \ .$$

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1 Laplacian and Kernels on Continuous and Discrete Tori

1.1 Laplacian and Kernels on the Continuous Torus

(a) For $n \in \mathbb{N}$, we denote by $\mathbb{T}^n := (\mathbb{R}/\mathbb{Z})^n$ the continuous n-torus. Where it seems helpful, one can always think of the torus $\mathbb{T}^n$ as the set $[0,1)^n \subset \mathbb{R}^n$. It inherits from $\mathbb{R}^n$ the additive group structure and the Lebesgue measure, denoted in the sequel by $d\mathcal{L}^n(x)$ or simply by dx. The distance on $\mathbb{T}^n$ is given by

$$d(x,y) := \left(\sum_{k=1}^n \big(|x_k - y_k| \wedge |1 - x_k + y_k|\big)^2 \right)^{1/2} \ .$$

(b) For $z \in \mathbb{Z}^n$ and $x \in \mathbb{T}^n$ put

$$\Phi_z(x) := \exp\Big(2\pi i\, z \cdot x\Big) \ .$$

The family $(\Phi_z)_{z \in \mathbb{Z}^n}$ is a complete ON basis of $L^2_{\mathbb{C}}(\mathbb{T}^n)$. It consists of eigenfunctions of the negative Laplacian $-\Delta = -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ on $\mathbb{T}^n$ with corresponding eigenvalues given by

$$\lambda_z := (2\pi|z|)^2 \ .$$

(c) The Fourier transform of the function $f \in L^2_{\mathbb{C}}(\mathbb{T}^n)$ is the function (or "sequence") $g \in \ell^2(\mathbb{Z}^n)$ given by

$$g(z) := \langle f, \Phi_z \rangle_{\mathbb{T}^n} := \int_{\mathbb{T}^n} f(x)\, \overline{\Phi}_z(x)\, dx \ .$$

Conversely, for g as above and a.e. $x \in \mathbb{T}^n$,

$$f(x) = \sum_{z \in \mathbb{Z}^n} g(z) \Phi_z(x) \ .$$

(d) To obtain a complete ON basis $(\varphi_z)_{z\in\mathbb{Z}^n}$ for the real L^2-space, choose a subset $\hat{\mathbb{Z}}^n$ of $\mathbb{Z}^n \setminus \{0\}$ with

$$\mathbb{Z}^n \setminus \{0\} = \hat{\mathbb{Z}}^n \mathbin{\dot\cup} \big(- \hat{\mathbb{Z}}^n \big) \ ,$$

and define

$$\varphi_z(x) := \begin{cases} \frac{1}{\sqrt{2}} \big(\Phi_z + \Phi_{-z}\big)(x) = \sqrt{2}\, \cos\big(2\pi\, z \cdot x\big) & \text{if } z \in \hat{\mathbb{Z}}^n \ , \\ \frac{1}{\sqrt{2}\, i} \big(\Phi_z - \Phi_{-z}\big)(x) = \sqrt{2}\, \sin\big(2\pi\, z \cdot x\big) & \text{if } z \in -\hat{\mathbb{Z}}^n \ , \\ \mathbf{1} & \text{if } z = 0 \ . \end{cases}$$

(e) Functions f on $\mathbb{T}^n$ will be called *grounded* if $\int f \, d\mathcal{L}^n = 0$. For $s \in \mathbb{R}$, the (grounded, real) Sobolev space $\mathring{H}^s(\mathbb{T}_n)$ can be identified with a set of formal series:

$$\mathring{H}^s(\mathbb{T}^n) = \left\{ f = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \alpha_z \varphi_z : \ \alpha_z \in \mathbb{R}, \ \sum_{z \in \mathbb{Z}^n \setminus \{0\}} |z|^{2s} \, |\alpha_z|^2 < \infty \right\} \ .$$

Then for all $f = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \alpha_z \varphi_z \in \mathring{H}^r(\mathbb{T}^n)$ and $g = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \beta_z \varphi_z \in \mathring{H}^s(\mathbb{T}^n)$ with $r + s \geq 0$,

$$\langle f, g \rangle_{\mathbb{T}^n} = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \alpha_z \, \beta_z.$$

The norm of $\mathring{H}^s(\mathbb{T}^n)$ is given by the square root of $\sum_{z \in \mathbb{Z}^n \setminus \{0\}} |z|^{2s} \, |\alpha_z|^2$. Equivalently it could be defined with λ_z^s in place of $|z|^{2s}$. This is the convention adopted in [5]. The two norms differ only by a factor $(2\pi)^s$.

(f) Given any function $u : \mathbb{Z}^n \to \mathbb{C}$ we define the *principal value along cubes* of the series $\sum_z u(z)$ by

$$\sum_{z \in \mathbb{Z}^n}^{\square} u(z) := \lim_{L \to \infty} \sum_{z \in \mathbb{Z}^n, \ \|z\|_\infty < L/2} u(z)$$

provided the latter limit exists in $\mathbb{C}$ or in $\mathbb{R} \cup \{\pm\infty\}$.

(g) Since M is compact, there exists a unique *grounded Green kernel* $\mathring{G}$ satysfying

$$\mathring{G}(x, y) \simeq |x - y|^{2-n} \ .$$

In particular, $\mathring{G} \in L^p(M \times M)$ for all $p < \frac{n}{n-2}$. We claim that we have:

$$\begin{aligned} \mathring{G}(x, y) &= \sum_{z \in \mathbb{Z}^n \setminus \{0\}}^{\square} \frac{1}{(2\pi|z|)^2} \, \varphi_z(x) \, \varphi_z(y) = \sum_{z \in \mathbb{Z}^n \setminus \{0\}}^{\square} \frac{1}{(2\pi|z|)^2} \, \Phi_z(x) \, \overline{\Phi}_z(y) \\ &= \sum_{z \in \mathbb{Z}^n \setminus \{0\}}^{\square} \frac{1}{(2\pi|z|)^2} \, \Phi_z(x - y) = \sum_{z \in \mathbb{Z}^n \setminus \{0\}}^{\square} \frac{1}{(2\pi|z|)^2} \, \cos\Big(2\pi\, z \cdot (x - y)\Big), \end{aligned}$$

where the convergence holds almost everywhere and in L^p for $p < n/(n - 2)$. Indeed consider the filtration $(\mathfrak{F}_L)$ where $\mathfrak{F}_L$ is the σ-algebra generated by the φ_z for $z \in \mathbb{Z}^n$, $\|z\|_\infty < L/2$, and the associated closed martingale $\mathring{G}_L = \mathbf{E}[\mathring{G}|\mathfrak{F}_L]$, where expectation is with respect to $\mathsf{vol} \otimes \mathsf{vol}$. Take $z \in \mathbb{Z}^n$ with $\|z\|_\infty < L/2$. Since φ_z is $\mathfrak{F}_L$-measurable, we get that

$$\int \mathring{G}_k(x,y)\varphi_z(y)dy = \int \mathring{G}(x,y)\varphi_z(y)dy = (-\Delta)^{-1}\varphi_z(x) = \lambda_z^{-1}\varphi_z(x).$$

On the other hand, when $\|z\|_\infty \geq L/2$, since the φ_z's form an orthonormal basis, we find that $\mathbf{E}[\varphi_z|\mathfrak{F}_L] = 0$, and thus

$$\int \mathring{G}_L(x,y)\varphi_z(y)dy = 0.$$

This shows that

$$\mathring{G}_L(x,y) = \sum_{z \in \mathbb{Z}_L^n \setminus \{0\}} \lambda_z^{-1}\varphi_z(x)\varphi_z(y),$$

and thus the almost everywhere convergence of the series follows by the martingale convergence theorem.

(h) The *polyharmonic operator* is defined as

$$a_n \cdot (-\Delta)^{n/2} \qquad \text{with} \qquad a_n := \frac{2}{\Gamma(n/2)\,(4\pi)^{n/2}}\ .$$

The inverse operator admits a kernel denoted by k.

As for the Green kernel, we have the following representation.

Lemma 1.1. *We have that*

$$k = \sum_{z \in \mathbb{Z}^n \setminus \{0\}}^{\square} \frac{1}{(2\pi|z|)^n}\,\varphi_z \otimes \varphi_z,$$

where the series converges in $L^2(\mathbb{T}^n \times \mathbb{T}^n)$ and almost-everywhere.

Remark 1.2. We conjecture that the convergence indeed holds everywhere but do not have a proof of this fact.

Proof. Since the series on the right-hand side is orthogonal, we find that

$$\left\| \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \lambda_z^{-n/2}\varphi_z \otimes \varphi_z \right\|_{L^2} = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} (2\pi z)^{-2n} < \infty.$$

This shows that the series actually converges in L^2. The rest of the claim is obtained by a martingale argument as for the previous lemma. $\square$

Lemma 1.3. *The function*

$$f \colon x \longmapsto k(x,0) = \frac{1}{\Gamma(n/2)} \int_0^\infty \mathring{p}_t(x,0)\, t^{n/2-1} dt \tag{1}$$

is differentiable at every $x \in \mathbb{T}^n \setminus \{0\}$, and, for every $k \leq n$,

$$\frac{\partial}{\partial x_k} f(x) = \frac{1}{\Gamma(n/2)} \int_0^\infty \frac{\partial}{\partial x_k} \mathring{p}_t(x,0)\, t^{n/2-1} dt\ . \tag{2}$$

Proof. The heat-kernel representation in (1) holds as in [6, Lem. 2.4]. For fixed $k \leq n$, standard Gaussian upper heat kernel estimates provide the summability of the right-hand side in (2), hence (2) follows by differentiation under integral sign. Since $x \mapsto \big(\frac{\partial}{\partial x_k}\mathring{p}_t\big)(x,0)$ is continuous for every k on the whole of $\mathbb{T}^n$, we have that $\frac{\partial}{\partial x_k} f$ is continuous away from 0, and the differentiability of f follows by standard arguments in multivariate calculus. $\square$

The constant a_n is chosen such that it leads to a precise logarithmic divergence of k.

Lemma 1.4 ([5])**.** $\exists C = C(n) : \forall x,y \in \mathbb{T}^n$*:*

$$\left| k(x,y) - \log \frac{1}{d(x,y)} \right| \leq C\ . \tag{3}$$

Proof. Note that the estimate in Proposition 2.13 in [5] for the kernel $G^{n/2}(x,y)$ of the $n/2$-power of the Green operator not only holds for even but also for odd n. $\square$

1.2 Laplacian and Kernels on the Discrete Torus

(a) For the sequel, fix $L \in \mathbb{N}$. For convenience, we assume that L is odd. Put

$$\mathbb{Z}^n_L := \{z \in \mathbb{Z}^n : \|z\|_\infty := \max_{k=1,\dots,n} |z_k| < L/2\} ,$$

and let

$$\mathbb{T}^n_L := \left(\tfrac{1}{L}\mathbb{Z}\right)^n / \mathbb{Z}^n$$

denote the discrete n-torus with edge length $\frac{1}{L}$. Where helpful, one can think of the discrete torus $\mathbb{T}^n_L$ as the set $\frac{1}{L}\mathbb{Z}^n_L = \{\frac{k}{L} : k \in \mathbb{Z}, 0 \le k < L\})^n \subset \mathbb{R}^n$. We always regard it as a subset of the continuous torus $\mathbb{T}^n$. Furthermore, let

$$m_L := \frac{1}{L^n} \sum_{z \in \mathbb{T}^n_L} \delta_z$$

denote the normalized counting measure on $\mathbb{T}^n_L$. Points $v, u \in \mathbb{T}^n_L$ are *neighbors*, in short $v \sim u$, if $d(v,u) = \frac{1}{L}$. Each point in $\mathbb{T}^n_L$ has $2n$ neighbors.

(b) We define the *discrete Laplacian* Δ_L acting on functions $f \in L^2(\mathbb{T}^n_L)$ by

$$\Delta_L f(v) := L^2 \cdot \sum_{u \sim v} \Big[f(u) - f(v) \Big] = 2nL^2 \big(\mathsf{p}_L f - f\big)(v)$$

with the transition kernel on $\mathbb{T}^n_L$ given by

$$p_L(v,u) := \frac{L^n}{2n} \mathbf{1}_{v \sim u}$$

and its action by $(\mathsf{p}_L f)(v) = L^{-n} \sum_u p_L(v,u) f(u)$. Furthermore, we define the grounded transition kernel by $\mathring{p}_L(v,u) := p_L(v,u) - 1$.

The *discrete Green operator* acting on grounded functions $f \in \mathring{L}^2(\mathbb{T}^n_L)$ is defined by

$$\mathring{\mathsf{G}}_L f := \frac{1}{2nL^2} \sum_{k=0}^{\infty} \mathsf{p}^k_L f = \frac{1}{2nL^2} \sum_{k=0}^{\infty} \mathring{\mathsf{p}}^k_L f .$$

In particular, the grounded discrete Green kernel is given by

$$\mathring{G}_L(v,u) = \frac{1}{2nL^2} \sum_{k=0}^{\infty} \mathring{p}^k_L(v,u)$$

and its action by $(\mathring{\mathsf{G}}_L f)(v) = L^{-n} \sum_y \mathring{G}_L(v,u) f(u)$.

(c) A complete ON basis of the complex $L^2_{\mathbb{C}}(\mathbb{T}^n_L, m_L)$ is given by $(\Phi_z)_{z \in \mathbb{Z}^n_L}$ with

$$\Phi_z(v) := \exp\Big(2\pi i \, z \cdot v\Big) \qquad (\forall v \in \mathbb{T}^n_L) .$$

The functions Φ_z are (normalized) eigenfunctions of the negative discrete Laplacian $-\Delta_L$ with eigenvalues

$$\lambda_{L,z} := 4L^2 \sum_{k=1}^{n} \sin^2\Big(\pi z_k / L\Big) . \tag{4}$$

Note that as $L \to \infty$, the RHS converges to $\lambda_z = (2\pi|z|)^2$ for any $z \in \mathbb{Z}^n$.

A complete ON basis of $L^2_{\mathbb{R}}(\mathbb{T}^n_L, m_L)$ is given by the functions φ_z for $z \in \mathbb{Z}^n_L$ where as before $\varphi_0 \equiv 1$ and $\varphi_z(v) = \sqrt{2}\cos\big(2\pi \, z \cdot v\big)$ if $z \in \hat{\mathbb{Z}}^n \cap \mathbb{Z}^n_L$ and $\varphi_z(v) = \sqrt{2}\sin\big(2\pi \, z \cdot v\big)$ if $z \in (-\hat{\mathbb{Z}}^n) \cap \mathbb{Z}^n_L$.

Remark 1.5. For even L, the previous definitions require some modifications. The set $\mathbb{Z}_L^n$ has to be re-defined as

$$\mathbb{Z}_L^n := \big\{ -L/2+1, \dots, L/2-1, L/2 \big\}^n \ .$$

Each $z \in \mathbb{Z}_L^n$ we decompose into $z' := (z_k)_{k\in\sigma_z}$ and $\tilde{z} := (z_k)_{k\in\tau_z}$ with $\sigma_z := \{k \in \{1,\dots,n\} : z_k = L/2\}$, $\tau_z := \{k \in \{1,\dots,n\} : z_k < L/2\}$. Similarly, for $v \in \mathbb{T}_L^n$ we put $v' := (v_k)_{k\in\sigma_z}$ and $\tilde{v} := (v_k)_{k\in\tau_z}$. Then

$$\Phi_z(v) = (-1)^{L\,|v'|_{\sigma_z}} \cdot \Phi_{\tilde{z}}(\tilde{v}) \quad \text{with } |v'|_{\sigma_z} := \sum_{k\in\sigma_z} v_k \ .$$

Thus a complete ON basis of $L^2_{\mathbb{R}}(\mathbb{T}_L^n, m_L)$ is given by the functions

$$\varphi_z(v) := (-1)^{L\,|v'|_{\sigma_z}} \cdot \varphi_{\tilde{z}}(\tilde{v}) \ , \qquad z \in Z_L^n \ , \tag{5}$$

where $\varphi_{\tilde{z}}$ for $\tilde{z} \in \mathbb{Z}^n$ with $\|\tilde{z}\|_\infty < L/2$ is defined as before.

(d) In terms of the discrete eigenfunctions, the *discrete grounded Green kernel*, the integral kernel of the inverse of $-\Delta_L$ acting on grounded L^2-functions, is given as

$$\begin{aligned}\mathring{G}_L(v,u) &= \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\lambda_{L,z}} \varphi_z(v)\,\varphi_z(u) = \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\lambda_{L,z}} \Phi_z(v)\,\overline{\Phi}_z(u) \\ &= \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{4L^2 \sum_{k=1}^n \sin^2\Big(\pi z_k/L\Big)} \cdot \cos\Big(2\pi\, z\cdot(v-u)\Big) \ ,\end{aligned}$$

and the *discrete polyharmonic kernel*, the integral kernel of the inverse of $a_n(-\Delta_L)^{n/2}$ acting on grounded L^2-functions, as

$$\begin{aligned}k_L(v,u) &= \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \varphi_z(v)\,\varphi_z(u) = \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \Phi_z(v)\,\overline{\Phi}_z(u) \\ &= \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\Big(4L^2 \sum_{k=1}^n \sin^2\big(\pi z_k/L\big)\Big)^{n/2}} \cdot \cos\Big(2\pi\, z\cdot(v-u)\Big) \ .\end{aligned} \tag{6}$$

1.3 Extensions and Projections

(a) Piecewise Constant Extension/Projection. Set $Q_L := [-\frac{1}{2L}, \frac{1}{2L})^n$ and $Q_L(v) := v + Q_L$ for $v \in \mathbb{T}_L^n$. Observe that

$$\mathbb{T}^n = \dot{\bigcup_{v\in\mathbb{T}_L^n}} Q_L(v) \ .$$

Functions on $\mathbb{T}^n$ are called *piecewise constant* if they are constant on each of the cubes $Q_L(v), v \in \mathbb{T}_L^n$. Every function f on the discrete torus $\mathbb{T}_L^n$ can be uniquely *extended* to a piecewise constant function $f_{L,\flat}(x)$ by setting $f_{L,\flat}(x) := f(v)$ if $x \in Q_L(v)$. In other words,

$$f_{L,\flat}(x) = (\dot{\mathsf{q}}_L f)(x) := \langle q_L(x,\,\cdot\,), f\rangle_{\mathbb{T}_L^n}$$

with the Markov kernel $q_L(x,v) := L^n\,\mathbf{1}_{v+Q_L}(x)$ on $\mathbb{T}^n \times \mathbb{T}_L^n$. The latter is the restriction of the Markov kernel

$$q_L = L^n \sum_{v\in\mathbb{T}_L^n} \mathbf{1}_{v+Q_L} \otimes \mathbf{1}_{v+Q_L} \quad \text{on } \mathbb{T}^n \times \mathbb{T}^n. \tag{7}$$

Note that $\int_{\mathbb{T}^n} q_L(x,y)dy = 1$ as well as $\int_{\mathbb{T}_L^n} q_L(x,v)dm_L(v) = 1$.

The *projection* from $L^2(\mathbb{T}^n)$ onto the set of piecewise constant functions on $\mathbb{T}^n$ is given by $f \mapsto f_{\flat,L}$ with

$$f_{\flat,L}(x) := (\mathsf{q}_L f)(x) := \langle q_L(x,\,\cdot\,), f\rangle_{\mathbb{T}^n} = L^n \sum_{v\in\mathbb{T}_L^n} \langle 1_{v+Q_L}, f\rangle_{\mathbb{T}^n} \cdot 1_{v+Q_L}(x) \ .$$

Here and in the sequel, the *integral operators* associated with kernels p, q, r will be denoted by $\mathsf{p}, \mathsf{q}, \mathsf{r}$, resp. In general, these are regarded as integral operators on $\mathbb{T}^n$. If we want to regard them as integral operators on $\mathbb{T}_L^n$, we write $\dot{\mathsf{p}}, \dot{\mathsf{q}}, \dot{\mathsf{r}}$ instead.

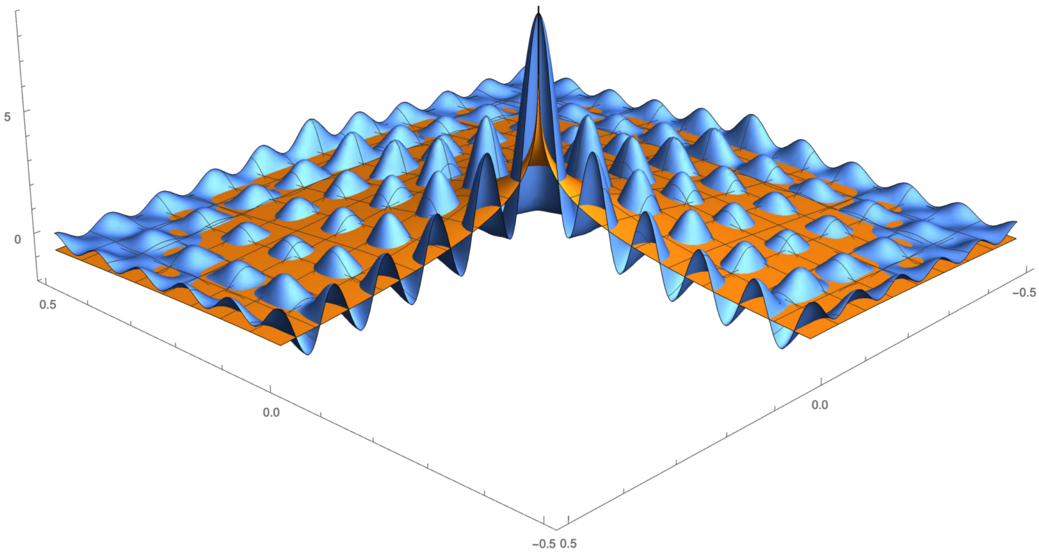


Figure 1: $k(0,y)$ (orange) and $k_{11}(0,y)$ (blue) for $y \in \mathbb{T}^2$ (sectional view with one quadrant removed).

(b) Fourier Extension/Projection. Let $\mathcal{D}_L$ denote the linear span of $\{\varphi_z : z \in \mathbb{Z}_L^n\}$. Every function f on the discrete torus $\mathbb{T}_L^n$ can be uniquely represented as $f = \sum_{z\in\mathbb{Z}_L^n} \alpha_z\varphi_z$ with suitable coefficients $\alpha_z \in \mathbb{R}$ for $z \in \mathbb{Z}_L^n$, and thus uniquely *extends* to a function $f_{L,\sharp} \in \mathcal{D}_L$ on the continuous torus $\mathbb{T}^n$. Formally,

$$f_{L,\sharp}(x) := (\dot{\mathsf{r}}_L f)(x) := \langle f, r_L(x,\cdot)\rangle_{\mathbb{T}_L^n} := \sum_{z\in\mathbb{Z}_L^n} \langle f, \varphi_z\rangle_{\mathbb{T}_L^n} \cdot \varphi_z(x)$$

with the kernel

$$r_L := \sum_{z\in\mathbb{Z}_L^n} \varphi_z \otimes \varphi_z \quad \text{on } \mathbb{T}^n \times \mathbb{T}^n. \tag{8}$$

Regarded as a kernel on $\mathbb{T}_L^n \times \mathbb{T}^n$, the latter defines the Fourier extension operator. As a kernel on $\mathbb{T}_L^n \times \mathbb{T}_L^n$ it indeed is the identity.

Conversely, the *projection* from $\bigcup_s H^s(\mathbb{T}^n)$ onto $\mathcal{D}_L$ is given by $f \mapsto f_{\sharp,L}$ with

$$f_{\sharp,L}(x) := (\mathsf{r}_L f)(x) := \langle f, r_L(x,\cdot)\rangle_{\mathbb{T}^n} := \sum_{z\in\mathbb{Z}_L^n} \langle f, \varphi_z\rangle_{\mathbb{T}^n}\, \varphi_z(x).$$

In particular, if $f = \sum_{z\in\mathbb{Z}^n} \alpha_z\varphi_z$ then $f_{\sharp,L} = \sum_{z\in\mathbb{Z}_L^n} \alpha_z\varphi_z$.

(c) Enhancement and Reduction. For $f = \sum_{z\in\mathbb{Z}_L^n} \alpha_z\varphi_z \in \bigcup_s H^s(\mathbb{T}^n)$ we define its *spectral reduction* and its *spectral enhancement*, resp., by

$$f_L^{-\circ} := \sum_{z\in\mathbb{Z}_L^n} \left(\frac{\lambda_{L,z}}{\lambda_z}\right)^{n/4} \alpha_z\varphi_z, \qquad f_L^{+\circ} := \sum_{z\in\mathbb{Z}_L^n} \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4} \alpha_z\varphi_z.$$

Note that

$$\frac{\lambda_{L,z}}{\lambda_z} = \frac{L^2}{\pi^2\,|z|^2} \sum_{k=1}^n \sin^2(\pi z_k/L) \;\in\; \left[(2/\pi)^2, 1\right] \quad \text{and} \;\to 1 \text{ as } L \to \infty. \tag{9}$$

Similarly, we define its *integral reduction* and its *integral enhancement*, resp., by

$$f_L^{\circ-} := \sum_{z\in\mathbb{Z}_L^n} \vartheta_{L,z}\alpha_z\varphi_z, \qquad f_L^{\circ+} := \sum_{z\in\mathbb{Z}_L^n} \frac{1}{\vartheta_{L,z}}\alpha_z\varphi_z$$

with

$$\vartheta_{L,z} := \prod_{k=1}^{n} \left(\frac{L}{\pi z_k} \sin\left(\frac{\pi z_k}{L}\right)\right) \in [(2/\pi)^n, 1] \quad \text{and } \to 1 \text{ as } L \to \infty. \tag{10}$$

In terms of integral operators this can be expressed as

$$f_L^{-\circ} = \mathsf{r}_L^{-\circ} f, \qquad f_L^{+\circ} = \mathsf{r}_L^{+\circ} f, \qquad f_L^{\circ -} = \mathsf{r}_L^{\circ -} f, \qquad f_L^{\circ +} = \mathsf{r}_L^{\circ +} f$$

with integral and enhancement kernels on $\mathbb{T}^n \times \mathbb{T}^n$ defined as follows

$$r_L^{+\circ} = \sum_{z\in\mathbb{Z}_L^n} \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4} \varphi_z \otimes \varphi_z, \qquad r_L^{-\circ} = \sum_{z\in\mathbb{Z}_L^n} \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{-n/4} \varphi_z \otimes \varphi_z$$

$$r_L^{\circ +} = \sum_{z\in\mathbb{Z}_L^n} \vartheta_{L,z}^{-1}\, \varphi_z \otimes \varphi_z, \qquad r_L^{\circ -} = \sum_{z\in\mathbb{Z}_L^n} \vartheta_{L,z}\, \varphi_z \otimes \varphi_z$$

$$r_L^{+} = \sum_{z\in\mathbb{Z}_L^n} \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4} \vartheta_{L,z}^{-1}\, \varphi_z \otimes \varphi_z, \qquad r_L^{-} = \sum_{z\in\mathbb{Z}_L^n} \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{-n/4} \vartheta_{L,z}\, \varphi_z \otimes \varphi_z$$

Lemma 1.6. *For $f \in \mathcal{D}_L$,*

$$\mathsf{q}_L f = f_L^{\circ -} \text{ on } \mathbb{T}_L^n \quad \text{and} \quad \mathsf{q}_L(f_L^{\circ +}) = f \text{ on } \mathbb{T}_L^n.$$

Proof. For $f = \Phi_z$ with $z \in \mathbb{Z}_L^n$, and for $v \in \mathbb{T}_L^n$,

$$\begin{aligned}\mathsf{q}_L f(v) &= L^n \int_{v+Q_L} \Phi_z(x)dx = \Phi_z(v)\cdot L^n \int_{Q_L} \Phi_z(x)dx \\ &= \Phi_z(v) \cdot \prod_{k=1}^n L \int_{-\frac{1}{2L}}^{\frac{1}{2L}} \cos(2\pi x_k z_k)dx_k = \Phi_z(v)\cdot \prod_{k=1}^n \left(\frac{L}{\pi z_k} \sin\left(\frac{\pi z_k}{L}\right)\right).\end{aligned}$$

Therefore, for $f = \varphi_z$ with $z \in \mathbb{Z}_L^n$, and for $v \in \mathbb{T}_L^n$,

$$\mathsf{q}_L f(v) = f(v)\cdot \vartheta_{L,z}.$$

Thus the claim follows. □

(d) Continuous vs. Discrete Scalar Product. For functions $f = \sum_{z\in\mathbb{Z}_L^n} \alpha_z\varphi_z$ and $g = \sum_{w\in\mathbb{Z}_L^n} \beta_w\varphi_w$, the scalar products in $\mathbb{T}_L^n$ and in $\mathbb{T}^n$ coincide:

$$\langle f, g\rangle_{\mathbb{T}_L^n} = \langle f, g\rangle_{\mathbb{T}^n} = \sum_{z\in\mathbb{Z}_L^n} \alpha_z\, \beta_z \ .$$

This simple identity, however, no longer holds if the Fourier representation of f and g also contains terms with higher frequencies.

Lemma 1.7. *(i) For $f = \sum_{z\in\mathbb{Z}_K^n} \alpha_z\varphi_z$ and $g = \sum_{w\in\mathbb{Z}_K^n} \beta_w\varphi_w$,*

$$\langle f, g\rangle_{\mathbb{T}_L^n} = \sum_{z\in\mathbb{Z}_K^n} \sum_{w\in\mathbb{Z}^n, \|z+Lw\|_\infty < K/2} \alpha_z\, \beta_{z+Lw}.$$

(ii) For any $\alpha : \mathbb{Z}^n \to \mathbb{R}$, the limit $f = \sum^{\square}_{z\in\mathbb{Z}^n} \alpha_z\varphi_z$ exists in $L^2(\mathbb{T}_L^n)$ if and only if

$$\sup_K \Big\| \sum_{z\in\mathbb{Z}_K^n} \alpha_z\varphi_z \Big\|^2_{\mathbb{T}_L^n} < \infty \tag{11}$$

(iii) For all $f = \sum^{\square}_{z\in\mathbb{Z}^n} \alpha_z\varphi_z$ and $g = \sum^{\square}_{w\in\mathbb{Z}^n} \beta_w\varphi_w$ in $L^2(\mathbb{T}_L^n)$,

$$\langle f, g\rangle_{\mathbb{T}_L^n} = \lim_{K\to\infty} \sum_{z\in\mathbb{Z}_K^n} \sum_{w\in\mathbb{Z}^n : \|z+Lw\|_\infty < K/2} \alpha_z\beta_{z+Lw} \ , \qquad \langle f, g\rangle_{\mathbb{T}^n} = \sum^{\square}_{z\in\mathbb{Z}^n} \alpha_z\, \beta_z.$$

Proof. We first prove (i). To that extent, we prove the analogous assertion in the complex Hilbert space: for all $f = \sum_{z\in\mathbb{Z}^n_K} a_z\Phi_z$ and $g = \sum_{w\in\mathbb{Z}^n_K} b_w\Phi_w$,

$$\begin{aligned}\langle f, g\rangle_{\mathbb{T}^n_L} &= \Big\langle \sum_{z\in\mathbb{Z}^n_K} a_z\Phi_z, \sum_{w\in\mathbb{Z}^n_K} b_w\Phi_w \Big\rangle_{\mathbb{T}^n_L}\\ &= \int_{\mathbb{T}^n_L} \sum_{z\in\mathbb{Z}^n_K}\sum_{w\in\mathbb{Z}^n_K} a_z\,\bar{b}_w \exp\big(2\pi i\, v(z-w)\big)\, dm_L(v)\\ &= \sum_{z\in\mathbb{Z}^n_K}\sum_{w\in\mathbb{Z}^n_K} a_z\,\bar{b}_w \cdot \int_{\mathbb{T}^n_L} \exp\big(2\pi i\, v(z-w)\big)\, dm_L(v)\\ &= \sum_{z\in\mathbb{Z}^n_K}\ \sum_{w\in\mathbb{Z}^n, \|z+Lw\|_\infty<K/2} a_z\,\bar{b}_{z+Lw}\end{aligned}$$

since for every $z\in\mathbb{Z}^n$

$$\int_{\mathbb{T}^n_L} \exp(2\pi i\, vz)\, dm_L(v) = \begin{cases} 1, & \text{if } z\in L\mathbb{Z}^n\\ 0, & \text{else}\end{cases}.$$

The claim for the real Hilbert space then follows choosing $a_z = \frac{1}{\sqrt{2}}(\alpha_z + i\alpha_{-z})$ and $a_{-z} = \frac{1}{\sqrt{2}}(\alpha_z - i\alpha_{-z})$ for $z\in\hat{\mathbb{Z}}^n$ and analogously b_z.

We prove (ii). Assume first that $f\in L^2(\mathbb{T}^n_L)$. Then $\sum_{z\in\mathbb{Z}^n_K}\alpha_z\varphi_z$ converges to f in $L^2(\mathbb{T}^n_L)$. This implies (11).

Conversely, assume that (11) holds. Then a martingale argument similar to that of Lemma 1.1 shows that f is the limit in $L^2(\mathbb{T}^n_L)$ of $\sum_{z\in\mathbb{Z}^n_K}\alpha_z\varphi_z$.

We prove (iii). We only need to show the first equation. By linearity it is sufficient to show it for $f = g$. In view of what precedes, we have

$$\|f\|^2_{\mathbb{T}^n_L} = \lim_{K\to\infty}\Bigg\|\sum_{z\in\mathbb{Z}^n_K}\alpha_z\varphi_z\Bigg\|^2 = \lim_{K\to\infty}\sum_{z\in\mathbb{Z}^n_K}\ \sum_{w\in\mathbb{Z}^n:\|z+Lw\|_\infty<K/2}\alpha_z\alpha_{z+Lw},$$

which proves the claim. The convergence of the series is ensured by (11). □

Remark 1.8. According to the previous lemma, in particular, for every $f = \sum^{\square}_{z\in\mathbb{Z}^n}\alpha_z\varphi_z$,

$$\|f\|^2_{\mathbb{T}^n_L} = \sum_{z\in\mathbb{Z}^n}\sum_{w\in\mathbb{Z}^n}\alpha_z\,\alpha_{z+Lw}$$

if the latter series is absolutely convergent.

One can show (cf. proof of Theorem 3.7) that the latter series is absolutely convergent if $f\in\bigcup_{s>n/2}H^s(\mathbb{T}^n)$. This is in accordance with the Sobolev Embedding Theorem which asserts that in this case $f\in\mathcal{C}(\mathbb{T}^n)$ and thus guarantees that the pointwise evaluation of f (at the lattice points of $\mathbb{T}^n_L$) is meaningful.

2 The Polyharmonic Gaussian Field on the Discrete Torus

2.1 Definition and Construction of the Field

Throughout the sequel, fix integers n and L. For convenience, we assume that L is odd, and we set $N := L^n$.

Definition 2.1. *A random field $h_L = (h_L(v))_{v\in\mathbb{T}^n_L}$ — defined on some probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ — is called* polyharmonic Gaussian field on the discrete n-torus *if it is a centered Gaussian field with covariance function k_L (see (6)).*

Proposition 2.2. *Given iid standard normals ξ_z for $z \in \mathbb{Z}_L^n \setminus \{0\}$ on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$, a polyharmonic Gaussian field on $\mathbb{T}_L^n$ is defined by*

$$h_L^\omega(v) := \frac{1}{\sqrt{a_n}} \sum_{z \in \mathbb{Z}_L^n \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/4}} \cdot \xi_z^\omega \cdot \varphi_z(v) \qquad (\forall v \in \mathbb{T}_L^n,\ \omega \in \Omega) \ . \tag{12}$$

Here $\lambda_{L,z} = 4L^2 \sum_{k=1}^n \sin^2\big(\pi z_k/L\big)$ for $z \in \mathbb{Z}_L^n$ are the eigenvalues of the discrete Laplacian, see (4), and the eigenvalue 0 is excluded in the representation of the random field.

Proof. For all $v, u \in \mathbb{T}_L^n$,

$$\mathbf{E}\big[h_L(v)\, h_L(u)\big] = \frac{1}{a_n} \sum_{z \in \mathbb{Z}_L^n \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \cdot \varphi_z(v)\, \varphi_z(u) = k_L(v, u) \ . \qquad \square$$

Alternatively the polyharmonic field on the discrete torus $\mathbb{T}_L^n$ can be defined in terms of the white noise on the discrete torus. Recall that a random field $\Xi = (\Xi_v)_{v \in \mathbb{T}_L^n}$ is called *white noise on* $(\mathbb{T}_L^n, m_L)$ if the Ξ_v for $v \in \mathbb{T}_L^n$ are independent centered Gaussian random variables with variance L^n. (This normalization guarantees that $\int_{\mathbb{T}_L^n} \Xi_v dm_L(v)$ is $\mathcal{N}(0,1)$ distributed.)

Proposition 2.3. *Given a white noise $\Xi = (\Xi_v)_{v \in \mathbb{T}_L^n}$ on $(\mathbb{T}_L^n, m_L)$, a polyharmonic Gaussian field on $\mathbb{T}_L^n$ is defined by*

$$h_L^\omega(v) = \frac{1}{\sqrt{a_n}\, L^n} \sum_{u \in \mathbb{T}_L^n} \mathring{G}_L^{n/4}(v, u)\, \Xi_u^\omega$$

with

$$\mathring{G}_L^{n/4}(v, u) = \sum_{z \in \mathbb{Z}_L^n \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/4}} \cdot \cos\big(2\pi z (v - u)\big) \ .$$

Proof. For all $v, w \in \mathbb{T}_L^n$,

$$\begin{aligned}\mathbf{E}\big[h_L(v)\, h_L(w)\big] &= \frac{1}{a_n\, L^{2n}} \sum_{u \in \mathbb{T}_L^n} \mathring{G}_L^{n/4}(v, u) \cdot \mathring{G}_L^{n/4}(w, u) \cdot L^n \\ &= \frac{1}{a_n} \mathring{G}_L^{n/2}(v, w) = k_L(v, w) \ . \qquad \square\end{aligned}$$

In other words,

$$h_L^\omega = \frac{1}{\sqrt{a_n}} \mathring{\mathsf{G}}_L^{n/4}\, \Xi^\omega \tag{13}$$

with $\Xi := (\Xi_v)_{v \in \mathbb{T}_L^n}$ being a white noise on $\mathbb{T}_L^n$. The latter is a Gaussian random variable on $\mathbb{R}^N \cong \mathbb{R}^{\mathbb{T}_L^n}$ — recall that $N = L^n$ — with distribution

$$d\mathbf{P}(\Xi) = \frac{1}{(2\pi N)^{N/2}}\, e^{-\frac{1}{2N} \|\Xi\|^2}\, d\mathcal{L}^N(\Xi) \ .$$

Here $\|\Xi\|$ denotes the Euclidean norm of $\Xi \in \mathbb{R}^N$, and thus under the identification $\mathbb{R}^N \cong \mathbb{R}^{\mathbb{T}_L^n}$,

$$\frac{1}{N} \|\Xi\|^2 = \|\Xi\|^2_{L^2(\mathbb{T}_L^n, m_L)} \ .$$

2.2 A Second Look on the Polyharmonic Gaussian Field on Discrete Tori

(a) Consider the orthogonal decomposition of $\mathbb{R}^N$ into the line $\mathbb{R} \cdot (1, \ldots, 1)$ and its orthogonal complement $\mathring{H} := \{\Xi \in \mathbb{R}^N : \sum_{v=1}^N \Xi_v = 0\}$. More precisely, consider the maps

$$\bar{A} : \mathbb{R}^N \longrightarrow \mathbb{R}, \quad \Xi \longmapsto \bar{\Xi} := \frac{1}{\sqrt{N}} \sum_{v=1}^N \Xi_v$$

and

$$\mathring{A}:\mathbb{R}^N\longrightarrow\mathring{H},\quad \Xi\longmapsto\mathring{\Xi}\quad\text{with}\quad \mathring{\Xi}_j:=\Xi_j-\frac{1}{\sqrt{N}}\bar{\Xi}\ .$$

Note that $A:=(\mathring{A},\bar{A}):\mathbb{R}^N\to\mathring{H}\times\mathbb{R}\subset\mathbb{R}^{1+N}$ is a bijective linear map with $A^T A=E_N$ and inverse given by

$$B:\mathring{H}\times\mathbb{R}\to\mathbb{R}^N,\quad (\mathring{\Xi},t)\mapsto\mathring{\Xi}+\frac{t}{\sqrt{N}}\cdot(1,\dots,1)\ .$$

Thus if $\mathcal{L}^{N-1}_H$ denotes the $(N-1)$-dimensional Lebesgue measure on the hyperplane $\mathring{H}$ then on $\mathbb{R}^N$,

$$\mathcal{L}^N=\mathcal{L}^{N-1}_{\mathring{H}}\otimes\mathcal{L}^1\ .$$

The push forward $\bar{A}_*\mathbf{P}$ is the normal distribution $\mathcal{N}(0,\sqrt{N})$ on the real line. The push forward $\mathring{\mathbf{P}}:=\mathring{A}_*\mathbf{P}$, called “law of the grounded white noise”, is a Gaussian measure on the hyperplane $\mathring{H}$ given explicitly as

$$d\mathring{\mathbf{P}}(\Xi)=\frac{1}{(2\pi N)^{\frac{N-1}{2}}}\,e^{-\frac{1}{2N}\|\Xi\|^2}\,d\mathcal{L}^{N-1}_{\mathring{H}}(\Xi)\ .$$

It can also be characterized as the conditional law $\mathbf{P}(\,\cdot\,|\bar{A}=0)$.

(b) Let us define a measure on $\mathbb{R}^N\cong\mathbb{R}^{\mathbb{T}^n_L}$ by

$$d\widehat{\boldsymbol{\nu}}(h):=c_n\,e^{-\frac{a_n}{2N}\|(-\Delta_L)^{n/4}h\|^2}\,d\mathcal{L}^N(h) \tag{14}$$

with a constant

$$c_n:=\left(\frac{a_n}{2\pi N}\right)^{\frac{N-1}{2}}\cdot\prod_{z\in\mathbb{Z}^n_L\setminus\{0\}}\left(4L^2\sum_{k=1}^n\sin^2\left(\pi z_k/L\right)\right)^{n/4}\ , \tag{15}$$

and consider the push forwards under the maps $\bar{A}$ and $\mathring{A}$ introduced above. Then

$$\bar{A}_*\widehat{\boldsymbol{\nu}}=c_n\,\mathcal{L}^1\quad\text{on }\mathbb{R}^1$$

and $\boldsymbol{\nu}:=\mathring{A}_*\widehat{\boldsymbol{\nu}}$ is a measure (actually, a probability measure as we will see below) on the hyperplane $\mathring{H}:=\{\Xi\in\mathbb{R}^N:\bar{\Xi}=0\}\cong\mathbb{R}^{N-1}$ given by

$$d\boldsymbol{\nu}(h):=c_n\,e^{-\frac{a_n}{2N}\|(-\Delta_L)^{n/4}h\|^2}\,d\mathcal{L}^{N-1}_{\mathring{H}}(h)\ .$$

(c) Now consider the map

$$T:\mathbb{R}^N\to\mathbb{R}^N\ ,\qquad h\mapsto\Xi=\sqrt{a_n}\,(-\Delta_L)^{n/4}h$$

as well as its restriction $\mathring{T}:\mathring{H}\to\mathring{H}$. The latter is bijective with inverse

$$\mathring{T}^{-1}:\mathring{H}\to\mathring{H}\ ,\qquad \Xi\mapsto h=\frac{1}{\sqrt{a_n}}\,\mathring{\mathsf{G}}^{n/4}_L\Xi\ ,$$

cf. (13), and with determinant

$$\det\mathring{T}=a_n^{\frac{N-1}{2}}\cdot\prod_{z\in\mathbb{Z}^n_L\setminus\{0\}}\lambda^{n/4}_{L,z}\ .$$

Theorem 2.4. *The distribution of the discrete polyharmonic field on $\mathbb{T}^n_L$ is given by the probability measure $\boldsymbol{\nu}$ on $\mathbb{R}^{\mathbb{T}^n_L}\cong\mathbb{R}^N$. (Indeed, it is supported there by the hyperplane of grounded fields.) Furthermore,*

$$\mathring{T}_*\boldsymbol{\nu}=\mathring{\mathbf{P}}\ .$$

Proof. For bounded measurable f on $\mathring{H}$,

$$\begin{aligned}
\int_{\mathring{H}} f(\Xi)\, d\mathring{T}_* \boldsymbol{\nu}(\Xi) &= \int_{\mathring{H}} f(\mathring{T}h)\, d\boldsymbol{\nu}(h) \\
&= c_n \int_{\mathring{H}} f(\mathring{T}h)\, e^{-\frac{a_n}{2N}\|(-\Delta_L)^{n/4}h\|^2}\, d\mathcal{L}^{N-1}_{\mathring{H}}(h) \\
&= c_n \int_{\mathring{H}} f(\mathring{T}h)\, e^{-\frac{1}{2N}\|\mathring{T}h\|^2}\, d\mathcal{L}^{N-1}_{\mathring{H}}(h) \\
&= c_n \det \mathring{T}^{-1} \int_{\mathring{H}} f(\Xi)\, e^{-\frac{1}{2N}\|\Xi\|^2}\, d\mathcal{L}^{N-1}_{\mathring{H}}(\Xi) \\
&= c_n \det \mathring{T}^{-1}\, (2\pi N)^{\frac{N-1}{2}} \int_{H} f(\Xi)\, d\mathring{\mathbf{P}}(\Xi).
\end{aligned}$$

Since $c_n \det \mathring{T}^{-1}\, (2\pi N)^{\frac{N-1}{2}} = 1$ according to our choice of c_n, this proves the claim. □

2.3 The Reduced Polyharmonic Gaussian Field on the Discrete Torus

Besides the polyharmonic Gaussian field h_L on the discrete torus, we occasionally consider two closely related random fields $h_L^{-\circ}$ and h_L^- in the defining properties of which the eigenvalues $\lambda_{L,z} = 4L^2 \sum_{k=1}^n \sin^2\big(\pi z_k/L\big)$ of the discrete Laplacian are replaced by the eigenvalues $\lambda_z = (2\pi|z|)^2$ of the continuous Laplacian or by $\lambda_z \cdot \vartheta_{L,z}^{-4/n}$, resp., with $\vartheta_{L,z}$ as in (10).

More precisely, a *spectrally reduced discrete polyharmonic Gaussian field* is a centered Gaussian field $h_L^{-\circ} = (h_L^{-\circ}(v))_{v\in\mathbb{T}_L^n}$ with covariance function

$$k_L^{-\circ}(v,u) := \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\lambda_z^{n/2}}\, \varphi_z(v)\, \varphi_z(u) = \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{(2\pi|z|)^n} \cdot \cos\Big(2\pi\, z\cdot(v-u)\Big)\ .$$

A *reduced discrete polyharmonic Gaussian field* is a centered Gaussian field $h_L^- = (h_L^-(v))_{v\in\mathbb{T}_L^n}$ with covariance function

$$k_L^{-}(v,u) := \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{\vartheta_{L,z}^2}{\lambda_z^{n/2}}\, \varphi_z(v)\, \varphi_z(u) = \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{\vartheta_{L,z}^2}{(2\pi|z|)^n} \cdot \cos\Big(2\pi\, z\cdot(v-u)\Big)\ .$$

Similarly as before for h_L, we obtain the following representation results.

Remark 2.5. (i) Given a polyharmonic Gaussian field h_L on $\mathbb{T}_L^n$, a reduced polyharmonic Gaussian field and a spectrally reduced polyharmonic Gaussian field on $\mathbb{T}_L^n$ are defined by

$$h_L^- := \mathsf{r}_L^-(h_L), \qquad h_L^{-\circ} := \mathsf{r}_L^{-\circ}(h_L)\ .$$

(ii) Given iid standard normals ξ_z for $z \in \mathbb{Z}_L^n \setminus \{0\}$, a reduced polyharmonic Gaussian field on $\mathbb{T}_L^n$ is defined by

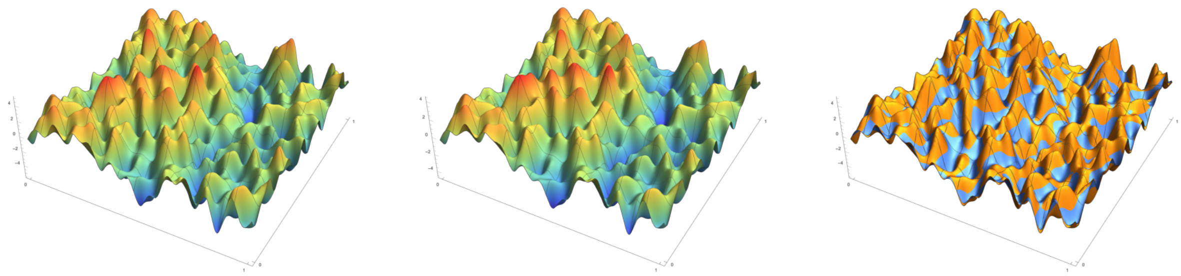
$$h_L^-(v) := \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{\vartheta_{L,z}}{\lambda_z^{n/4}} \cdot \xi_z \cdot \varphi_z(v)\ . \tag{16}$$

and a spectrally reduced polyharmonic Gaussian field by

$$h_L^{-\circ}(v) := \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \frac{1}{\lambda_z^{n/4}} \cdot \xi_z \cdot \varphi_z(v)\ . \tag{17}$$

3 The Polyharmonic Gaussian Field on the Continuous Torus and its (Semi-) Discrete Approximations

This section is devoted to the analysis of approximation properties for the polyharmonic field on the continuous torus in terms of Gaussian fields on the discrete torus and semi-discrete extensions of the latter on the continuous torus.



(a) $h_{25,\sharp}$ **(b)** $h_{\sharp,25}$ **(c)** $h_{25,\sharp}$ (blue) and $h_{\sharp,25}$ (orange)

Figure 2: Fourier extension/projection of h on $\mathcal{D}_{25}$ with same realization of the randomness.

- The basic objects are the polyharmonic field h on the continuous torus and its discrete counterpart, the polyharmonic field h_L on the discrete torus.
- Starting from the field h on $\mathbb{T}^n$, we define its Fourier projection (aka eigenfunction approximation) $h_{\sharp,L}$, its piecewise constant projection $h_{\flat,L}$, its natural projection $h_{\circ,L}$, and its enhanced projection $h_{+,L}$. All of them are Gaussian random fields on $\mathbb{T}^n$.
- Starting from the field h_L on $\mathbb{T}^n_L$, we define its Fourier extension $h_{L,\sharp}$ and its piecewise constant extension $h_{L,\flat}$. Analogous extensions are defined for the so-called spectrally reduced discrete field $h_L^{-\circ}$ and the reduced discrete field h_L^- on $\mathbb{T}^n_L$. All these extensions are Gaussian random fields on $\mathbb{T}^n$.

To summarize

- $\flat$ stands for piecewise constant extension/projection, $\sharp$ for Fourier extension/restriction;
- $h_{L,*}$ with $* \in \{\flat, \sharp\}$ denotes the respective extension of the discrete field h_L; similarly for $h_L^{-\circ}$ and h_L^-;
- $h_{*,L}$ with $* \in \{\flat, \sharp, \circ, +\}$ denotes the projection of the continuous field h onto the respective class of fields of order L on the continuous torus.

3.1 The Polyharmonic Gaussian Field on the Continuous Torus and Convergence Properties of its Projections

3.1.1 The Polyharmonic Gaussian Field h on the Continuous Torus

Definition 3.1. *A random field $h = (\langle h|f\rangle)_{f\in H^{n/2}(\mathbb{T}^n)}$ on the continuous n-torus is called* polyharmonic Gaussian field *if it is a centered Gaussian field with covariance function k in the sense that*

$$\mathbf{E}\big[\langle h|f\rangle \cdot \langle h|g\rangle\big] = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} f(x)k(x,y)g(y)\,dy\,dx \qquad (\forall f,g \in H^{n/2}(\mathbb{T}^n)) .$$

Proposition 3.2. *(i) The polyharmonic Gaussian field exists.*

(ii) It can be realized in $\mathring{H}^{-\epsilon}(\mathbb{T}^n)$.

(iii) The pairing $\langle h|f\rangle$ continuously extends to all $f \in \mathring{H}^{-n/2}(\mathbb{T}^n)$.

Proof. For even n, the polyharmonic field to be considered here is just a particular case of the co-polyharmonic field considered in [5] on large classes of Riemannian manifolds. For flat spaces like the torus, the arguments for proving Thm. 3.1 and Rmks. 3.4+3.5 there obviously also apply to odd n. □

3.1.2 Fourier Projection of h

Given a polyharmonic Gaussian field h on the continuous torus, we define its projection onto the space $\mathcal{D}_L$ (see subsection 1.3(a)) by

$$h_{\sharp,L}(x) := \langle h | r_L(x,\,\cdot\,)\rangle, \qquad r_L := \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \varphi_z \otimes \varphi_z \tag{18}$$

This is a centered Gaussian field on $\mathbb{T}^n$ with covariance function

$$k_{\sharp,L}(x,y) := \frac{1}{a_n} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda_z^{n/2}} \cdot \cos\Big(2\pi\, z\cdot(x-y)\Big) \tag{19}$$

where $\lambda_z = (2\pi|z|)^2$.

Proposition 3.3 ([5])**.** *For all $f \in \mathring{H}^{-n/2}(\mathbb{T}^n)$,*

$$\langle h_{\sharp,L}, f\rangle_{\mathbb{T}^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \text{in } L^2(\mathbf{P}) \text{ as } L\to\infty$$

and for every $\epsilon > 0$,

$$h_{\sharp,L} \to h \quad \text{in } L^2(H^{-\epsilon}(\mathbb{T}^n), \mathbf{P}) \text{ as } L\to\infty\ .$$

Proof. For the convenience of the reader, we summarize briefly the argument from [5] for the first assertion:

$$\begin{aligned}\mathbf{E}\Big[\langle h - h_{\sharp,L}, f\rangle^2\Big] &= \frac{1}{a_n}\mathbf{E}\left[\left|\sum_{z\in\mathbb{Z}^n\setminus\mathbb{Z}^n_L} \frac{1}{(2\pi|z|)^{n/2}}\,\xi_z\,\langle\varphi_z, f\rangle\right|^2\right] \\ &= \frac{1}{a_n}\sum_{z\in\mathbb{Z}^n\setminus\mathbb{Z}^n_L} \frac{1}{(2\pi|z|)^{n}}\,\langle\varphi_z, f\rangle^2 \to 0\end{aligned}$$

as $L\to\infty$ since

$$\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{(2\pi|z|)^{n}}\,\langle\varphi_z, f\rangle^2 = \big\|(-\Delta)^{-n/2} f\big\|^2_{L^2} = \big\|f\big\|^2_{\mathring{H}^{-n/2}} < \infty\ . \qquad \square$$

3.1.3 Piecewise Constant Projection of h

Given a polyharmonic Gaussian field h on the continuous torus, we define its piecewise constant projection (cf. subsection 1.3(b)) by

$$h_{\flat,L}(x) := \langle h | q_L(x,\,\cdot\,)\rangle, \qquad q_L := L^n \sum_{v\in\mathbb{T}^n_L} \mathbf{1}_{v+Q_L} \otimes \mathbf{1}_{v+Q_L}\ . \tag{20}$$

Then $(h_{\flat,L}(x))_{x\in\mathbb{T}^n}$ is a centered Gaussian field with covariance function

$$k_{\flat,L}(x,y) := \mathbb{E}\big[h_{\flat,L}(x)\, h_{\flat,L}(y)\big] = L^{2n} \sum_{v,w\in\mathbb{T}^n_L} \mathbf{1}_{v+Q_L}(x)\,\mathbf{1}_{w+Q_L}(y) \int_{v+Q_L}\int_{w+Q_L} k(x',y')\,dy'\,dx'\ . \tag{21}$$

Proposition 3.4. *For all $f \in L^2(\mathbb{T}^n)$,*

$$\langle h_{\flat,L}, f\rangle_{\mathbb{T}^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \text{in } L^2(\mathbf{P}) \text{ as } L\to\infty$$

and for every $s > n/2$,

$$\big\|h_{\flat,L} - h\big\|_{H^{-s}} \to 0 \quad \text{in } L^2(\mathbf{P}) \text{ as } L\to\infty\ .$$

Proof. Since $\langle h_{\flat,L}, f\rangle_{\mathbb{T}^n} = \langle h, \mathsf{q}_L f\rangle_{\mathbb{T}^n}$ with $(\mathsf{q}_L f)(x) := \int_{\mathbb{T}^n} q_L(x,y) f(y)\, dy$, we obtain

$$\begin{aligned}\mathbf{E}\Big[\langle h_{\flat,L} - h, f\rangle^2\Big] = \mathbf{E}\Big[\langle h, \mathsf{q}_L f - f\rangle^2\Big] &= \big\|\mathsf{q}_L f - f\big\|^2_{H^{-n/2}} \\ &\leq \big\|\mathsf{q}_L f - f\big\|^2_{L^2} \to 0 \quad \text{as } L\to\infty.\end{aligned}$$

To prove the second assertion, let h be given as

$$h = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \xi_z \frac{\varphi_z}{\lambda_z^{n/4}}$$

from which we get

$$h_{\flat,L} = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \xi_z \frac{\mathsf{q}_L\varphi_z}{\lambda_z^{n/4}} = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{\xi_z}{\lambda_z^{n/4}} \sum_{w\in\mathbb{Z}^n\setminus\{0\}} \langle \varphi_w, \mathsf{q}_L\varphi_z\rangle\varphi_w.$$

Consequently

$$\begin{aligned}\sqrt{a_n}\,(-\Delta)^{-s/2}\,(h - h_{\flat,L}) = &\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \xi_z\lambda_z^{-n/4-s/2}\,(\varphi_z - \langle\varphi_z, \mathsf{q}_L\varphi_z\rangle\varphi_z) \\ &- \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \xi_z\lambda_z^{-n/4} \sum_{w\in\mathbb{Z}^n\setminus\{0,z\}} \lambda_w^{-s/2}\langle\varphi_w, \mathsf{q}_L\varphi_z\rangle\varphi_w.\end{aligned}$$

Finally,

$$\begin{aligned}a_n\mathbf{E}\big[\big\|h - h_{\flat,L}\big\|^2_{H^{-s}}\big] = &\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \lambda_z^{-n/2-s}\,\big(1 - \langle\varphi_z, \mathsf{q}_L\varphi_z\rangle\big)^2 \\ &+ \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \lambda_z^{-n/2} \sum_{w\in\mathbb{Z}^n\setminus\{0,z\}} \lambda_w^{-s}\langle\varphi_w, \mathsf{q}_L\varphi_z\rangle^2.\end{aligned} \tag{22}$$

Since $|\langle\varphi_z, \mathsf{q}_L\varphi_z\rangle| \leq 1$ for all L and $\langle\varphi_z, \mathsf{q}_L\varphi_z\rangle \to 1$ as $L\to\infty$ and

$$\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \lambda_z^{-n/2-s}\,\big(1 - \langle\varphi_z, \mathsf{q}_L\varphi_z\rangle\big)^2 \leq \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \lambda_z^{-n/2-s} < \infty,$$

we find that the first term on the RHS of (22) vanishes as $L\to\infty$. By Parseval's identity and the fact that $\lambda_z \geq 1$ for all $z\neq 0$, we get that

$$\begin{aligned}\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \lambda_z^{-n/2} \sum_{w\in\mathbb{Z}^n\setminus\{0,z\}} \lambda_w^{-s}\langle\varphi_w, \mathsf{q}_L\varphi_z\rangle^2 &\leq \sum_{w\in\mathbb{Z}^n\setminus\{0\}} \lambda_w^{-s} \sum_{z\in\mathbb{Z}^n\setminus\{0,w\}} \langle\varphi_w, \mathsf{q}_L\varphi_z\rangle^2 \\ &= \sum_{w\in\mathbb{Z}^n\setminus\{0\}} \lambda_w^{-s}\,\Big[\|\mathsf{q}_L\varphi_w\|^2_{L^2} - \langle\varphi_w, \mathsf{q}_L\varphi_w\rangle^2\Big] \\ &\leq \sum_{w\in\mathbb{Z}^n\setminus\{0\}} \lambda_w^{-s}\end{aligned}$$

which converges since $s > n/2$. Moreover, $0 \leq \|\mathsf{q}_L\varphi_w\|^2_{L^2} - \langle\varphi_w, \mathsf{q}_L\varphi_w\rangle^2 \leq 1$ for all w and L, and $\|\mathsf{q}_L\varphi_w\|^2_{L^2} - \langle\varphi_w, \mathsf{q}_L\varphi_w\rangle^2 \to 0$ for all w as $L\to\infty$. Thus also the second term on the RHS of (22) vanishes as $L\to\infty$. □

3.1.4 Enhanced Projection of h

Given a polyharmonic Gaussian field h on the continuous torus, we define its *enhanced piecewise constant projection* or briefly *enhanced projection* by

$$h_{+,L}(x) := \langle h | p_{+,L}(x,\,\cdot\,)\rangle, \qquad p_{+,L} := q_L \circ r_L^+ \tag{23}$$

where as before $r_L^+ := \sum_{z\in\mathbb{Z}_L^n} \frac{1}{\vartheta_{L,z}} \cdot \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4} \cdot \varphi_z \otimes \varphi_z$ and $q_L := L^n \sum_{v\in\mathbb{T}_L^n} \mathbf{1}_{v+Q_L} \otimes \mathbf{1}_{v+Q_L}$. It is a centered Gaussian field with covariance function

$$k_{+,L}(x,y) := \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n} \frac{1}{\vartheta_{L,z}^2} \cdot \frac{1}{\lambda_{L,z}^{n/2}} \cdot \mathsf{q}_L\varphi_z(x) \cdot \mathsf{q}_L\varphi_z(y). \tag{24}$$

More precisely, it can be regarded as a piecewise constant random field on the continuous torus, or equivalently as a random field on the discrete torus.

Lemma 3.5. *(i) For all $f \in L^2(\mathbb{T}^n)$,*

$$\mathsf{p}_{+,L} f \to f \quad \text{in } L^2 \text{ as } L \to \infty.$$

(ii) On $\mathbb{T}^n \times \mathbb{T}^n$,

$$k_{+,L} \to k \quad \text{in } L^0 \text{ as } L \to \infty.$$

Proof. (i) Recall that $\mathsf{p}_{+,L} = \mathsf{q}_L \circ \mathsf{r}_L^+$. From (the proof of) Example 3.12 in [5] we know that $\|\mathsf{q}_L f - f\|_{L^2} \to 0$ as $L \to \infty$ and, by Jensen's inequality,

$$\left\|\mathsf{q}_L f - \mathsf{q}_L \mathsf{r}_L^+ f\right\|_{L^2} \le \left\|f - \mathsf{r}_L^+ f\right\|_{L^2}.$$

Moreover, the latter goes to 0 as $L \to \infty$ according to

$$\begin{aligned}\left\|f - \mathsf{r}_L^+ f\right\|_{L^2}^2 &= \left\|\sum_{\|z\|_\infty < L/2} \left[1 - \frac{1}{\vartheta_{L,z}} \cdot \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4}\right] \cdot \langle f, \varphi_z\rangle\, \varphi_z + \sum_{\|z\|_\infty > L/2} \langle f, \varphi_z\rangle\, \varphi_z\right\|_{L^2}^2 \\ &= \sum_{\|z\|_\infty < L/2} \left[1 - \frac{1}{\vartheta_{L,z}} \cdot \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4}\right]^2 \cdot \langle f, \varphi_z\rangle^2 + \sum_{\|z\|_\infty > L/2} \langle f, \varphi_z\rangle^2 \;\to\; 0.\end{aligned}$$

The convergence of the last term here follows from the finiteness of $\sum_z \langle f, \varphi_z\rangle^2 = \|f\|_{L^2}^2$. The convergence of the first term in the last displayed formula follows from the facts that $\left|1 - \frac{1}{\vartheta_{L,z}} \cdot \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4}\right| \le C := (\frac{\pi}{2})^{3n/2}$ for all L and z, that $\left|1 - \frac{1}{\vartheta_{L,z}} \cdot \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4}\right| \to 0$ for all z as $L \to \infty$, and that $\sum_z \langle f, \varphi_z\rangle^2 < \infty$.

(ii) Denote by $\mathsf{q}_L^{\otimes 2}$ the twofold action of q_L on functions of two variables, and put

$$k_L^+(x,y) := \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n} \frac{1}{\vartheta_{L,z}^2} \cdot \frac{1}{\lambda_{L,z}^{n/2}} \cdot \varphi_z(x) \cdot \varphi_z(y). \tag{25}$$

Then $k_{+,L} = \mathsf{q}_L^{\otimes 2} k_L^+$. The claimed convergence will follow from the three subsequent convergence assertions:

(1) $\mathsf{q}_L^{\otimes 2} k(x,y) \to k(x,y)$ locally uniformly for all $x \neq y$

(2) $\iint \left|\mathsf{q}_L^{\otimes 2} k_L^+ - \mathsf{q}_L^{\otimes 2} k\right|^2 dxdy \le \iint \left|k_L^+ - k\right|^2 dxdy$

(3) $\iint \left|k_L^+ - k\right|^2 dxdy \to 0$.

Assertion (1) here is trivial since k is smooth outside the diagonal and since q_L acts with bounded support $\le 1/L \to 0$. Assertion (2) follows from a simple application of Jensen's inequality. To see assertion (3), observe that

$$\begin{aligned}\iint \left|k_L^+ - k\right|^2 dxdy &= \frac{1}{a_n} \iint \left|\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \left(\frac{1}{\vartheta_{L,z}^2 \lambda_{L,z}^{n/2}} \mathbf{1}_{\{\|z\|_\infty < L/2\}} - \frac{1}{\lambda_z^{n/2}}\right) \varphi_z(x)\, \varphi_z(y)\right|^2 dx\, dy \\ &= \frac{1}{a_n} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \left(\frac{1}{\vartheta_{L,z}^2 \lambda_{L,z}^{n/2}} \mathbf{1}_{\{\|z\|_\infty < L/2\}} - \frac{1}{\lambda_z^{n/2}}\right)^2 \\ &= \frac{1}{a_n} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{(2\pi|z|)^{2n}} \left(\frac{\lambda_z^{n/2}}{\vartheta_{L,z}^2 \lambda_{L,z}^{n/2}} \mathbf{1}_{\{\|z\|_\infty < L/2\}} - 1\right)^2.\end{aligned}$$

The latter converges to 0 as $L \to \infty$ since the term $\left(\frac{\lambda_z^{n/2}}{\vartheta_{L,z}^2 \lambda_{L,z}^{n/2}} \mathbf{1}_{\{\|z\|_\infty < L/2\}} - 1 \right)$ is bounded uniformly in L and z, since it converges to 0 for every z as $L \to \infty$, and since the sum $\sum_{z \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{(2\pi|z|)^{2n}}$ is finite. □

3.1.5 Natural Projection of h

Given a polyharmonic Gaussian field h on the continuous torus, we define its *natural projection* as the piecewise constant projection of its Fourier projection:

$$h_{\circ,L} := \mathsf{q}_L(\mathsf{r}_L(h)). \tag{26}$$

In other words, $h_{\circ,L}(x) := \langle h | p_{\circ,L}(x, \cdot) \rangle$ where $p_{\circ,L} := q_L \circ r_L$ and as before $r_L := \sum_{z \in \mathbb{Z}_L^n} \varphi_z \otimes \varphi_z$ and $q_L := L^n \sum_{v \in \mathbb{T}_L^n} \mathbf{1}_{v+Q_L} \otimes \mathbf{1}_{v+Q_L}$. It is a centered Gaussian field with covariance function

$$k_{\circ,L}(x,y) := \frac{1}{a_n} \sum_{z \in \mathbb{Z}_L^n} \frac{1}{\lambda_z^{n/2}} \cdot \mathsf{q}_L \varphi_z(x) \cdot \mathsf{q}_L \varphi_z(y). \tag{27}$$

More precisely, it can be regarded as a piecewise constant random field on the continuous torus and equivalently as a random field on the discrete torus.

Lemma 3.6. *(i) For all $f \in L^2(\mathbb{T}^n)$,*

$$\mathsf{p}_{\circ,L} f \to f \quad \textit{in } L^2 \textit{ as } L \to \infty.$$

(ii) On $\mathbb{T}^n \times \mathbb{T}^n$,

$$k_{\circ,L} \to k \quad \textit{in } L^0 \textit{ as } L \to \infty.$$

Proof. Analogously to (but simpler than) Lemma 3.5. □

3.2 Convergence Properties of the Polyharmonic Gaussian Field on the Discrete Torus and of its Extensions

3.2.1 Polyharmonic Gaussian Field h_L on the Discrete Torus

Theorem 3.7. *For all $f \in \bigcup_{s > n/2} H^s(\mathbb{T}^n)$,*

$$\langle h_L, f \rangle_{\mathbb{T}_L^n} \to \langle h, f \rangle_{\mathbb{T}^n} \quad \textit{in } L^2(\mathbf{P}) \textit{ as } L \to \infty \ .$$

Proof. Given $f = \sum_{z \in \mathbb{Z}^n} \alpha_z \varphi_z \in \bigcup_{s > n/2} H^s(\mathbb{T}^n)$, according to Lemma 1.7,

$$\begin{aligned} \langle h, f \rangle_{\mathbb{T}^n} - \langle h_L, f \rangle_{\mathbb{T}_L^n} = & \frac{1}{\sqrt{a_n}} \sum_{z \in \mathbb{Z}_L^n \setminus \{0\}} \xi_z \cdot \left[\frac{1}{\lambda_z^{n/4}} \alpha_z - \frac{1}{\lambda_{L,z}^{n/4}} \sum_{w \in \mathbb{Z}^n} \alpha_{z+Lw} \right] \\ & + \frac{1}{\sqrt{a_n}} \sum_{z \in \mathbb{Z}^n, \, \|z\|_\infty \geq L/2} \xi_z \cdot \frac{1}{\lambda_z^{n/4}} \alpha_z \ . \end{aligned}$$

Thus

$$
\begin{aligned}
&a_n \cdot \mathbf{E}\Big[\big|\langle h, f\rangle_{\mathbb{T}^n} - \langle h_L, f\rangle_{\mathbb{T}^n_L}\big|^2\Big] \\
&= \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left[\frac{1}{\lambda_z^{n/4}}\alpha_z - \frac{1}{\lambda_{L,z}^{n/4}} \sum_{w\in\mathbb{Z}^n} \alpha_{z+Lw}\right]^2 + \sum_{z\in\mathbb{Z}^n,\, \|z\|_\infty\geq L/2} \frac{1}{\lambda_z^{n/2}}\alpha_z^2 \\
&\leq 2 \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left[\frac{1}{\lambda_z^{n/4}} - \frac{1}{\lambda_{L,z}^{n/4}}\right]^2 \alpha_z^2 \\
&\quad + 2 \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} \alpha_{z+Lw}\right]^2 + \sum_{z\in\mathbb{Z}^n,\, \|z\|_\infty\geq L/2} \frac{1}{\lambda_z^{n/2}}\alpha_z^2 \\
&\leq \frac{2}{(2\pi)^n} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left[1 - \frac{\lambda_z^{n/4}}{\lambda_{L,z}^{n/4}}\right]^2 \frac{1}{|z|^n}\alpha_z^2 \\
&\quad + \frac{2}{4^n} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{|z|^n} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} \alpha_{z+Lw}\right]^2 + \frac{1}{(2\pi)^n} \sum_{z\in\mathbb{Z}^n,\, \|z\|_\infty\geq L/2} \frac{1}{|z|^n}\alpha_z^2 .
\end{aligned}
$$

Now as $L \to \infty$, the last term vanishes (since f in particular lies in $H^{-n/2}(\mathbb{T}^n)$) and also the first term vanishes, see the proof of Theorem 3.9 below. To estimate the second term, choose $s > n/2$ with $\sum_{z\in\mathbb{Z}^n\setminus\{0\}} |z|^{2s}|\alpha_z|^2 < \infty$, which exists by definition of f. Then,

$$
\begin{aligned}
&\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{|z|^n} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} \alpha_{z+Lw}\right]^2 \leq \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} \alpha_{z+Lw}\right]^2 \\
&\leq \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|z+Lw|^{2s}}\right] \cdot \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} |z+Lw|^{2s} \cdot |\alpha_{z+Lw}|^2\right] .
\end{aligned}
$$

Estimating the term in the first bracket by

$$
\sum_{w\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|z+Lw|^{2s}} \leq \sum_{u\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|u|^{2s}} < \infty ,
$$

we then obtain that

$$
\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{|z|^n} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} \alpha_{z+Lw}\right]^2 \leq \left[\sum_{u\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|u|^{2s}}\right] \cdot \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} |z+Lw|^{2s} \cdot |\alpha_{z+Lw}|^2\right]
$$

and it therefore suffices to show that the second factor vanishes as $L \to \infty$. Since $z, w \neq 0$, we have that $|z + Lw| \geq L/2$, thus, relabelling $v := z + Lw$,

$$
\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \left[\sum_{w\in\mathbb{Z}^n\setminus\{0\}} |z+Lw|^{2s} \cdot |\alpha_{z+Lw}|^2\right] = \left[\sum_{v\in\mathbb{Z}^n,\, \|v\|_\infty\geq L/2} |v|^{2s} \cdot |\alpha_v|^2\right] \to 0 \qquad \text{as } L \to \infty
$$

being the remainder of a convergent series. □

3.2.2 Piecewise Constant Extension of h_L

Given a polyharmonic Gaussian field h_L on the discrete torus, recall that its piecewise constant extension to the continuous torus by $h_{L,\flat}(x) := h_L(v)$ if $x \in v + Q_L$ where $Q_L = [-\frac{1}{2L}, \frac{1}{2L})^n$. Then

$$
\langle h_{L,\flat}, f\rangle_{\mathbb{T}^n} = \langle h_L, \mathsf{q}_L f\rangle_{\mathbb{T}^n_L}
$$

with $\mathsf{q}_L f \in L^2(\mathbb{T}^n_L)$ for $f \in L^2(\mathbb{T}^n)$. Note that $\mathsf{q}_L f(v) = L^n \int_{v+Q_L} f(y)dy$ for $v \in \mathbb{T}^n_L$.

Theorem 3.8. *For all* $f \in \bigcup_{s>n/2} H^s(\mathbb{T}^n)$,

$$\langle h_{L,\flat}, f\rangle_{\mathbb{T}^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \text{in } L^2(\mathbf{P}) \text{ as } L \to \infty \, . \tag{28}$$

Furthermore, for every $s > n/2$,

$$\left\|h_{L,\flat} - h\right\|^2_{\mathring{H}^{-s}} \to 0 \quad \text{in } L^2(\mathbf{P}) \text{ as } L \to \infty \, . \tag{29}$$

Proof. By construction,

$$\langle h_{L,\flat}, f\rangle_{\mathbb{T}^n} - \langle h, f\rangle_{\mathbb{T}^n} = \langle h_L, \mathsf{q}_L f - f\rangle_{\mathbb{T}^n_L} + \langle h_L, f\rangle_{\mathbb{T}^n_L} - \langle h, f\rangle_{\mathbb{T}^n}$$

and according to the previous Theorem 3.7, $\langle h_L, f\rangle_{\mathbb{T}^n_L} - \langle h, f\rangle_{\mathbb{T}^n} \to 0$ as $L \to \infty$. The first claim thus follows from

$$\begin{aligned}\mathbf{E}\Big[\langle h_L, \mathsf{q}_L f - f\rangle^2_{\mathbb{T}^n_L}\Big] &= \frac{1}{a_n} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda^{n/2}_{L,z}} \langle \varphi_z, \mathsf{q}_L f - f\rangle^2_{\mathbb{T}^n_L} \\ &\le \frac{1}{4^n\, a_n} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{|z|^n} \langle \varphi_z, \mathsf{q}_L f - f\rangle^2_{\mathbb{T}^n_L} \\ &= \frac{1}{4^n\, a_n} \left\|\mathsf{q}_L f - f\right\|^2_{H^{-n/2}} \\ &\le \frac{1}{4^n\, a_n} \left\|\mathsf{q}_L f - f\right\|^2_{L^2} \to 0 \quad \text{as } L \to \infty\end{aligned}$$

since by Sobolev embedding

$$\bigcup_{s>n/2} H^s(\mathbb{T}^n) \subset \mathcal{C}(\mathbb{T}^n) \, .$$

For the second claim, observe that

$$h_{L,\flat} = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \varphi_z \sum_{w\in\mathbb{Z}^n_L\setminus\{0\}} \lambda^{-n/4}_{w,L} \xi_w \langle \varphi_w, \mathsf{q}_L \varphi_z\rangle.$$

From this, we get

$$a_n \left\|h_{L,\flat}\right\|^2_{H^{-s}} = \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \lambda^{-s}_z \left(\sum_{w\in\mathbb{Z}^n_L\setminus\{0\}} \lambda^{-n/4}_{w,L} \xi_w \langle \varphi_w, \mathsf{q}_L \varphi_z\rangle \right)^2 .$$

And thus

$$a_n \mathbf{E} \left\|h_{L,\flat}\right\|^2_{H^{-s}} = \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \lambda^{-s}_z \sum_{w\in\mathbb{Z}^n_L\setminus\{0\}} \lambda^{-n/2}_{w,L} \langle \varphi_w, \mathsf{q}_L \varphi_z\rangle^2$$

On the one hand, as $L \to \infty$, the summand converges to $\lambda_z^{-s-n/2} 1_{z=w}$. On the other hand, by Parseval's indentity, we find that

$$\sum_{w\in\mathbb{Z}^n_L\setminus\{0\}} \lambda^{-n/2}_{w,L} \langle \varphi_w, \mathsf{q}_L \varphi_z\rangle^2 = \left\|(-\Delta_L)^{-n/2} \mathsf{q}_L \varphi_z\right\|^2_{L^2(\mathbb{T}^n_L)} \le 1.$$

Thus the sum is uniformly bounded since $s > n/2$. This shows that

$$\mathbf{E} \left\|h_{L,\flat}\right\|^2_{H^{-s}} \to \mathbf{E} \left\|h\right\|^2_{H^{-s}} .$$

A similar computation shows that

$$\mathbf{E} \langle h, h_{L,\flat}\rangle_{H^{-s}} \to \mathbb{E} \left\|h\right\|^2_{H^{-s}} .$$

This concludes the proof of the second claim. □

3.2.3 Fourier Extension of h_L

Given a polyharmonic Gaussian field h_L on the discrete torus, we define its Fourier extension to the continuous torus as

$$h_{L,\sharp}(x) := \dot{\mathsf{r}}_L h(x) := \langle h, r_L(x,\cdot)\rangle_{\mathbb{T}^N_L} \qquad (\forall x \in \mathbb{T}^n) .$$

In other words, for every ω the function $h^\omega_{L,\sharp}$ is the unique function in $\mathcal{D}_L$ with $h^\omega_{L,\sharp} = h^\omega_L$ on $\mathbb{T}^n_L$, cf. (12). If h_L is given as

$$h_L(v) := \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda^{n/4}_{L,z}} \cdot \xi_z \cdot \varphi_z(v) \qquad (\forall v \in \mathbb{T}^n_L)$$

then the same formula provides the representation of $h_{L,\sharp}$:

$$h_{L,\sharp}(x) := \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda^{n/4}_{L,z}} \cdot \xi_z \cdot \varphi_z(x) \qquad (\forall x \in \mathbb{T}^n) . \tag{30}$$

This is a centered Gaussian field on $\mathbb{T}^n$ with covariance function

$$k_{L,\sharp}(x,y) = \frac{1}{a_n} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda^{n/2}_{L,z}} \cdot \cos\Big(2\pi\, z\cdot(x-y)\Big) \qquad (\forall x,y \in \mathbb{T}^n) \tag{31}$$

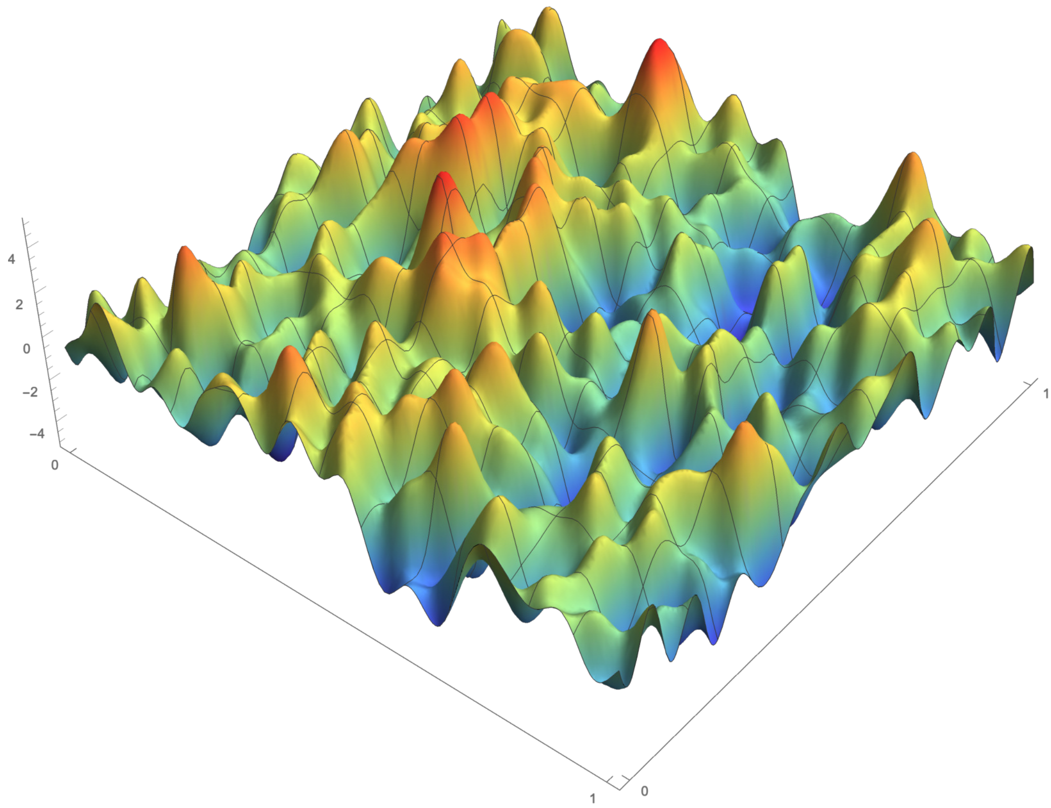


Figure 3: A realization of $h_{25,\sharp}$

Theorem 3.9. *For all* $f \in \mathring{H}^{-n/2}(\mathbb{T}^n)$,

$$\langle h_{L,\sharp}, f\rangle_{\mathbb{T}^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \textit{in } L^2(\mathbf{P}) \textit{ as } L\to\infty , \tag{32}$$

and for all $\epsilon > 0$,

$$\big\| h_{L,\sharp} - h \big\|^2_{H^{-\epsilon}} \to 0 \quad \textit{in } L^2(\mathbf{P}) \textit{ as } L\to\infty . \tag{33}$$

Proof. According to the Proposition 3.3, we already know that $\langle h_{\sharp,L}, f\rangle \to \langle h, f\rangle$. Thus it suffices to prove $\langle h_{L,\sharp} - h_{\sharp,L}, f\rangle \to 0$. This follows according to

$$\begin{aligned}
&\mathbf{E}\Big[\langle h_{L,\sharp} - h_{\sharp,L}, f\rangle^2\Big] \\
&= \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \left(\frac{1}{\lambda_{L,z}^{n/2}} - \frac{1}{\lambda_z^{n/2}}\right) \langle\varphi_z, f\rangle^2 \\
&= \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \left(\left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/2} - 1\right) \frac{\langle\varphi_z, f\rangle^2}{\lambda_z^{n/2}} \to 0
\end{aligned}$$

as $L \to \infty$ by the Dominated Convergence Theorem since

$$\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{\lambda_z^{n/2}} \langle\varphi_z, f\rangle^2 < \infty \,,$$

$$0 \le \left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/2} - 1 \le 2^n$$

for all z and L, and

$$\left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/2} - 1 \to 0$$

as $L \to \infty$ for every z.

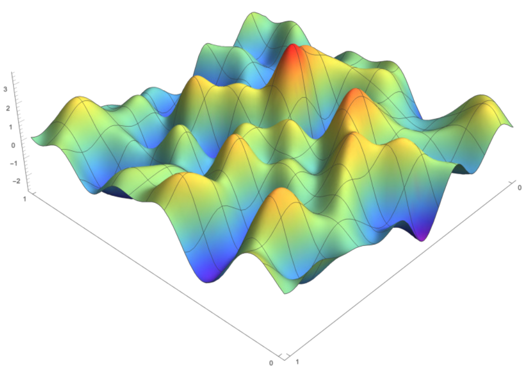
For the second claim, we use the fact that again by Proposition 3.3 for every $\epsilon > 0$,

$$\mathbf{E}\Big[\big\|h - h_{\sharp,L}\big\|^2_{\mathring{H}^{-\epsilon}}\Big] \to 0 \quad \text{as } L \to \infty \,.$$

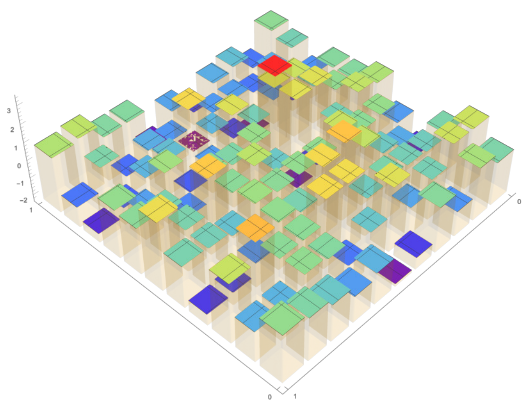
Furthermore,

$$\begin{aligned}
\mathbf{E}\Big[\big\|h_{L,\sharp} - h_{\sharp,L}\big\|^2_{\mathring{H}^{-\epsilon}}\Big] &= \frac{1}{a_n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \left(\frac{1}{\lambda_{L,z}^{n/4}} - \frac{1}{(2\pi|z|)^{n/2}}\right)^2 \frac{1}{|z|^{2\epsilon}} \\
&= \frac{1}{a_n(2\pi)^n} \sum_{z\in\mathbb{Z}_L^n\setminus\{0\}} \left(\left(\frac{\lambda_z}{\lambda_{L,z}}\right)^{n/4} - 1\right)^2 \frac{1}{|z|^{n-2\epsilon}} \to 0
\end{aligned}$$

as $L \to \infty$ by the same Dominated Convergence arguments as for the first claim. □



(a) $h_{11,\sharp}$



(b) $h_{11,\flat}$

Figure 4: Two different approximations/extensions of the field h with same realization of the randomness.

3.2.4 Reduced and Spectrally Reduced Polyharmonic Gaussian Fields on the Discrete Torus

The previous assertions Theorem 3.7, 3.8, and 3.9 on the polyharmonic field and its (Fourier or piecewise constant, resp.) extensions — as well as their proofs — hold true in analogous form for the reduced and spectrally reduced polyharmonic fields h_L^- and $h_L^{-\circ}$ and their (Fourier or piecewise constant, resp.) extensions $h_{L,\sharp}^-$, $h_{L,\flat}^-$ and $h_{L,\sharp}^{-\circ}$, $h_{L,\flat}^{-\circ}$.

Proposition 3.10. *(i) For all* $f \in \bigcup_{s>n/2} H^s(\mathbb{T}^n)$,

$$\langle h_L^{-\circ}, f\rangle_{\mathbb{T}_L^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \text{in } L^2(\mathbf{P}) \text{ as } L \to \infty$$

as well as

$$\langle h_{L,\flat}^{-\circ}, f\rangle_{\mathbb{T}^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \text{in } L^2(\mathbf{P}) \text{ as } L \to \infty\ .$$

(ii) For all $f \in \mathring{H}^{-n/2}(\mathbb{T}^n)$,

$$\langle h_{L,\sharp}^{-\circ}, f\rangle_{\mathbb{T}^n} \to \langle h, f\rangle_{\mathbb{T}^n} \quad \text{in } L^2(\mathbf{P}) \text{ as } L \to \infty\ .$$

(iii) Analogously with h_L^-, $h_{L,\flat}^-$, $h_{L,\sharp}^-$ *in the place of* $h_L^{-\circ}$, $h_{L,\flat}^{-\circ}$, $h_{L,\sharp}^{-\circ}$.

3.2.5 Related Approaches and Results

Remark 3.11 (Log-correlated Gaussian fields in the continuum). For $n = 1$, the field h is the lower limiting case of the fractional Brownian motion with (regularity) parameter in $(\frac{1}{2}, \frac{3}{2})$, see e.g. [12, 7]. For $n = 2$, it is the celebrated *Gaussian Free Field* (GFF) on $\mathbb{T}^2$, surveyed in [16]. For $n = 1$, it coincides in distribution with the restriction of a GFF to a line ($\mathbb{T}^1 \subset \mathbb{T}^2$). For $n \geq 3$, it is a log-correlated Gaussian field surveyed in [7]. The conformal invariance of h on $\mathbb{T}^n$ is a consequence of the conformal invariance of the Laplace–Beltrami operator on flat geometries. The correct (i.e., conformally covariant) construction of log-correlated Gaussian fields in general non-flat geometries may be found in [5].

Remark 3.12 (Discrete-to-continuum approximation). To the best of our knowledge, no discretization/discrete approximation results are available for log-correlated Gaussian fields in dimension $n \geq 5$. In small dimension however, Gaussian fields analogous to h are known to be scaling limits of different discrete models. For $n = 2$, in light of the celebrated universality property of GFFs, the field h is a scaling limit for a huge number of different discrete Gaussian and non-Gaussian fields defined in various settings, from lattices to random environments, as, e.g., the random conductance model [1]. For $n = 4$, the field h is generated by the Neumann bi-Laplacian; the analogous field generated by the Dirichlet bi-Laplacian on $[0, 1]^4$ is the scaling limit of the membrane model [13, 10], see [2, 14], as well as of the odometer for the divisible sandpile model [11], see [4]. We stress that our convergence results for different discretizations of h hold in $\mathring{H}^{-s}$ with $s > 2$, thus matching the same range of exponents as for the scaling limit of the sandpile odometer, see [3, Prop. 14]. On the other hand, the analogous scaling limit for the membrane model has so far been proven only in H^{-s} for $s > 6$, see [2, Thm. 3.11].

3.3 Identifications

Lemma 3.13. *The spectral enhancement* $h_{\sharp,L}^{+\circ}$ *of the Fourier projection* $h_{\sharp,L}$ *of the polyharmonic field* h *coincides in distribution with the Fourier extension* $h_{L,\sharp}$ *of the discrete polyharmonic field* h_L.

Similarly, the Fourier approximation $h_{\sharp,L}$ *of the polyharmonic field* h *coincides in distribution with the spectral reduction* $h_{L,\sharp}^{-\circ}$ *of the Fourier extension* $h_{L,\sharp}$ *of the discrete polyharmonic field* h_L *(which in turn coincides with the Fourier extension of the spectrally reduced discrete polyharmonic field). Or, in other terms, the restriction of* $h_{\sharp,L}$ *onto* $\mathbb{T}_L^n$ *defines a spectrally reduced polyharmonic field* $h_L^{-\circ}$.

Lemma 3.14. *Restricted to the discrete torus, the enhanced projection* $h_{+,L}$ *of the polyharmonic Gaussian field* h *on the continuous torus coincides in distribution with the polyharmonic Gaussian field* h_L *on the discrete torus.*

Proof. Assume without restriction that h is given in terms of standard iid normal variables $(\xi_z)_{z\in\mathbb{Z}^n\setminus\{0\}}$ as

$$h = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{1}{\lambda_z^{n/4}}\, \xi_z\, \varphi_z.$$

Then

$$\mathsf{r}_L^+(h) = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{1}{\vartheta_{L,z}} \frac{1}{\lambda_{L,z}^{n/4}}\, \xi_z\, \varphi_z$$

on $\mathbb{T}^n$. Thus for $v \in \mathbb{T}^n_L$,

$$\begin{aligned} h_{+,L}(v) &= \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{1}{\vartheta_{L,z}} \frac{1}{\lambda_{L,z}^{n/4}}\, \xi_z \cdot L^n \int_{v+Q_L} \varphi_z(y)dy \\ &= \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{1}{\lambda_{L,z}^{n/4}}\, \xi_z\, \varphi_z(v) \end{aligned}$$

according to Lemma 1.6. Therefore, by Proposition 2.2, $h_{+,L}$ is distributed according to the polyharmonic Gaussian field on the discrete torus $\mathbb{T}^n_L$. $\square$

Lemma 3.15. *Let h be a polyharmonic Gaussian field on the continuous torus $\mathbb{T}^n$ and let*

$$h_{\circ,L} \coloneqq \mathsf{q}_L(\mathsf{r}_L(h))$$

denote the piecewise constant projection of its Fourier projection, called natural projection of h*. Then this field on $\mathbb{T}^n$ coincides in distribution with the piecewise constant extension of the reduced discrete polyharmonic field $h^-_{L,\flat}$. In particular, the field $h_{\circ,L}$ coincides on $\mathbb{T}^n_L$ in distribution with the reduced discrete polyharmonic field h^-_L.*

Proof. Assume without restriction that

$$h = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{1}{\lambda_z^{n/4}}\, \xi_z\, \varphi_z.$$

Then

$$h_{\circ,L}(x) = \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{1}{\lambda_z^{n/4}}\, \xi_z \cdot \sum_{v\in\mathbb{T}^n_L} \mathbf{1}_{v+Q_L}(x) \cdot L^n \int_{v+Q_L} \varphi_z(y)dy,$$

and thus for $v \in \mathbb{T}^n_L$,

$$\begin{aligned} h_{\circ,L}(v) &= \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{1}{\lambda_z^{n/4}}\, \xi_z \cdot L^n \int_{v+Q_L} \varphi_z(y)dy \\ &= \frac{1}{\sqrt{a_n}} \sum_{z\in\mathbb{Z}^n\setminus\{0\}}^{\square} \frac{\vartheta_{L,z}}{\lambda_z^{n/4}}\, \xi_z\, \varphi_z(v) \end{aligned}$$

according to Lemma 1.6. Therefore, $h_{\circ,L}$ is distributed on $\mathbb{T}^n_L$ according to the reduced discrete polyharmonic Gaussian field. $\square$

4 LQG Measures on Discrete and Continuous Tori

We will introduce and analyze *Liouville Quantum Gravity (= LQG) measures* on discrete and continuous tori. Our main result in this section will be that as $L \to \infty$ the LQG measures on the discrete tori $\mathbb{T}^n_L$ will converge to the LQG measure on the continuous torus $\mathbb{T}^n$.

An analogous convergence assertion in greater generality will be proven for the so-called *reduced LQG measures*, random measures on the discrete tori $\mathbb{T}^n_L$ defined in terms of the discrete polyharmonic fields h_L.

4.1 LQG Measure on the Continuous Torus and its Approximations

For $\gamma \in \mathbb{R}$, define the random measure $\mu_{\sharp,L}$ on $\mathbb{T}^n$ by

$$d\mu_{\sharp,L}(x) = \exp\Big(\gamma h_{\sharp,L}(x) - \frac{\gamma^2}{2} k_{\sharp,L}(x,x)\Big)\, d\mathcal{L}^n(x)$$

where $h_{\sharp,L}$ denotes the *Fourier projection* (or *eigenfunction approximation*) of the polyharmonic field h and $k_{\sharp,L}$ the associated covariance function (which has constant value $\frac{1}{a_n}\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}}\frac{1}{(2\pi|z|)^n}$ on the diagonal) as introduced in (18) and (19).

Proposition 4.1 ([5, Thms. 4.1+4.15])**.** *Assume $|\gamma| < \gamma^* := \sqrt{2n}$. Then for $\mathbf{P}$-a.e. ω, a unique Borel measure μ^ω on $\mathbb{T}^n$ exists with*

$$\mu^\omega_{\sharp,L} \to \mu^\omega \qquad \text{as } L\to\infty$$

in the sense of weak convergence of measures on $\mathbb{T}^n$ (i.e. tested against $f \in \mathcal{C}(\mathbb{T}^n)$). Even more, for every $f \in L^1(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n} f\, d\mu_{\sharp,L} \to \int_{\mathbb{T}^n} f\, d\mu \qquad \text{as } L\to\infty \quad \mathbf{P}\text{-a.s. and in } L^1(\mathbf{P})\ .$$

For $|\gamma| < \sqrt{n}$, the latter convergence also holds in $L^2(\mathbf{P})$.

Definition 4.2. *The random measure μ on $\mathbb{T}^n$ constructed and characterized above is called* polyharmonic LQG measure *or* polyharmonic Gaussian multiplicative chaos.

The random measures $\mu_{\sharp,L}$ on $\mathbb{T}^n$ are called Fourier approximations of the polyharmonic LQG measure.

Piecewise Constant Approximation. For $\gamma \in \mathbb{R}$, define the random measure $\mu_{\flat,L}$ on $\mathbb{T}^n$ by

$$d\mu_{\flat,L}(x) = \exp\Big(\gamma h_{\flat.L}(x) - \frac{\gamma^2}{2} k_{\flat,L}(x,x)\Big)\, d\mathcal{L}^n(x)$$

where $h_{\flat,L}$ denotes the *piecewise constant projection* of the polyharmonic field h and $k_{\flat,L}$ the associated covariance function (which is constant on the diagonal) as introduced in (20) and (21).

Proposition 4.3 ([5, Thm. 4.14])**.** *Assume $|\gamma| < \sqrt{2n}$. Then in $L^1(\mathbf{P})$,*

$$\mu_{\flat,L} \to \mu \qquad \text{as } L\to\infty.$$

Proof. This theorem is proven in [5, Thm. 4.14] using Kahane's convexity inequality. □

Enhanced Approximation. For $\gamma \in \mathbb{R}$, define the random measure $\mu_{+,L}$ on $\mathbb{T}^n$ by

$$d\mu_{+,L}(x) = \exp\Big(\gamma h_{+,L}(x) - \frac{\gamma^2}{2} k_{+,L}(x,x)\Big)\, d\mathcal{L}^n(x)$$

where $h_{+,L}$ denotes the *enhanced projection* of the polyharmonic field h and $k_{+,L}$ the associated covariance function.

Proposition 4.4. *Assume $|\gamma| < \sqrt{n/e}$. Then in $L^1(\mathbf{P})$,*

$$\mu_{+,L} \to \mu \qquad \text{as } L\to\infty.$$

Proof. To show the convergence of the LQG measures $\mu_{+,L}$ associated with the enhanced projections of the polyharmonic field on the torus $\mathbb{T}^n$ to the LQG measure μ on $\mathbb{T}^n$, we verify the necessary assumptions in [5, Lem. 4.5], a rewriting in the present setting of the general construction of Gaussian Multiplicative Chaoses by A. Shamov, [15]. Lemma 3.5 provides the convergence results for the regularizing kernel $p_{+,L}$ and for the covariance kernel $k_{+,L}$. The uniform integrability of the approximating sequence of random measures $\mu_{+,L}$ will be proven as Theorem 4.11 in the last section.

□

Natural Approximation. For $\gamma \in \mathbb{R}$, define the random measure $\mu_{\circ,L}$ on $\mathbb{T}^n$ by

$$d\mu_{\circ,L}(x) = \exp\Big(\gamma h_{\circ,L}(x) - \frac{\gamma^2}{2} k_{\circ,L}(x,x)\Big)\, d\mathcal{L}^n(x)$$

where $h_{\circ,L}$ denotes the *natural projection* of the polyharmonic field h and $k_{\circ,L}$ the associated covariance function.

Proposition 4.5. *Assume* $|\gamma| < \sqrt{n}$. *Then in* $L^1(\mathbf{P})$,

$$\mu_{\circ,L} \to \mu \qquad \textit{as } L \to \infty.$$

Proof. To show the convergence of the LQG measures $\mu_{\circ,L}$ associated with the natural projections of the polyharmonic field h to the LQG measure μ on $\mathbb{T}^n$, we again verify the necessary assumptions in [5, Lem. 4.5]. Lemma 3.6 provides the convergence results for the regularizing kernel $p_{\circ,L}$ and for the covariance kernel $k_{\circ,L}$. The uniform integrability — even L^2-boundedness — of the approximating sequence of random measures $\mu_{\circ,L}$ follows from the L^2-boundedness of the sequence of random measures $\mu_{\sharp,L}$ as stated in Proposition 4.1 and a straightforward application of Jensen's inequality with the Markov kernel q_L:

$$\begin{aligned}
\sup_L \mathbb{E}\Big[|\mu_{\circ,L}(\mathbb{T}^n)|^2\Big] &= \sup_L \mathbb{E}\Bigg[\Big|\int_{\mathbb{T}^n} \exp\Big(\gamma h_{\circ,L}(x) - \frac{\gamma^2}{2} k_{\circ,L}(x)\Big)\, dx\Big|^2\Bigg] \\
&= \sup_L \iint_{\mathbb{T}^n\times\mathbb{T}^n} \exp\big(\gamma^2\, k_{\circ,L}(x,y)\big)\, dy\, dx \\
&\le \sup_L \iint_{\mathbb{T}^n\times\mathbb{T}^n} \iint_{\mathbb{T}^n\times\mathbb{T}^n} \exp\big(\gamma^2\, k_{\sharp,L}(x',y')\big)\, q_L(x,x')\, q_L(y,y')\, dy'\, dx'\, dy\, dx \\
&= \sup_L \iint_{\mathbb{T}^n\times\mathbb{T}^n} \exp\big(\gamma^2\, k_{\sharp,L}(x,y)\big)\, dy\, dx \\
&= \sup_L \mathbb{E}\Big[|\mu_{\sharp,L}(\mathbb{T}^n)|^2\Big] = \mathbb{E}\Big[|\mu(\mathbb{T}^n)|^2\Big] < \infty\ .
\end{aligned}$$

Alternatively, we can also use Jensen's inequality directly at the level of $h_{\circ,L}$. Namely, we find that

$$\begin{aligned}
\exp\Big(\gamma h_{\circ,L}(x) - \frac{\gamma^2}{2} k_{\circ,L}(x,x)\Big) &= \exp\Big(\int \Big(\gamma h_{\sharp,L}(x') - \frac{\gamma^2}{2} k_{\sharp,L}(x',x'')\Big)\, q_L(x,x')q_L(x,x'')dxdx'dx''\Big) \\
&\le \int \exp\Big(\gamma h_{\sharp,L}(x') - \frac{\gamma^2}{2} k_{\sharp,L}(x',x'')\Big)\, q_L(x,x')q_L(x,x'').
\end{aligned}$$

From which, we get

$$\mu_{\circ,L}(\mathbb{T}^n) \le \mu_{\sharp,L}(\mathbb{T}^n).$$

□

4.2 LQG Measures on the Discrete Tori and their Convergence

In this section, we will prove our main result: the convergence of the discrete LQG measures μ_L to the LQG measure μ on $\mathbb{T}^n$.

Whereas this convergence only holds for a restricted range of parameters γ, the convergence of the so-called reduced discrete LQG measures μ_L to the LQG measure μ on $\mathbb{T}^n$ will hold in greater generality.

Definition 4.6. *For given* $\gamma \in \mathbb{R}$, *the* polyharmonic LQG measure on the discrete torus $\mathbb{T}^n_L$ *or* discrete LQG measure *is the random measure* μ_L *on* $\mathbb{T}^n_L$ *defined by*

$$d\mu_L(v) = \exp\Big(\gamma h_L(v) - \frac{\gamma^2}{2} k_L(v,v)\Big)\, dm_L(v)\ .$$

Here h_L is the polyharmonic Gaussian field on the discrete torus $\mathbb{T}^n_L$, k_L its covariance function as introduced in (12) and (6), and m_L the normalized counting measure $\frac{1}{L^n}\sum_{u\in\mathbb{T}^n_L}\delta_u$ on the discrete torus. Recall that $k_L(v,v)=\frac{1}{a_n}\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}}\frac{1}{\lambda^{n/2}_{L,z}}$ for all $v\in\mathbb{T}^n_L$.

For proving convergence of the random measures μ_L on the discrete tori $\mathbb{T}^n_L$ as $L\to\infty$, we will restrict ourselves to subsequences for which the discrete tori are hierarchically ordered, say $L=a^\ell$ as $\ell\to\infty$ for some fixed integer $a\geq 2$ and $\ell\in\mathbb{N}$. For convenience, we will assume that a is odd.

Theorem 4.7. *Assume $|\gamma|<\gamma_*$, and let a be an odd integer ≥ 2. Then in $L^1(\mathbf{P})$,*

$$\mu_{a^\ell}\to\mu \qquad \text{as } \ell\to\infty.$$

Proof. Given a as above, let us call a function f on $\mathbb{T}^n$ *piecewise constant* if it is constant on all cubes $v+Q_L$, $v\in\mathbb{T}^n_L$, for some $L=a^{\ell'}$. For such f and all $\ell\geq\ell'$,

$$\int f\,d\mu_{a^\ell}=\int f\,d\mu_{+,a^\ell}. \tag{34}$$

Indeed, the field h_{+,a^ℓ} is constant all cubes $v+Q_{a^\ell}$, $v\in\mathbb{T}^n_{a^\ell}$, and Lemma 3.14 yields that the fields h_{a^ℓ} and h_{+,a^ℓ} coincide (in distribution) on the discrete torus $\mathbb{T}^n_{a^\ell}$. Thus also the associated LQG measures of all cubes $v+Q_{a^\ell}$, $v\in\mathbb{T}^n_{a^\ell}$, coincide.

Hence, for piecewise constant functions f, the convergence

$$\int f\,d\mu_{a^\ell}\to\int f\,d\mu \qquad \text{as } \ell\to\infty$$

follows from the previous Proposition 4.4.

For continuous f, the claim follows by approximation of f by piecewise constant f_j, $j\in\mathbb{N}$. Indeed,

$$\mathbf{E}\Big[\Big|\int f\,d\mu_{a^\ell}-\int f_j\,d\mu_{a^\ell}\Big|\Big]\leq\mathbf{E}\Big[\int\Big|f-f_j\Big|\,d\mu_{a^\ell}\Big]=\int\Big|f-f_j\Big|\,dx\ \to\ 0$$

as $j\to\infty$, uniformly in $\ell\in\mathbb{N}$, and similarly with μ in the place of μ_{a^ℓ}. ☐

4.3 Reduced LQG Measures on the Discrete Tori and their Convergence

Recall that if h_L is a polyharmonic field on the discrete torus then

$$h^-_L:=\mathsf{r}^-_L(h_L)$$

defines a reduced polyharmonic field on the discrete torus. If h_L is given as

$$h_L(v)=\frac{1}{\sqrt{a_n}}\sum_{z\in\mathbb{Z}^n_L}\frac{1}{\lambda^{n/4}_{L,z}}\,\xi_z\,\varphi_z(v)$$

then

$$h^-_L(v)=\frac{1}{\sqrt{a_n}}\sum_{z\in\mathbb{Z}^n_L}\frac{\vartheta_{L,z}}{\lambda^{n/4}_z}\,\xi_z\,\varphi_z(v).$$

For $\gamma\in\mathbb{R}$, define the random measure μ^-_L on $\mathbb{T}^n_L$, called *reduced discrete LQG measure*, by

$$d\mu^-_L(v)=\exp\Big(\gamma h^-_L(v)-\frac{\gamma^2}{2}k^-_L(v,v)\Big)\,dm_L(v)$$

Theorem 4.8. *Assume $|\gamma|<\sqrt{n}$ and let a be an odd integer ≥ 2. Then in $L^1(\mathbf{P})$,*

$$\mu^-_{a^\ell}\to\mu \qquad \text{as } \ell\to\infty.$$

Proof. Let a be given as above and let f be a function on $\mathbb{T}^n$ which is constant on all cubes $v+Q_L$, $v\in\mathbb{T}^n_L$, for some $L=a^{\ell'}$. Then according to Lemma 3.15 for all $\ell\geq\ell'$,

$$\int f\,d\mu^-_{a^\ell}=\int f\,d\mu_{\circ,a^\ell}. \tag{35}$$

Hence, for piecewise constant functions f, the convergence

$$\int f \, d\mu^{-}_{a^{\ell}} \to \int f \, d\mu \qquad \text{as } \ell \to \infty$$

follows from the previous Proposition 4.5. For continuous f, the claim follows by approximation of f by piecewise constant f_j, $j \in \mathbb{N}$. $\square$

4.4 Convergence Results for Semi-discrete LQG Measures

So far, we have studied the LQG measure and the reduced LQG measure on the discrete torus $\mathbb{T}^n_L$ and their convergence properties as $L \to \infty$. In terms of the polyharmonic field h_L on the discrete torus, we can also define the so-called semi-discrete LQG measure as well as the spectrally reduced semi-discrete LQG measure on the continuous torus. These are the LQG measures associated with the Fourier extension of the discrete field h_L and of the spectrally reduced discrete field $h^{-\circ}_L$. All these random measures are functions of the discrete field h_L.

4.4.1 Semi-discrete LQG Measure

For $\gamma \in \mathbb{R}$, define the random measure $\mu_{L,\sharp}$ on $\mathbb{T}^n$, called *semi-discrete LQG measure*, by

$$d\mu_{L,\sharp}(x) = \exp\Big(\gamma h_{L,\sharp}(x) - \frac{\gamma^2}{2} k_{L,\sharp}(x,x)\Big) \, d\mathcal{L}^n(x)$$

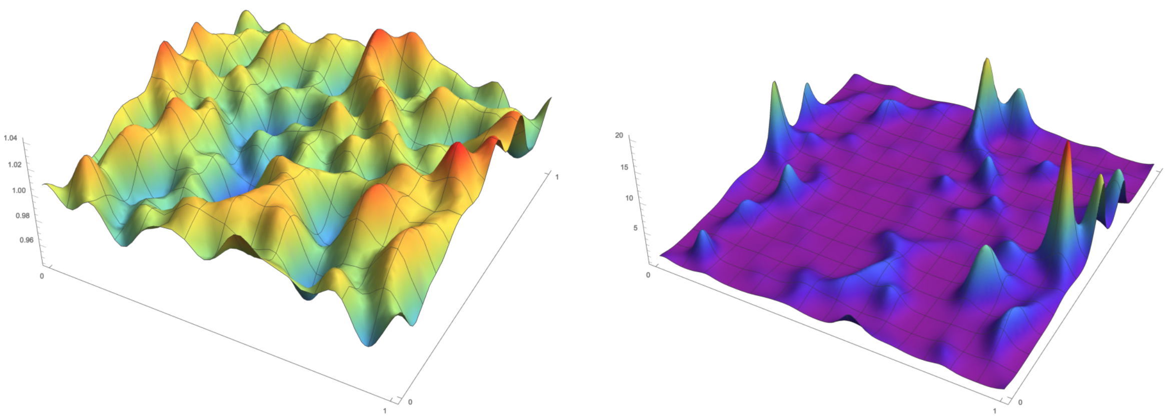
where $h_{L,\sharp}$ denotes the *Fourier extension* of the discrete polyharmonic field h_L and $k_{L,\sharp}$ the associated covariance function as introduced in §3.2.3.

Theorem 4.9. *Assume $|\gamma| < \gamma_* := \sqrt{\frac{n}{e}}$. Then in $L^1(\mathbf{P})$,*

$$\mu_{L,\sharp} \to \mu \qquad \textit{as } L \to \infty.$$

Proof. According to Remark 3.13, the Fourier extension $h_{L,\sharp}$ of the discrete random field h_L coincides in distribution with the field obtained from the continuous field h by regularization with the kernel $r^{+\circ}_L$.

To show the convergence of the LQG measures $\mu_{L,\sharp}$ associated with the Fourier extensions of the polyharmonic field on the torus $\mathbb{T}^n$ to the LQG measure μ on $\mathbb{T}^n$, we again verify the necessary assumptions in [5, Lem. 4.5]. Criteria (ii) and (iii) of Lemma 4.5 in [5], can be verified exactly as in the proof of Lemma 3.5. The uniform integrability of the approximating sequence of random measures follows from Theorem 4.11 in the next section. $\square$



(a) $\gamma = 0.01$ **(b)** $\gamma = 1$

Figure 5: The Gaussian Multiplicative Chaos $\mu_{L,\sharp}$ on $\mathbb{T}^2$ for $L = 15$, different values of γ and same realization of the randomness.

4.4.2 Spectrally Reduced Semi-discrete LQG Measure

For $\gamma \in \mathbb{R}$, define the random measure $\mu_{L,\sharp}^{-\circ}$ on $\mathbb{T}^n$, called *spectrally reduced semi-discrete LQG measure*, by

$$d\mu_{L,\sharp}^{-\circ}(x) = \exp\Big(\gamma h_{L,\sharp}^{-\circ}(x) - \frac{\gamma^2}{2} k_{L,\sharp}^{-\circ}(x,x)\Big)\, d\mathcal{L}^n(x)$$

where $h_{L,\sharp}^{-\circ}$ denotes the *Fourier extension* of the spectrally reduced discrete polyharmonic field $h_L^{-\circ}$ and $k_{L,\sharp}^{-\circ}$ the associated covariance function as introduced in §3.2.4. As a corollary to Lemma 3.13 and Proposition 4.1, we directly obtain

Theorem 4.10. *Assume* $|\gamma| < \sqrt{2n}$. *Then in* $\mathbf{P}$*-probability and in* $L^1(\mathbf{P})$*,*

$$\mu_{L,\sharp}^{-\circ} \to \mu \qquad \text{as } L \to \infty.$$

4.5 Uniform Integrability of Discrete and Semi-discrete LQG Measures

Finally, we address the question of uniform integrability of approximating sequences of LQG measures. We provide a self-contained argument for L^2-boundedness, independent of Kahane's work [9].

Theorem 4.11. *Assume* $|\gamma| < \gamma_* := \sqrt{\frac{n}{e}}$. *Then*

(i)

$$\sup_L \int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{L,\sharp}(0,y)\Big)\, d\mathcal{L}^n(y) < \infty\ , \tag{36}$$

(ii)

$$\sup_L \int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{L,\flat}(0,y)\Big)\, d\mathcal{L}^n(y) < \infty\ . \tag{37}$$

(iii)

$$\sup_L \int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{+,L}(0,y)\Big)\, d\mathcal{L}^n(y) < \infty\ . \tag{38}$$

Proof. (i) Recall from (31) that for $x, y \in \mathbb{T}^n$,

$$k_{L,\sharp}(x,y) = \frac{1}{a_n} \sum_{z \in \mathbb{Z}_L^n \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \cdot \exp\Big(2\pi i\, z \cdot (x-y)\Big)$$

Here as before, $\lambda_{L,z} = 4L^2 \sum_{k=1}^n \sin^2\big(\pi z_k/L\big)$. Given $\epsilon > 0$, choose $R > 2$ such that $|z| + \frac{1}{2}\sqrt{n} \le (1+\epsilon)|z|$ for all $z \in \mathbb{Z}^n$ with $\|z\|_\infty \ge R/2$, and $S > 1$ such that $t \le (1+\epsilon)\sin(t)$ for all $t \in [0, \frac{\pi}{2S}]$. Decompose $k_{L,\sharp}$ for $L > RS$ into $k_{L,R,S} + g_{L,R} + f_{L,S}$ with

$$f_{L,S}(x,y) = \frac{1}{a_n} \sum_{z \in \mathbb{Z}_L^n \setminus \mathbb{Z}_{L/S}^n} \frac{1}{\lambda_{L,z}^{n/2}} \cdot e^{2\pi i\, z\cdot(x-y)}\ , \qquad g_{L,R}(x,y) = \frac{1}{a_n} \sum_{z \in \mathbb{Z}_R^n \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \cdot e^{2\pi i\, z\cdot(x-y)}$$

and

$$k_{L,R,S}(x,y) = \frac{1}{a_n} \sum_{z \in \mathbb{Z}_{L/S}^n \setminus \mathbb{Z}_R^n} \frac{1}{\lambda_{L,z}^{n/2}} \cdot e^{2\pi i\, z\cdot(x-y)}\ .$$

For fixed R, obviously $g_{L,R}(x,y)$ is uniformly bounded in L, x, y. Similarly, since $\sin(t) \ge \frac{2}{\pi} t$ for $t \in [0, \pi/2]$ we have that for fixed S,

$$\begin{aligned}
\big|f_{L,S}\big|(x,y) &\le \frac{1}{a_n} \sum_{z \in \mathbb{Z}_L^n \setminus \mathbb{Z}_{L/S}^n} \frac{1}{\lambda_{L,z}^{n/2}} \le \frac{1}{a_n} \sum_{z \in \mathbb{Z}_L^n \setminus \mathbb{Z}_{L/S}^n} \frac{1}{4^n |z|^n} \\
&\le \frac{1}{4^n\, a_n} \int_{B_{\sqrt{n}L/2}(0) \setminus B_{L/(2S)}(0)} \frac{1}{|x|^n} d\mathcal{L}^n(x) \\
&= C\Big[\log(\sqrt{n}L/2) - \log(L/(2S))\Big] = C' < \infty\ .
\end{aligned}$$

For (36) to hold, it thus suffices to prove that

$$\sup_L \int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{L,R,S}(0,y)\Big)\, d\mathcal{L}^n(y) < \infty \ , \tag{39}$$

for some $R > 2$ as above.

For proving the latter, we follow the argument of the proof of Lemma 2, p. 611 in [17]. To start with, we use the multi-dimensional Hausdorff–Young inequality, which can be found in [8, p. 248]:

$$\text{For } p \geq 2 \quad \int_{\mathbb{T}^n} |k_{L,R,S}(0,y)|^p\, dy \leq \left(\sum_{z\in\mathbb{Z}^n_{L/S}\setminus\mathbb{Z}^n_R} |c(z)|^{p'} \right)^{p-1} ,$$

where $c(z) = \frac{1}{a_n}\big(4L^2 \sum_{k=1}^n \sin^2(\pi z_k/L)\big)^{-n/2}$ and $p' \in [1,2]$ is the Hölder-conjugate. Since $\pi|z_k|/L \leq (1+\epsilon)|\sin(\pi z_k/L)|$ for all z_k/L under consideration, we have that

$$\int_{\mathbb{T}^n} |k_{L,R,S}(0,y)|^p\, dy \leq (1+\epsilon)^{np'(p-1)} \left(\frac{1}{a_n^{p'}} \sum_{z\in\mathbb{Z}^n_{L/S}\setminus\mathbb{Z}^n_R} \frac{1}{(2\pi|z|)^{np'}} \right)^{p-1} .$$

Let $Q_1(z)$ be the unit cube $\prod_{i=1}^n [z-\frac{1}{2}e_i, z+\frac{1}{2}e_i]$ around $z\in\mathbb{Z}^n$. Since by assumption

$$|x| \leq |z| + \frac{1}{2}\sqrt{n} \leq (1+\frac{1}{2}\sqrt{n})|z| \leq (1+\epsilon)|z|$$

for all $x \in Q_1(z)$ and all z with $\|z\|_\infty \geq R/2$, we estimate

$$\begin{aligned}\sum_{z\in\mathbb{Z}^n_{L/S}\setminus\mathbb{Z}^n_R} \int_{Q_1(z)} \frac{1}{|z|^{np'}}\, dx &\leq \sum_{z\in\mathbb{Z}^n\setminus\mathbb{Z}^n_R} (1+\epsilon)^{np'} \int_{Q_1(z)} \frac{1}{|x|^{np'}}\, dx \\ &\leq (1+\epsilon)^{np'} \int_{\mathbb{R}^n\setminus B_{R/2}(0)} \frac{1}{|x|^{np'}}\, dx.\end{aligned}$$

With Cavalieri's principle the integral can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{R/2}(0)} \frac{1}{|x|^{np'}}\, dx &= \int_{R/2}^\infty \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} \frac{1}{r^{np'}}\, dr \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{n(p'-1)} \left(\frac{R}{2}\right)^{n(1-p')} \leq \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{p}{n}\end{aligned}$$

since $p' > 1$ and $R \geq 2$. Hence we obtain for $k_{L,R,S}$:

$$\int_{\mathbb{T}^n} |k_{L,R,S}(0,y)|^p\, dy \leq \left(\frac{(1+\epsilon)^2}{2\pi}\right)^{np} \frac{1}{a_n^p} \left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{p}{n}\right)^{p-1} . \tag{40}$$

Summing these terms over all $p \in \mathbb{N}\setminus\{1\}$ yields

$$\begin{aligned}\sum_{p\geq 2} \frac{\gamma^{2p}}{p!} \int_{\mathbb{T}^n} |k_{L,R,S}(0,y)|^p\, dy &\leq \sum_{p\geq 2} \frac{\gamma^{2p}}{p!} \left(\frac{(1+\epsilon)^{2n}}{(2\pi)^n a_n}\right)^p \left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{p}{n}\right)^{p-1} \\ &= \frac{n\Gamma(n/2)}{2\pi^{n/2}} \sum_{p\geq 2} \frac{\gamma^{2p}}{p!p} \left(\frac{(1+\epsilon)^{2n}}{(2\pi)^n a_n} \frac{2\pi^{n/2}p}{n\Gamma(n/2)}\right)^p \\ &\sim \frac{n\Gamma(n/2)}{2\pi^{n/2}} \sum_{p\geq 2} \frac{1}{p\sqrt{2\pi p}} \left(\frac{(1+\epsilon)^{2n} 2\pi^{n/2} e\gamma^2}{(2\pi)^n a_n n\Gamma(n/2)}\right)^p ,\end{aligned}$$

where we used Stirling's formula $p! \sim \sqrt{2\pi p}(\frac{p}{e})^p$. The last sum is finite if

$$(1+\epsilon)^{2n}\gamma^2 < \frac{(2\pi)^n a_n n\Gamma(n/2)}{2\pi^{n/2}e} = \frac{n}{e} = \gamma_*^2 \ , \tag{41}$$

where we inserted $a_n = \frac{2}{(4\pi)^{n/2}\Gamma(n/2)}$. Since by assumption $|\gamma| < \gamma_*$ and since $\epsilon > 0$ was arbitrary, by appropriate choice of the latter, (41) is satisfied.

To treat the cases $p = 0$ and $p = 1$, observe that $(\int_{\mathbb{T}^n} |k_{L,R,S}(0,y)|\, dy)^2 \leq \int_{\mathbb{T}^n} |k_{L,R,S}(0,y)|^2\, dy$. Thus, there exists a constant $C^R_{n,\gamma}$ such that

$$\sum_{p\geq 0} \frac{\gamma^{2p}}{p!} \int_{\mathbb{T}^n} |k_{L,R,S}(0,y)|^p\, dy \leq C^R_{n,\gamma},$$

uniformly in L, and thus in turn there exists a constant $C^\sharp_{n,\gamma}$ such that

$$\sup_L \sum_{p\geq 0} \frac{\gamma^{2p}}{p!} \int_{\mathbb{T}^n} |k_{L,\sharp}(0,y)|^p\, dy \leq C^\sharp_{n,\gamma},$$

which proves (36).

(ii) To show (37), note that

$$\int_{\mathbb{T}^n} \exp\left(\gamma^2 k_{L,\flat}(0,y)\right) d\mathcal{L}^n(y) = \int_{\mathbb{T}^n_L} \exp\left(\gamma^2 k_L(0,v)\right) dm_L(v)$$

for every L. Furthermore, for $p \geq 2$,

$$\int_{\mathbb{T}^n_L} |k_L(0,v)|^p\, dm_L(v) \leq \left(\frac{1}{a_n^{p'}} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} |c(z)|^{p'}\right)^{p-1}.$$

Indeed, for $p = 2$ this is due to Parseval's identity, and for $p = \infty$ this holds since $|\exp(2\pi i z(x-y))| = 1$. The estimate holds for all intermediate $p \in (2,\infty)$ by virtue of the Riesz–Thorin theorem. Then, the proof of (37) follows the lines above.

(iii) Recall that $k_{+,L} = q_L \circ k_L^+$ with k_L given is (25). Thus by Jensen's inequality, (38) will follow from

$$\sup_L \int_{\mathbb{T}^n} \exp\left(\gamma^2 k_L^+(0,y)\right) d\mathcal{L}^n(y) < \infty\ . \tag{42}$$

To prove this, we argue as before in (i), now with

$$k_L^+(x,y) = \frac{1}{a_n} \sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\vartheta^2_{L,z}\, \lambda^{n/2}_{L,z}} \cdot \exp\left(2\pi i\, z\cdot(x-y)\right)$$

in the place of $k_{L,\sharp}(x,y)$. For given $\epsilon > 0$, choose $R > 2$ and $S > 1$ as before. In particular, then $t \leq (1+\epsilon)\sin(t)$ for all $t \in [0, \frac{\pi}{2S}]$ and thus

$$1 \geq \vartheta_{L,z} \geq (1+\epsilon)^{-n} \qquad \forall z \in \mathbb{Z}^n_{L/S}.$$

Thus decomposing k_L^+ intro three factors as before and then arguing as before will prove the claim. □

Corollary 4.12. *If $|\gamma| < \gamma_*$, then for each $f \in L^2(\mathbb{T}^n)$,*

(i) the family $\left(\int_{\mathbb{T}^n} f\, d\mu_{L,\sharp}\right)_{L\in\mathbb{N}}$ is $L^2(\mathbf{P})$-bounded,

(ii) the family $\left(\int_{\mathbb{T}^n} f\, d\mu_{L,\flat}\right)_{L\in\mathbb{N}}$ is $L^2(\mathbf{P})$-bounded.

(iii) the family $\left(\int_{\mathbb{T}^n} f\, d\mu_{+,L}\right)_{L\in\mathbb{N}}$ is $L^2(\mathbf{P})$-bounded.

Proof. (i) Given $f \in L^2(\mathbb{T}^n)$ and γ as above, consider the Gaussian variables

$$Y_{L,\sharp} := \int_{\mathbb{T}^n} f\, d\mu_{L,\sharp} = \int_{\mathbb{T}^n} \exp\left(\gamma h_{L,\sharp}(x) - \frac{\gamma^2}{2} k_{L,\sharp}(x,x)\right) f(x)\, d\mathcal{L}^n(x).$$

Then

$$\sup_L \mathbb{E}\Big[\big|Y_{L,\sharp}\big|^2\Big] = \sup_L \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{L,\sharp}(x,y)\Big)\, f(x)\, f(y)\, dm_L(y)\, d\mathcal{L}^n(x)$$
$$\leq \sup_L \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{L,\sharp}(x,y)\Big)\, f^2(x)\, dm_L(y)\, d\mathcal{L}^n(x)$$
$$\leq \|f\|^2 \cdot \sup_L \int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{L,\sharp}(0,y)\Big)\, dm_L(y) \leq \|f\|^2 \cdot C^{\sharp}_{n,\gamma} < \infty.$$

(ii) Similarly,

$$\sup_L \mathbb{E}\Big[\big|Y_{L,\flat}\big|^2\Big] \leq \|f\|^2 \cdot \sup_L \int_{\mathbb{T}^n} \exp\Big(\gamma^2 k_{L,\flat}(0,y)\Big)\, dm_L(y) \leq \|f\|^2 \cdot C^{\flat}_{n,\gamma} < \infty$$

for $Y_{L,\flat} := \int_{\mathbb{T}^n} f\, d\mu_{L,\flat} = \int_{\mathbb{T}^n} \exp\Big(\gamma h_{L,\flat}(x) - \frac{\gamma^2}{2} k_{L,\flat}(x,x)\Big)\, f(x)\, d\mathcal{L}^n(x)$.

(iii) Analogously. □

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