

# CONFORMALLY INVARIANT RANDOM FIELDS, LIOUVILLE QUANTUM GRAVITY MEASURES, AND RANDOM PANEITZ OPERATORS ON RIEMANNIAN MANIFOLDS OF EVEN DIMENSION

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## Abstract

For large classes of *even-dimensional* Riemannian manifolds  $(M, g)$ , we construct and analyze conformally invariant random fields. These centered Gaussian fields  $h = h_g$ , called *co-polyharmonic Gaussian fields*, are characterized by their covariance kernels  $k$  which exhibit a precise logarithmic divergence:  $|k(x, y) - \log \frac{1}{d(x, y)}| \leq C$ . They share the fundamental quasi-invariance property under conformal transformations: if  $g' = e^{2\varphi}g$ , then

$$h_{g'} \stackrel{(d)}{=} e^{n\varphi} h_g - C \cdot \text{vol}_{g'}$$

with an appropriate random variable  $C = C_\varphi$ .

In terms of the co-polyharmonic Gaussian field  $h$ , we define the *Liouville Quantum Gravity measure*, a random measure on  $M$ , heuristically given as

$$d\mu_g^h(x) := e^{\gamma h(x) - \frac{\gamma^2}{2} k(x, x)} d\text{vol}_g(x),$$

and rigorously obtained as almost sure weak limit of the right-hand side with  $h$  replaced by suitable regular approximations  $h_\ell, \ell \in \mathbb{N}$ . These measures share a crucial quasi-invariance property under conformal transformations: if  $g' = e^{2\varphi}g$ , then

$$d\mu_{g'}^{h'}(x) \stackrel{(d)}{=} e^{F^h(x)} d\mu_g^h(x)$$

for an explicitly given random variable  $F^h(x)$ .

In terms on the Liouville Quantum Gravity measure, we define the *Liouville Brownian motion* on  $M$  and the *random GJMS operators*. Finally, we present an approach to a

conformal field theory in arbitrary even dimensions with an ansatz based on Branson’s  $Q$ -curvature: we give a rigorous meaning to the *Polyakov–Liouville measure*

$$d\nu_g^*(h) = \frac{1}{Z_g^*} \exp\left(-\int \Theta Q_g h + m e^{\gamma h} d\text{vol}_g\right) \exp\left(-\frac{a_n}{2} \mathfrak{p}_g(h, h)\right) dh$$

for suitable positive constants  $\Theta, m, \gamma$  and  $a_n$ , and we derive the corresponding *conformal anomaly*.

The set of *admissible* manifolds is conformally invariant. It includes all compact 2-dimensional Riemannian manifolds, all compact non-negatively curved Einstein manifolds of even dimension, and large classes of compact hyperbolic manifolds of even dimension. However, not every compact even-dimensional Riemannian manifold is admissible.

Our results concerning the logarithmic divergence of the kernel  $k$  — defined as the Green kernel for the GJMS operator on  $(M, g)$  — rely on new sharp estimates for heat kernels and higher order Green kernels on arbitrary compact manifolds.

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## Introduction

Conformally invariant random objects on the complex plane or on Riemannian surfaces are a central topic of current research and play a fundamental role in many mathematical theories. The last two decades have seen an impressive wave of fascinating constructions, deep insights and spectacular results for various conformally (quasi-) invariant random objects, most prominently the Gaussian Free Field, the Liouville quantum measure, the Brownian map, and the SLE curves.

In this paper, we use ideas from conformal geometry in higher dimension to establish the foundations for a mathematical theory of conformally invariant random fields and Liouville Quantum Gravity on compact Riemannian manifolds of even dimension.

**Co-polyharmonic Gaussian Fields.** We construct conformally quasi-invariant random Gaussian fields  $h$  on admissible Riemannian manifolds  $(M, g)$  of arbitrary even dimension. The covariance kernels of these centered Gaussian fields, naively interpreted as  $k(x, y) = \mathbf{E}[h(x)h(y)]$ , exhibits a logarithmic divergence

$$\left| k(x, y) - \log \frac{1}{d(x, y)} \right| \leq C.$$

As for the Gaussian Free Field, these random fields, called *co-polyharmonic Gaussian fields*, are not classical functions on  $M$  but rather distributions in  $\mathfrak{D}'$ , the dual space of  $\mathfrak{D} = C^\infty(M)$ . They also can be regarded as elements in the Sobolev space  $H^s(M)$  of any negative order  $s < 0$ . By construction, they annihilate constants, that is  $\langle h | \mathbf{1} \rangle = 0$ .

We prove (Thm. 3.13) that co-polyharmonic Gaussian fields are conformally quasi-invariant: let  $h_g$  denote the co-polyharmonic Gaussian field for  $(M, g)$  and  $h_{g'}$  that for  $(M, g')$  with  $g' = e^{2\varphi}g$  and  $\varphi$  smooth, then

$$h_{g'} \stackrel{(d)}{=} e^{n\varphi} h_g - C \cdot \text{vol}_{g'}, \quad (1)$$

where  $C$  is an appropriate random variable that ensures that the right-hand side annihilates constants. Here and in all the paper we use  $\stackrel{(d)}{=}$  to indicate that two random variables have the same law.

**Co-polyharmonic operators.** For a given manifold  $(M, g)$  of even dimension  $n$ , the covariance kernel  $k_g$  is — up to a multiplicative constant  $a_n = 2(4\pi)^{-n/2}/\Gamma(n/2)$  — the integral kernel of an operator  $K_g$  that is inverse to the operator  $P_g$  on the ‘grounded’  $L^2$ -space  $\dot{H} := \{u \in L^2(M, \text{vol}_g) : \int u \, d\text{vol}_g = 0\}$ . Here,

$$P_g = (-\Delta_g)^{n/2} + \text{low order terms} \quad (2)$$

denotes the *co-polyharmonic operator* or *Graham–Jenne–Mason–Sparling operator of maximal order*. The operator  $P_g$  plays the role of a conformally invariant power of the Laplacian and has been first defined in [GJMS92]. For  $n = 2$ , the non-negative operator  $P_g$  is just  $-\Delta_g$ , the negative of the Laplacian, and for  $n = 4$  it is the celebrated *Paneitz operator* [Pan83].

The co-polyharmonic Gaussian field  $h$  on  $(M, g)$  can easily be constructed in terms of the eigenbasis  $(\psi_j)_{j \in \mathbb{N}_0}$  of  $P_g$ : with  $(\nu_j)_{j \in \mathbb{N}_0}$  the corresponding eigenvalues and any sequence  $(\xi_j)_{j \in \mathbb{N}}$  of independent standard normal random variables, then (Prop. 3.9)

$$h = \lim_{\ell \rightarrow \infty} (h_\ell \text{vol}_g), \quad h_\ell(x) := \sum_{j=1}^{\ell} \frac{\psi_j(x) \xi_j}{\sqrt{a_n \nu_j}}. \quad (3)$$

In the above expression, we see  $h_\ell \text{vol}_g$  as a random distribution and the convergence holds in quadratic mean.

**Liouville Quantum Gravity measures.** We then define the *Liouville Quantum Gravity measure*  $\mu^h$  on  $(M, g)$  for every parameter  $\gamma \in \mathbb{R}$  with  $|\gamma| < \sqrt{2n}$  as a random finite measure. Employing Kahane’s idea of *Gaussian multiplicative chaos*, we define (Thm. 4.1) the measure  $\mu^h$  as the almost sure limit (in the usual sense of weak convergence of measures) of the sequence  $(\mu^{h_\ell})_{\ell \in \mathbb{N}}$  of finite measures on  $M$  given by

$$d\mu^{h_\ell}(x) := e^{\gamma h_\ell(x) - \frac{\gamma^2}{2} k_\ell(x, x)} d\text{vol}_g(x), \quad (4)$$

with  $h_\ell$  as in (3) and  $k_\ell(x, x) := \mathbf{E}[h_\ell(x)^2] = \sum_{j=1}^{\ell} \frac{\psi_j(x)^2}{a_n \nu_j}$ .

We establish that almost surely the measure  $\mu^h$  is a finite measure on  $M$  with full topological support, and for every  $s > \gamma^2/4$ , it does not charge sets of vanishing  $H^s$ -capacity (Thm. 4.21). In particular, it does not charge sets of vanishing  $H^{n/2}$ -capacity since  $|\gamma| < \sqrt{2n}$  throughout. If, moreover,  $|\gamma| < 2$  then, almost surely,  $\mu^h$  does not charge sets of vanishing  $H^1$ -capacity.

The Liouville Quantum Gravity measure has a crucial quasi-invariance property (Thm. 4.16). To formulate it, let  $\mu_g^h$  denote the Liouville Quantum Gravity measure on  $(M, g)$  and  $\mu_{g'}^{h'}$  the one on  $(M, g')$  for  $g' = e^{2\varphi}g$ . Then,

$$\mu_{g'}^{h'} \stackrel{(d)}{=} e^{F^h} \mu_g^h, \quad (5)$$

with a random variable  $F^h$  that is given explicitly.

We also define Liouville Quantum Gravity measures in two other flavors: the *refined* and *adjusted Liouville Quantum Gravity measures*, respectively denoted by  $\tilde{\mu}$  and  $\bar{\mu}$ . They are equal, up to a (possibly random) multiplicative constant, to the *plain* Liouville Quantum Gravity measure defined above and thus share many properties with it. The adjusted measure exhibits a simpler quasi-invariance property (Thm. 4.31):

$$\bar{\mu}_{g'}^{h'} = e^{-\gamma\xi} e^{\left(n + \frac{\gamma^2}{2}\right)\varphi} \bar{\mu}_g^h, \quad (6)$$

where  $\xi$  is a normal random variable.

**Random quadratic forms.** With respect to the Liouville Quantum Gravity measure, we can define a variety of random objects which play a fundamental role in geometric analysis, spectral theory, and probabilistic potential theory.

Restricting to the range  $\gamma \in (-2, 2)$ , we construct (Thm. 5.1) a *random Dirichlet form* on  $L^2(M, \mu^h)$  by:

$$\mathcal{E}^h(u, u) := \int_M |\nabla u|^2 d\text{vol}_g, \quad \mathcal{D}(\mathcal{E}^h) := H^1(M) \cap L^2(M, \mu^h).$$

The associated reversible and continuous Markov process is the *Liouville Brownian motion* (see [GRV14, GRV16] and [Ber15] for two independent constructions on the plane). It is obtained from the standard Brownian motion on  $(M, g)$  through time change. The new time scale is given as the right inverse of the additive functional

$$A_t^h = \lim_{\ell \rightarrow \infty} \int_0^t \exp\left(\gamma h_\ell(X_s) - \frac{\gamma^2}{2} k_\ell(X_s, X_s)\right) ds. \quad (7)$$

In dimension  $n > 2$ , however, this Liouville Brownian motion has no canonical invariance property under conformal transformations.

To obtain conformally quasi-invariant random objects in higher dimensions, our starting point, in Theorem 5.6, is the random co-polyharmonic form

$$\mathfrak{p}^h(u, v) := \int_M u \mathfrak{P} v d\text{vol}, \quad \mathcal{D}(\mathfrak{p}^h) := H^{n/2}(M) \cap L^2(M, \mu^h),$$

(rather than the random Dirichlet form) which in the full range  $\gamma \in (-\sqrt{2n}, \sqrt{2n})$  is, almost surely, a well-defined non-negative closed symmetric bilinear form on  $L^2(M, \mu^h)$ . It allows us to define *random co-polyharmonic operators*  $\mathfrak{P}^h$ . The associated random co-polyharmonic heat flow  $e^{-t\mathfrak{P}^h}$  is the gradient flow for the deterministic quadratic functional  $\frac{1}{2}\mathfrak{p}$  in the random landscape  $L^2(M, \mu^h)$  (Prop. 5.9).

In Theorem 5.11, we show that the random co-polyharmonic operators share the fundamental quasi-invariance property

$$\mathfrak{P}_{g'}^{h'} \stackrel{(d)}{=} e^{-F^h} \mathfrak{P}_g^h, \quad (8)$$

with  $F^h$  as in (5).

**Polyakov–Liouville measure.** Finally, we propose an ansatz for a conformal field theory on compact manifolds of arbitrary even dimension. Our approach, based on Branson’s  $Q$ -curvature, provides a rigorous meaning to the Polyakov–Liouville measure  $\nu_g^*$ , informally given as

$$\nu_g^*(dh) = \frac{1}{Z_g} \exp(-S_g(h)) dh$$

with the (non-existing) uniform distribution  $dh$  on the set of fields and the action

$$S_g(h) := \int_M \left( \frac{a_n}{2} h \mathsf{P}_g h + \Theta Q_g h + m e^{\gamma h} \right) d\text{vol}_g ,$$

where  $m, \Theta, \gamma > 0$  are parameters (subjected to some restrictions). To rigorously define the *adjusted Polyakov–Liouville measure*  $\bar{\nu}_g^*$ , we interpret it as

$$d\bar{\nu}_g^*(h+a) := \exp \left( -\Theta \langle h+a | Q_g \rangle - m e^{\gamma a} \bar{\mu}_{g,\gamma}^h(M) \right) da d\nu_g(h) , \quad (9)$$

where  $\nu_g$  denotes the law of the co-polyharmonic Gaussian field, informally understood as  $\nu_g(dh) = \frac{1}{Z_g} \exp \left( -\frac{a_n}{2} \langle h | \mathsf{P}_g h \rangle \right) dh$ , and where  $\bar{\mu}_{g,\gamma}^h$  denotes the adjusted Liouville Quantum Gravity measure. We prove (Thm. 5.21) that for admissible manifolds of negative total  $Q$ -curvature, the measure  $\bar{\nu}_g^*$  is finite. That is, in terms of the *partition function*  $\bar{Z}_g^* := \int d\bar{\nu}_g^* < \infty$ . Moreover, for the particular choice  $\Theta := a_n \left( \frac{n}{\gamma} + \frac{\gamma}{2} \right)$ , the adjusted Polyakov–Liouville measure is quasi-invariant modulo shifts (Thm. 5.22) with *conformal anomaly*

$$\bar{Z}_{e^{2\varphi}g}^* / \bar{Z}_g^* = \exp \left( \frac{a_n}{2} \left( \frac{n}{\gamma} + \frac{\gamma}{2} \right)^2 \left[ 2 \int_M \varphi Q_g d\text{vol}_g + \mathfrak{p}_g(\varphi, \varphi) \right] \right) . \quad (10)$$

**Admissible manifolds.** Co-polyharmonic Gaussian fields do *not* exist on *every* compact Riemannian manifold. A compact even-dimensional Riemannian manifold  $(M, g)$  is called *admissible* if  $\mathsf{P}_g > 0$  on  $\dot{H}$ . Admissibility is a conformal invariance. All compact, non-negatively curved Einstein manifolds are admissible, and so are all compact hyperbolic manifolds with spectral gap  $\lambda_1 > \frac{n(n-2)}{4}$ . Of course, all compact 2-dimensional Riemannian manifolds are admissible.

One of our main results (Thm. 2.18) states that for every admissible manifold, the inverse of  $\mathsf{P}_g$  on  $\dot{H}$  has an integral kernel  $K_g$  which annihilates constants and satisfies

$$\left| K_g(x, y) - a_n \log \frac{1}{d(x, y)} \right| \leq C .$$

with  $a_n$  as above.

**The two-dimensional case.** Even in the case of surfaces, our approach provides new insights for the study of two-dimensional random objects. It applies to closed Riemannian surfaces of arbitrary genus and thus some of our results are new in the two-dimensional setting. Furthermore, by focussing on the *grounded* random field — which by definition annihilates constants — we gain a more precise transformation rule (cf. Thm.s 3.13, 4.16) than the ‘usual’ one which holds for the random field obtained by factoring out the constants (cf. Remark 3.17). Overall, our approach recovers many of the famous results concerning the Gaussian Free Field and the associated Liouville Quantum Gravity measure in dimension 2, and for the first time it provides an intrinsic Riemannian, conformally quasi-invariant extension to higher dimensions.

**Probabilistic context.** In dimension 2, conformally invariant random objects appear naturally in the study of continuum statistical models. The celebrated *Gaussian Free Field* naturally arises as the scaling limit of various discrete models of random surfaces, for instance discrete Gaussian Free Fields or harmonic crystals [She07]. A planar conformally invariant random object of fundamental importance is the *Schramm–Loewner evolution* [Law05, Sch07, Law18]. It plays a central role in many problems in statistical physics and satisfies some conformal invariance. The Schramm–Loewner evolution and the two-dimensional Gaussian Free Field are deeply related. For instance, level curves of the Discrete Gaussian Free Field converge to  $\text{SLE}_4$  [SS09], and zero contour lines of the Gaussian Free Field are well-defined random curves distributed according to  $\text{SLE}_4$  [SS13]. The work [MS16], and subsequent works in its series, thoroughly study the relation between the Schramm–Loewner evolution and Gaussian free field on the plane. Motivated by Polyakov’s informal formulation of Bosonic string theory [Pol81a, Pol81b], the papers

[DS11, DKRV16, GRV19] construct mathematically the *Liouville Quantum Gravity* on some surfaces and study its conformal invariance properties. Formally speaking, the Liouville Quantum Gravity is a random random surface obtained by random conformal transform of the Euclidean metric, where the conformal weight is the Gaussian Free Field. Since the Gaussian Free Field is only a distribution, we do not obtain a random Riemannian manifold but rather a random metric measure space. The aforementioned works construct the random measure based on a renormalization procedure due to Kahane [Kah85]. This renormalization depends on a roughness parameter  $\gamma$  and works only for  $|\gamma| < 2$ . In [MS20] and subsequent work in its series, J. Miller and S. Sheffield prove that for the value  $\gamma = \sqrt{8/3}$  the Liouville Quantum Gravity coincides with the Brownian map, that is a random metric measure space arising as a universal scaling limit of random trees and random planar graphs (see [LM12, Le 19] and the references therein). More recently, [DDDF20, GM21] establish the existence of the Liouville Quantum Gravity metric for  $\gamma \in (0, 2)$ . We also note that the case where  $\gamma$  is complex valued is studied in [GHPR20, Pfe21].

**Geometric context.** Despite the fact that the main attention of the probability community has focused so far on the two-dimensional case, (non-random) conformal geometry in dimensions  $n > 2$  is a fascinating field of research. Earlier results by [Tru68, Aub76, Sch84] completely solve the Yamabe problem [Yam60] on compact manifolds: every compact Riemannian manifold is conformally equivalent to a manifold with constant scalar curvature. In the general case, despite ground-breaking results by [ESS6] using the conformal Laplacian, a complete picture is still far from reach. On surfaces, the works [OPS88, OPS89] initiate an approach to the problem based on Polyakov’s variational formulation for the determinant of  $\Delta_g$  [Pol81a, Pol81b]: they show that constant curvature metrics have maximum determinant. In dimension 4, [BØ91] derives an equivalent of Polyakov’s formula for a conformal version of  $\Delta_g^2$ , known as the Paneitz operator and [CY95] finds extremal metrics associated to some functionals of the conformal Laplacian and the Paneitz operator. [GJMS92] constructs higher order equivalent of Paneitz operators, that is conformally invariant powers of  $\Delta_g$ , based on [FG85], see also [GZ03]. In particular in dimension 4, remarkable spectral properties, sharp functional inequalities and rigidity results have been derived in [CY95], [Gur99], [CGY02], and [CGY03]. See also [DHL00] for various such results in higher dimensions.

**Higher dimensional random geometry.** So far, conformally (quasi-)invariant extension for any of these random objects to higher dimensions were discussed only in [LO18] and [Cer19]. Indeed, until we finished and circulated a first version of our paper, we were not aware of any of these contributions. The ansatz of B. Cerclé [Cer19] is similar to ours, limited, however, to the sphere in  $\mathbb{R}^{n+1}$  and relying on an extrinsic approach, based on stereographic projections of the Euclidean space, whereas ours is an intrinsic, Riemannian approach. In particular, our approach also applies to huge classes of manifolds with negative total  $Q$ -curvature, a necessary condition for finiteness of the partition function and for well-definedness of the (normalized) Polyakov–Liouville measure. The approach by T. Levy & Y. Oz [LO18] is more on a heuristic level, not taking care, for instance, of the necessary positivity of the respective GJMS operators. Our intrinsic Riemannian approach also has the advantage that it canonically provides approximations by discrete polyharmonic fields and associated Liouville measures [DHKS]. Our construction of Liouville Brownian motion in higher dimensions and random GJMS operators is not anticipated so far, even not for the sphere or other particular cases.

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## 1 Co-polyharmonic operators on even-dimensional manifolds

Throughout the sequel, without explicitly mentioning it, all manifolds under consideration are assumed to be smooth, connected and without boundary. In particular, we use the terms *closed manifold* and *compact manifold* interchangeably.

### 1.1 Riemannian manifolds and conformal classes

Given a compact Riemannian manifold  $(M, g)$ , we denote its dimension by  $n$ , its volume measure by  $\text{vol} = \text{vol}_g$ , its scalar curvature by  $\text{scal}$  or by  $R$ , its Ricci curvature tensor by  $\text{Ric} = \{\text{Ric}_{ij} : i, j = 0, \dots, n\}$ , and its Laplace-Beltrami operator by  $\Delta = \Delta_g$ , the latter being a negative operator. The spectral gap (or in other words, the first non-trivial eigenvalue) of  $-\Delta_g$  on  $(M, g)$  is denoted by  $\lambda_1 > 0$ .

For  $u \in L^1(M, \text{vol}_g)$ , we set  $\langle u \rangle_g := \frac{1}{\text{vol}_g(M)} \int_M u \, d\text{vol}_g$ , and  $\pi_g(u) := u - \langle u \rangle_g$ . We define the usual *Sobolev spaces*  $H := L^2(M, \text{vol}_g)$  and  $H^s = H_g^s(M) := (1 - \Delta_g)^{-\frac{s}{2}} H$  for  $s \in \mathbb{R}$ . Moreover, we define the *grounded Sobolev spaces*

$$\mathring{H} = \mathring{H}_g^0(M) := \{u \in H : \langle u \rangle_g = 0\}$$

and  $\mathring{H}^s = \mathring{H}_g^s(M) := (-\Delta_g)^{-\frac{s}{2}} \mathring{H}$  for  $s \in \mathbb{R}$ . We remark that for  $s \geq 0$ ,  $\mathring{H}^s = \{u \in H^s : \langle u \rangle_g = 0\}$ .

The space of test functions  $\mathfrak{D} := C^\infty(M)$ , endowed with its usual Fréchet topology, is a nuclear space, see, for instance, the comments preceding [Gro66, Ch. II, Thm. 10, p. 55]. We denote by  $\mathfrak{D}'$  the topological dual of  $\mathfrak{D}$ , endowed with the Borel  $\sigma$ -algebra induced by the weak\* topology, and by  $\langle \cdot | \cdot \rangle = \mathfrak{D}' \langle \cdot | \cdot \rangle_{\mathfrak{D}}$  the standard duality pairing.

**Definition 1.1.** (i) *Two Riemannian metrics  $g$  and  $g'$  on a manifold  $M$  are conformally equivalent if there exists a (‘weight’) function  $\varphi \in C^\infty(M)$  such that  $g' = e^{2\varphi} g$ . The class of metrics which are conformally equivalent to a given metric  $g$  is denoted by  $[g]$ .*

(ii) *Two Riemannian manifolds  $(M, g)$  and  $(M', g')$  are conformally equivalent if there exists a  $C^\infty$ -diffeomorphism  $\Phi : M \rightarrow M'$  and a function  $\varphi \in C^\infty(M)$  such that the pull back of  $g'$  is conformally equivalent to  $g$  with weight  $\varphi$ , that is*

$$\Phi^* g' = e^{2\varphi} g .$$

*In other words, if  $(M', g')$  is isometric to  $(M, g'')$ , and  $g''$  and  $g$  are conformally equivalent. The class of Riemannian manifolds which are conformally equivalent to a given Riemannian manifold  $(M, g)$  is denoted by  $[(M, g)]$ .*

(iii) *A family of operators  $A_g$  on a family of conformally equivalent Riemannian manifolds  $(M, g)$  is called conformally quasi-invariant if for every pair  $(M, g)$  and  $(M', g')$  of conformally equivalent manifolds and associated maps  $\Phi$  and  $\varphi$  as in (ii) there exists a function  $f_\varphi$  on  $M$  such that*

$$e^{f_\varphi} \cdot (A_{e^{2\varphi} g} u) \circ \Phi = A_g(u \circ \Phi) , \quad u \in C^\infty(M) . \quad (11)$$



In conformal geometry, such an operator is usually called conformally covariant. However, in this paper, the notion covariance is already used for the key quantity for characterizing probabilistic dependencies.

The study of conformal mappings as in (ii) above is of particular interest in dimension 2 as powerful uniformization results are available. For instance, Riemann's mapping theorem [Osg00] states that every non-empty simply connected open strict subset of  $\mathbb{C}$  is conformally equivalent to the open unit disk. More generally, the uniformization theorem [Poi07] asserts that every simply connected Riemann surface is conformally equivalent either to the sphere, the plane, or the disc (each of them equipped with its standard metric).

In contrast, the class of conformal mappings in higher dimensions is very limited. According to Liouville's theorem [Lio50], conformal mappings of Euclidean domains in dimension  $\geq 3$  can be expressed as a finite number of compositions of translations, homotheties, orthonormal transformations, and inversions.

*Example 1.2.* Let  $(M', g')$  be the complex plane and  $(M, g)$  be the 2-sphere without north pole  $n$ , regarded as a punctured Riemann sphere. Then they are conformally equivalent in the sense of Definition 1.1 (ii). The conformal map  $\Phi$  is given by the stereographic projection (that is, for all  $x$  on the sphere  $\Phi(x)$  is the stereographic projection of the point  $x$ ), and the weight  $\varphi$  is given by  $\varphi(x) = -2 \log(\sqrt{2} \sin(d_{g^2}(n, x)/2))$ . This example, however, does not fit the setting of this work in two respects: (1) the manifold  $M'$  is non-compact, (2) the weight  $\varphi$  is non-smooth on the completion of  $M$  (it has a singularity at the north pole).

## 1.2 Co-polyharmonic operators

Henceforth,  $n$  denotes an *even* number and  $(M, g)$  is a *compact* Riemannian manifold of dimension  $n$ .

Our interest is primarily in the case  $n \geq 4$ . The case  $n = 2$  is widely studied with celebrated, deep and fascinating results. It serves here as a guideline. In this case, most of the following constructions and results are (essentially) well-known.

The fundamental object for our subsequent considerations are the *co-polyharmonic operators*  $P_g$ , also called *conformally invariant powers of the Laplacian* or *Graham–Jenne–Mason–Sparling operators of maximal order* (i.e. of order  $n/2$ ) as introduced in [GJMS92]. The co-polyharmonic operators are companions of the polyharmonic operators  $(-\Delta_g)^{n/2}$ , coming with *correction terms* which make them conformally invariant. The construction of the co-polyharmonic operators  $P_g$  is quite involved. We outline this construction in Section 1.3. Before we get into that, let us first summarize the crucial properties of the operators  $P_g$  that is relevant for the sequel. We stress that, together with the sign convention  $\Delta_g \leq 0$ , our definition (2) implies that  $P_g$  always has non-negative principal part.

**Theorem 1.3.** *For every compact manifold  $(M, g)$  of even dimension  $n$ ,*

- (i) *the co-polyharmonic operator  $P_g$  is a differential operator of order  $n$ ,*
- (ii) *the leading order is  $(-\Delta_g)^{n/2}$ , the zeroth order vanishes,*
- (iii) *the coefficients are  $C^\infty$  functions of the curvature tensor and its derivatives,*
- (iv) *it is symmetric and extends to a self-adjoint operator (denoted by the same symbol) on  $L^2(M, \text{vol}_g)$  with domain  $H_g^n$ ,*
- (v) *it is conformally quasi-invariant: if  $g' = e^{2\varphi}g$  for some  $\varphi \in C^\infty(M)$ , then*

$$P_{g'} = e^{-n\varphi} P_g. \quad (12)$$

*More generally, assume that  $(M, g)$  and  $(M', g')$  are conformally equivalent with  $C^\infty$ -diffeomorphism  $\Phi : M \rightarrow M'$  and weight  $\varphi \in C^\infty(M)$  such that  $\Phi^*g' = e^{2\varphi}g$ . Then,*

- (vi) *for all  $u \in C^\infty(M')$ :*
- $$(P_{g'} u) \circ \Phi = e^{-n\varphi} P_g (u \circ \varphi). \quad (13)$$



*Proof.* Most properties are due to [GJMS92], and re-stated in [GZ03]; self-adjointness is proven in [GZ03, Corollary, p. 91].  $\square$

*Remark 1.4.* (a) Some authors work directly with a Laplacian defined as a non-negative operator (for instance, [GZ03]). Other authors work with the usual Laplacian and consider the operator  $P_g$  with leading term  $\Delta_g^{n/2}$  (for instance, [Bra95, Gov06, Juh13]); this would correspond to  $(-1)^{n/2}P_g$  in our convention.

(b) In general, no closed expressions exist for the operators  $P_g$ . However, recursive formulas for the expression of  $P_g$  are known and a priori allow to explicitly compute  $P_g$  for any even  $n$ , [Juh13]. As the dimension increases, these formulas become more and more involved; the complexity of lower-order terms grows exponentially with  $n$ .

**Proposition 1.5.** *The most prominent cases are:*

(i) If  $n = 2$ , then  $P_g = -\Delta_g$ .

(ii) If  $n = 4$ , then  $P_g = \Delta_g^2 + \operatorname{div} \left( 2\operatorname{Ric}_g - \frac{2}{3}\operatorname{scal}_g \right) \nabla$  is the celebrated Paneitz operator, see [Pan83]. Here the curvature term  $2\operatorname{Ric}_g - \frac{2}{3}\operatorname{scal}_g$  should be viewed as an endomorphism of the tangent bundle, acting on the gradient of a function. In coordinates:

$$P_g u = \sum_{i,j} \nabla_i \left[ \nabla^i \nabla^j + 2\operatorname{Ric}_g^{ij} - \frac{2}{3}\operatorname{scal}_g \cdot g^{ij} \right] \nabla_j u, \quad \forall u \in C^\infty(M).$$

(iii) If  $(M, g)$  is an Einstein manifold with  $\operatorname{Ric}_g = kg$  (for some  $k \in \mathbb{R}$ ) and even dimension  $n$ , then

$$P_g = \prod_{j=1}^{n/2} \left[ -\Delta_g + \frac{k}{n-1} \nu_j^{(n)} \right] \quad (14)$$

with  $\nu_j^{(n)} := \frac{n}{2} \left( \frac{n}{2} - 1 \right) - j(j-1) = \left( \frac{n-1}{2} \right)^2 - \left( \frac{2j-1}{2} \right)^2$  for  $j = 1, \dots, n/2$ .

*Proof.* For (i) & (ii) see [GJMS92] or [CEØY08, p. 122]; for (iii) see [Gov06, Thm. 1.2]. All formulas above appear in these references up to a factor  $(-1)^{n/2}$ , due to the sign convention in the definition of  $P_g$ .  $\square$

*Example 1.6.* If  $(M, g)$  is flat, then  $P_g = (-\Delta_g)^{n/2}$  is the positive *poly-Laplacian*.

*Example 1.7.* If  $(M, g)$  is the round sphere  $\mathbb{S}^n$ , then  $P_g = \prod_{j=1}^{n/2} \left[ -\Delta_g + \nu_j^{(n)} \right]$  with  $\nu_j^{(n)}$  as above (this formula already appears in [Bra95]). In particular,  $P_g = \Delta_g^2 - 2\Delta_g$  in the case  $n = 4$ , and  $P_g = -\Delta_g^3 + 10\Delta_g^2 - 24\Delta_g$  in the case  $n = 6$ .

Conformally invariant operators with leading term a power of the Laplacian  $\Delta_g$  have been a focus in mathematics and physics for decades. For instance, Dirac [Dir36] constructs a conformally invariant wave operator on a four-dimensional surface in the five-dimensional projective plane in order to show that Maxwell equations are conformally invariant in a curved space-time (Lorentzian manifolds). In the case of Riemannian manifolds the Yamabe operator

$$\Delta_g - \frac{n-2}{4(n-1)} \operatorname{scal}_g,$$

encodes the behaviour of the Ricci curvature under conformal change and has proved of uttermost importance in the resolution of the Yamabe problem on compact Riemannian manifolds [Yam60, Tru68, Aub76, Sch84]. [Pan83] constructs a conformally invariant operator with leading term  $\Delta_g^2$ , and sixth-order analogues are constructed in [Bra85, Wü86].

### 1.3 Construction of the co-polyharmonic operators

Now let us outline the construction of the co-polyharmonic operators, introduced by C.R. Graham, R. Jenne, L.J. Mason, and G.A.J. Sparling [GJMS92]. They base their original construction on the *ambient metric*, introduced by C. Fefferman and C.R. Graham [FG85], a Lorentzian metric on a suitable manifold of dimension  $n + 2$ . In an alternative approach, proposed by C.R. Graham and M. Zworski [GZ03], the manifold  $(M, g)$

is regarded as the *boundary at infinity* of an asymptotically hyperbolic Einstein manifold  $(N, h)$  of dimension  $n + 1$ . Our presentation follows [GZ03], focussing on manifolds of even dimension  $n$  and on operators with maximal degree  $k = n/2$ .

### 1.3.1 The Poincaré metric associated to $(M, [g])$

Consider a compact Riemannian manifold  $(M, g)$  with conformal class  $[g]$  and even dimension  $n$ . Choose an  $n + 1$ -dimensional Riemannian manifold  $(N, h)$  with boundary such that  $\partial N = M$ , for instance,  $N = [0, \infty) \times M$ . A Riemannian metric  $h$  on  $N$  is called *conformally compact metric with conformal infinity  $[g]$*  if

$$h = \frac{\bar{h}}{x^2}, \quad \bar{h}|_{T\partial N} \in [g] \quad (15)$$

where  $\bar{h}$  is a smooth metric on  $N$ , and  $x : N \rightarrow \mathbb{R}_+$  is a smooth function such that  $\{x = 0\} = \partial N$  and  $dx|_{\partial N} \neq 0$ . We say that the metric  $h$  is *asymptotically even* if it is given as

$$h = \frac{1}{x^2} \left[ dx^2 + \sum_{i,j=1}^n \bar{h}_{ij}(x, \xi) d\xi^i d\xi^j \right]$$

where  $x$  is as above,  $(\xi^1, \dots, \xi^n)$  forms a coordinate system on  $M$ , and  $\bar{h}_{ij}$  for  $1 \leq i, j \leq n$  is an even function of  $x$ .

**Definition 1.8.** *A Poincaré metric associated to  $[g]$  is a conformally compact metric  $h$  with conformal infinity  $[g]$  which is asymptotically even and satisfies*

$$\text{Ric}_g + ng = \mathcal{O}(x^{n-2}), \quad \text{tr}_g(\text{Ric}_g + ng) = \mathcal{O}(x^{n+2}). \quad (16)$$

**Lemma 1.9** ([FG85, Theorem 2.3]). *For every compact  $(M, [g])$  of even dimension  $n$ , there exists a Poincaré metric  $h$ . It is uniquely determined up to addition of terms vanishing to order  $n - 2$  and up to a diffeomorphism fixing  $M$ .*

The prime example for this construction is provided by the Poincaré model of hyperbolic space: the  $n$ -dimensional round sphere  $M = \mathbb{S}^n$  is the boundary of the unit ball  $N = B_1(0) \subset \mathbb{R}^{n+1}$  equipped with the hyperbolic metric  $dh(r, \xi) = \frac{4}{(1-r^2)^2} [dr^2 + dg(\xi)]$ .

### 1.3.2 The generalized Poisson operator on $(N, h)$

Consider a compact  $(M, [g])$  of even dimension  $n$  and let  $h$  be the Poincaré metric associated to it on a suitable  $N$  with  $\partial N = M$ . Extending the traditional Landau notation, for a function  $v$  on  $N$  we say that  $v = \mathcal{O}(x^\infty)$  if  $v = \mathcal{O}(x^n)$  as  $x \rightarrow 0$  for every  $n \in \mathbb{N}$ .

**Lemma 1.10** ([GZ03, Props. 4.2, 4.3]). *(i) For every  $f \in \mathcal{C}^\infty(M)$  there exists a solution to*

$$\Delta_h u = \mathcal{O}(x^\infty) \quad (17)$$

*of the form*

$$u = F + G x^n \log x, \quad F, G \in \mathcal{C}^\infty(N), \quad F|_M = f. \quad (18)$$

*Here  $F$  is uniquely determined mod  $\mathcal{O}(x^n)$  and  $G$  is uniquely determined mod  $\mathcal{O}(x^\infty)$ .*

*(ii) Put  $\sigma_n := (-1)^{n/2} 2^n (n/2)! (n/2 - 1)!$  and*

$$P_g f := -2\sigma_n G|_M. \quad (19)$$

*Then,  $P_g$  is a differential operator on  $M$  with principal part  $(-\Delta_g)^{n/2}$ . It only depends on  $g$  and defines a conformally invariant operator which agrees with the operator constructed in [GJMS92].*

(No sign adjustment is required in comparison with [GZ03] since the convention there is that the Laplace–Beltrami operator is non-negative.)

*Remark 1.11.* Given  $(M, [g])$  and  $(N, h)$  as above, the co-polyharmonic operator  $P_g$  can alternatively be defined as residue at  $s = n$  of the meromorphic family of scattering matrix operators  $S(s)$ ,  $s \in \mathbb{C}$ , on  $(N, h)$ ,

$$P_g = -2\sigma_n \operatorname{Res}_{s=n} S(s) \quad (20)$$

with  $\sigma_n$  as above, [GZ03, Thm. 1].

## 1.4 Branson's $Q$ -curvature

The co-polyharmonic operators are closely related to *Branson's  $Q$ -curvature* in even dimensions, another important notion in conformal geometry.

The notion of  $Q$ -curvature was introduced on arbitrary even dimensions by T. Branson [Bra85, p. 11]. Its construction and properties have since been studied by many authors. In 4 dimensions, explicit computations for the  $Q$ -curvature are due to T. Branson and B. Ørsted [BØ91]. Its properties are very much akin to those of scalar curvature in 2 dimensions. C. Fefferman and K. Hirachi [FH03] presented an approach based on the ambient Lorentzian metric of [FG85]. The definition of  $Q$ -curvature may differ in the literature up to a sign or to the factor 2. Following [GZ03], with the notation of Remark 1.11, we have  $Q_g = -2\sigma_n S(n)\mathbf{1}$ .

The crucial property of  $Q$ -curvature is its behavior under conformal transformations.

**Proposition 1.12** ([Bra95, Corollary 1.4]). *If  $g' = e^{2\varphi}g$  then*

$$e^{n\varphi}Q_{g'} = Q_g + P_g\varphi. \quad (21)$$

Our sign convention for  $Q_g$  comes from our sign convention for  $P_g$  together with the validity of equation (21).

**Corollary 1.13.** *The total  $Q$ -curvature  $Q(M, g) := \int_M Q_g d\operatorname{vol}_g$  is a conformal invariant.*

*Proof.* By the previous Proposition,

$$\begin{aligned} Q(M, g') &= \int_M Q_{g'} e^{n\varphi} d\operatorname{vol}_g = Q(M, g) + \int_M P_g \varphi d\operatorname{vol}_g \\ &= Q(M, g) + \int_M \varphi P_g \mathbf{1} d\operatorname{vol}_g = Q(M, g) \end{aligned}$$

due to the self-adjointness of  $P_g$  and the fact that it annihilates constants.  $\square$

Again, explicit formulas are only known in low dimensions or for Einstein manifolds.

*Example 1.14.* Important cases are

- (i) If  $n = 2$ , then  $Q_g = \frac{1}{2}R_g = \frac{1}{2}\operatorname{scal}_g$  is half of the negative *scalar curvature*, see e.g. [CEØY08, Eqn. 3.1, up to a factor  $(-1)^{n/2}$ ].
- (ii) If  $n = 4$  then  $Q_g = -\frac{1}{6}\Delta_g \operatorname{scal}_g - \frac{1}{2}|\operatorname{Ric}_g|^2 + \frac{1}{6}\operatorname{scal}_g^2$  with  $|\operatorname{Ric}_g|^2 = \sum_{i,j} \operatorname{Ric}^{ij} \operatorname{Ric}_{ij}$ .
- (iii) If  $(M, g)$  is an Einstein manifold with  $\operatorname{Ric}_g = k g$  and even dimension, then [Gov06, Thm. 1.1, up to a factor  $(-1)^{n/2}$ ]

$$Q_g = (n-1)! \left( \frac{k}{n-1} \right)^{n/2}. \quad (22)$$

In particular, for the round sphere,  $Q_g = (n-1)!$ . For instance, if  $n = 4$ , then  $Q = 6$ .

Recall that a Riemannian manifold is called *conformally flat* if it is conformally equivalent to a flat manifold.

**Proposition 1.15.** *Let  $\chi(M)$  denote the Euler characteristic of  $(M, g)$ .*

- (i) *In the case  $n = 2$ ,*

$$Q(M, g) = 2\pi \chi(M).$$

(ii) In the case  $n = 4$ ,

$$Q(M, g) = 8\pi^2 \chi(M) - \frac{1}{4} \int_M |W|^2 d\text{vol}_g,$$

where  $W$  is the Weyl tensor, and  $|W|^2 = \sum_{a,b,c,d} W^{abcd} W_{abcd}$ . In particular,

$$Q(M, g) = 8\pi^2 \chi(M) \iff (M, g) \text{ is conformally flat.}$$

(iii) For any even  $n$ , if  $(M, g)$  is conformally flat, then with  $c_n = \frac{1}{2}(2n-1)!(4\pi)^{n/2}$ ,

$$Q(M, g) = c_n \chi(M).$$

*Proof.* The two-dimensional claim follows from Example 1.14 and the Gauss-Bonnet Theorem. See [BØ91, p. 673] for the case of dimension four, and [GZ03, p. 3] for the case of conformally flat manifolds in even dimension. Alternatively, see [CEØY08, pp. 122f.], up to a factor  $(-1)^{n/2}$ .  $\square$

*Remark 1.16.* Recall that for 2-dimensional oriented Riemannian manifolds,  $\chi(M) = 2 - 2g$  where  $g$  denotes the genus of  $M$ . Furthermore, for the sphere in even dimension,  $\chi(\mathbb{S}^n) = 2$ .

### 1.4.1 Some rigidity and equilibration results in $n = 4$

In dimension 4, the conformal invariant integral of the  $Q$ -curvature leads to remarkable rigidity and equilibration results, resembling famous analogous results in dimension 2. To formulate them, let us introduce another important conformal invariance, the Yamabe constant

$$Y(M, g) := \inf_{h \in [g]} \frac{\int_M \text{scal}_h d\text{vol}_h}{\sqrt{\text{vol}_h(M)}}.$$

**Proposition 1.17** ([Gur99],[CY95]). *Assume  $n = 4$ .*

- (i) *There exist compact hyperbolic manifolds with  $Y(M, g) < 0$  and  $Q(M, g) > 16\pi^2$ .*
- (ii) *If  $Y(M, g) \geq 0$ , then  $Q(M, g) \leq 16\pi^2$  with equality if and only if  $M = \mathbb{S}^4$ .*
- (iii) *If  $Y(M, g) \geq 0$  and  $Q(M, g) \geq 0$ , then  $P_g \geq 0$  and  $P_g u = 0 \iff u$  is constant.*
- (iv) *If  $Q(M, g) \leq 16\pi^2$  and  $P_g \geq 0$  with  $P_g u = 0 \iff u$  is constant, then there exists a conformal metric  $g'$  with constant  $Q$ -curvature.*

**Proposition 1.18** ([MS06, Theorem 4.1]). *For any  $g_0 = e^{2\varphi_0} g$  on  $M = \mathbb{S}^4$ , the  $Q$ -curvature flow*

$$\frac{\partial}{\partial t} g_t = -2(Q_{g_t} - \bar{Q}_{g_t})g_t$$

*(with  $\bar{Q}_{g_t} := \langle Q_{g_t} \rangle_{g_t}$  the mean value of  $Q_{g_t}$  on the round  $\mathbb{S}^4$ ) converges exponentially fast to a metric  $g_\infty = e^{2\varphi_\infty} g$  of constant  $Q$ -curvature 6 in the sense that  $\|\varphi_t - \varphi_\infty\|_{H^4} \leq C e^{-\delta t}$  for some constants  $C$  and  $\delta > 0$ .*

## 2 Admissible manifolds

**Definition 2.1.** *We say that a Riemannian manifold  $(M, g)$  is admissible if it is compact and of even dimension, and if the co-polyharmonic operator  $P_g$  is positive definite on  $L^2(M, \text{vol}_g)$ .*

As an immediate consequence of Theorem 1.3 (v) we obtain

**Corollary 2.2.** *Admissibility of a Riemannian manifold  $(M, g)$  is a conformal invariance, or in other words, it is a property of the conformal class  $(M, [g])$ .*

More generally, admissibility of  $(M, g)$  implies admissibility of any  $(N, h)$  conformally equivalent to  $(M, g)$  in the sense of Definition 1.1 (ii).

*Example 2.3.* Every 2-dimensional compact manifold is admissible.

Having at hands the explicit representation formula for the co-polyharmonic operators on Einstein manifolds from Lemma 1.5, we easily conclude

**Proposition 2.4.** *Every even dimensional compact Einstein manifold with nonnegative Ricci curvature is admissible.*

More generally, we obtain:

**Proposition 2.5.** *A compact Einstein manifold of even dimension  $n$  and of Ricci curvature  $-(n-1)\kappa$  is admissible if and only if  $\lambda_1 > \frac{n(n-2)}{4}\kappa$ .*

*Proof.* Since  $M$  has constant Ricci curvature  $-(n-1)\kappa$ , according to Proposition 1.5,  $P_g = \prod_{j=1}^{n/2} [-\Delta_g - \kappa\nu_j^{(n)}]$  with  $\nu_j$  ranging between 0 and  $\frac{n}{2}(\frac{n}{2}-1)$ . Thus  $P_g > 0$  on  $\hat{H}$  if and only if  $\lambda_1 > \frac{n}{2}(\frac{n}{2}-1)\kappa$ .  $\square$

*Remark 2.6.* (a) The number  $\frac{n(n-2)}{4}$  is strictly smaller than  $\frac{(n-1)^2}{4}$  which plays a prominent role as threshold for the spectral gap of hyperbolic manifolds (and which is also the spectral bound for the simply connected hyperbolic space). Many results in hyperbolic geometry deal with the question whether  $\lambda_1$  is close to  $\frac{(n-1)^2}{4}$ .

(b) The *Elstrodt–Patterson–Sullivan Theorem*, [Sul87, Thm. (2.17)], provides a lower bound for  $\lambda_1$  for a hyperbolic manifold  $M = \mathbb{H}^n/\Gamma$  in terms of the *critical exponent*  $\delta(\Gamma)$  of the Kleinian group  $\Gamma$  acting on the simply connected hyperbolic space  $\mathbb{H}^n$  of dimension  $n$  and curvature  $-1$ . More precisely,

$$\lambda_1 > \frac{n(n-2)}{4} \quad \text{if (and only if) } \delta(\Gamma) < \frac{n}{2}, \quad (23)$$

and, moreover,  $\lambda_1 = \frac{(n-1)^2}{4}$  if (and only if) even  $\delta(\Gamma) \leq \frac{n-1}{2}$ . Here  $\delta(\Gamma)$  denotes the infimal value for which the Poincaré series for  $\Gamma$  converges, that is,

$$\delta(\Gamma) := \inf \left\{ s \in \mathbb{R} : \sum_{\gamma \in \Gamma} \exp(-s d(x, \gamma y)) < \infty \right\},$$

the latter being independent of the choice of  $x, y \in M$ .

(c) Similar estimates for  $\lambda_1$  exist in terms of the Hausdorff dimension  $D$  of the *limit set* of  $\Gamma$  provided  $\Gamma$  is *geometrically finite without cusps*, see [Sul87, Thm. (2.21)]. More precisely,

$$\lambda_1 > \frac{n(n-2)}{4} \quad \text{if (and only if) } D < \frac{n}{2}, \quad (24)$$

and, moreover,  $\lambda_1 = \frac{(n-1)^2}{4}$  if (and only if) even  $D \leq \frac{n-1}{2}$ .

**Proposition 2.7.** *For every even dimension  $n \geq 4$ , there exist compact Einstein manifolds that are not admissible. They can be constructed, for instance, as  $M = M_1 \times M_2$  where  $M_1$  denotes any compact manifold of dimension  $n-2$  and of constant curvature  $-\frac{1}{n-3}$ , and where  $M_2$  denotes any compact hyperbolic Riemannian surface with  $\lambda_1(M_2) \leq 2/3$ .*

*Remark 2.8.* According to [Bus77, Satz 1], for every  $\varepsilon > 0$  there exist compact hyperbolic Riemannian surfaces with genus 2 and  $\lambda_1 < \varepsilon$ .

*Proof.* By construction,  $M$  is an Einstein manifold with constant Ricci curvature  $-g$ . Thus by the previous Proposition,  $M$  is admissible if and only if  $\lambda_1(M) > \frac{n(n-2)}{4(n-1)} \geq \frac{2}{3}$ . On the other hand, by construction  $\lambda_1(M) \leq \lambda_1(M_2) \leq \frac{2}{3}$ .  $\square$

Our main result in the section, Theorem 2.18, provides a sharp asymptotic estimate for  $K_g$ , the integral kernel for the inverse of  $P_g$  on  $\hat{H}$ .

## 2.1 Estimates for heat kernels and resolvent kernels

Deriving the exact asymptotic behaviour for the co-polyharmonic Green kernel  $K_g$  requires precise estimates on the integral kernel of the operators  $(\alpha - \Delta)^{n/2}$  for  $\alpha > -\lambda_1$ . These estimates depend on sharp heat kernel estimates, the upper one of which is new.

**Proposition 2.9.** *Let  $(M, g)$  be a compact  $n$ -dimensional manifold and let  $p_t$  denote its heat kernel, the integral kernel for the heat operator  $P_t := e^{t\Delta}$ .*

- (i) *Assume that  $\text{Ric} \geq -(n-1)a^2 g$  and set  $\lambda_* := \frac{(n-1)^2}{4}a^2$  if  $n \neq 2$  and  $\lambda_* = \frac{1}{6}a^2$  if  $n = 2$ . Then, for all  $t > 0$  and all  $x, y \in M$ ,*

$$p_t(x, y) \geq \frac{1}{(4\pi t)^{n/2}} \left( \frac{a d(x, y)}{\sinh(ad(x, y))} \right)^{\frac{n-1}{2}} e^{-\frac{d^2(x, y)}{4t}} e^{-\lambda_* t}. \quad (25)$$

- (ii) *Let a ball  $B = B_R(x) \subset M$  be given, assume that  $\text{sec} \leq b^2$  on  $B$  and that  $\text{inj}_x \geq R$ , and let  $p_t^0$  denote the heat kernel on  $B$  with Dirichlet boundary conditions. Moreover,*

- *in the case  $n \neq 2$ , assume that  $R \leq \frac{\pi}{b}$ , and set  $\lambda^* := \frac{n(n-1)}{6}b^2$ ,*
- *in the case  $n = 2$ , assume that  $R \leq \frac{\pi}{2b}$ , and set  $\lambda^* := \frac{1}{2}b^2$ .*

*Then, for all  $t > 0$  and all  $y \in B$ ,*

$$p_t^0(x, y) \leq \frac{1}{(4\pi t)^{n/2}} \left( \frac{b d(x, y)}{\sin(b d(x, y))} \right)^{\frac{n-1}{2}} e^{-\frac{d^2(x, y)}{4t}} e^{+\lambda^* t}. \quad (26)$$

*Proof.* (i) follows from [Stu92, Cor. 4.2 and Rmk. 4.4(a)] (we work with the geometric heat semigroup  $e^{t\Delta}$  rather than with the probabilistic semigroup  $e^{t\frac{\Delta}{2}}$  as in [Stu92]).

(ii) Let  $M = \mathbb{S}^{b,n}$  denote the round sphere of dimension  $n$  and radius  $1/b$  (which has constant curvature  $b^2$ ), fix a point  $\bar{x} \in M$ , and let  $\bar{B}$  denote the ball around  $\bar{x}$  of radius  $R$  in  $M$ . Denote by  $\bar{p}_t^0$  the heat kernel on  $\bar{B}$  with Dirichlet boundary conditions. By rotational invariance,

$$\bar{p}_t^0(\bar{x}, \bar{y}) = \bar{p}_t^0(\bar{d}(\bar{x}, \bar{y}))$$

for some function  $r \mapsto \bar{p}_t^0(r)$ . According to the celebrated heat kernel comparison theorem of Debiard–Gaveau–Mazet [DGM76],

$$p_t^0(x, y) \leq \bar{p}_t^0(d(x, y)) \quad (27)$$

for all  $t > 0$  and all  $y \in B$ .

We treat the case  $n \neq 2$  first. Following the strategy for deriving the lower bound (25) in [Stu92], define

$$\hat{p}_t^0(r) := \frac{1}{(4\pi t)^{n/2}} \left( \frac{br}{\sin(br)} \right)^{\frac{n-1}{2}} e^{-\frac{r^2}{4t}} e^{\lambda^* t} = g_t(r) \left( \frac{br}{\sin(br)} \right)^{\frac{n-1}{2}} e^{\lambda^* t}.$$

where  $\lambda^*$  as defined above and  $g_t(r) = (4\pi t)^{-n/2} e^{-r^2/4t}$  is the Gaussian kernel. We show that the function  $(t, \bar{y}) \mapsto H(t, \bar{y}) := \hat{p}_t^0(\bar{d}(\bar{x}, \bar{y}))$  is space-time super-harmonic on  $(0, \infty) \times \bar{B}$ . Indeed, a direct computation yields:

$$\begin{aligned} \partial_t \log \hat{p}^0 &= \partial_t \log g + \lambda^*; \\ \partial_r \log \hat{p}^0 &= \partial_r \log g + \frac{n-1}{2} \left( \frac{1}{r} - b \frac{\cos br}{\sin br} \right); \\ \partial_{rr}^2 \log \hat{p}^0 &= \partial_{rr}^2 \log g + \frac{n-1}{2} \left( \frac{b^2}{\sin^2 br} - \frac{1}{r^2} \right). \end{aligned}$$

Now using that  $H$  is a radial function and the chain rule we find that

$$\frac{1}{H} (\partial_t - \bar{\Delta}) H = \partial_t \log \hat{p}^0 - (n-1)b \frac{\cos br}{\sin br} \partial_r \log \hat{p}^0 - \partial_{rr}^2 \log \hat{p}^0 - (\partial_r \log \hat{p}^0)^2, \quad (28)$$

where the left-hand side is evaluated at  $(t, \bar{y})$  and the right-hand side at  $(t, \bar{d}(\bar{x}, \bar{y}))$ . We easily verify that  $g$  satisfies:

$$\partial_t \log g - \partial_{rr}^2 \log g - (\partial_r \log g)^2 - \frac{n-1}{r} \partial_r \log g = 0 .$$

We thus see that in (28) all the appearances of  $\log g$  cancel out, and we get:

$$\begin{aligned} \frac{1}{H}(\partial_t - \bar{\Delta})H &= \lambda^* - \frac{n-1}{2} \left( \frac{b^2}{\sin^2 br} - \frac{1}{r^2} \right) \\ &\quad - \frac{(n-1)^2}{2} b \frac{\cos br}{\sin br} \left( \frac{1}{r} - b \frac{\cos br}{\sin br} \right) - \frac{(n-1)^2}{4} \left( \frac{1}{r} - b \frac{\cos br}{\sin br} \right)^2 \\ &= \lambda^* - \frac{(n-1)(n-3)}{2} \frac{1}{r^2} + b^2 \frac{\cos^2 br}{\sin^2 br} \frac{(n-1)^2}{4} - \frac{n-1}{2} b^2 \frac{1}{\sin^2 br} \\ &= \lambda^* + \frac{(n-1)(n-3)}{4} \left[ \frac{b^2}{\sin^2 br} - \frac{1}{r^2} \right] - b^2 \frac{(n-1)^2}{4} \\ &\geq \lambda^* + \frac{(n-1)(n-3)}{4} \frac{b^2}{3} - b^2 \frac{(n-1)^2}{4} = 0 . \end{aligned}$$

On the other hand,  $\bar{p}$  is harmonic and by a comparison principle for solution of parabolic equations we thus have that  $p^0 \leq H$ . In order to properly justify the comparison principle, instead of working with  $p^0$  and  $H$  that have singular initial condition, we work instead with  $p_{R'}^0$  the solution to the heat equation with initial condition  $\mathbf{1}_{B_{R'}(y)}$  and  $H_{R'}$  which has the same expression as  $H$  except that we choose

$$g_{R'}(t, r) = \int_{B_{R'}^{\mathbb{R}^n}(r)} (4\pi t)^{-\frac{n}{2}} e^{-|y|^2/4t} dy ,$$

instead of  $g$ . The same computation yields that  $H_{R'}$  is a super-solution to the heat equation. Then, we argue as in [Stu92].

Now, for the case  $n = 2$ , defining  $\hat{p}^0$  and  $H$  as above we still find that

$$\begin{aligned} \frac{1}{H}(\partial_t - \bar{\Delta})H &= \lambda^* - \frac{1}{4} \left[ \frac{b^2}{\sin^2 br} - \frac{1}{r^2} + b^2 \right] \\ &\geq \lambda^* - b^2 \left[ \frac{1}{2} - \frac{1}{\pi^2} \right] \geq 0 , \end{aligned}$$

where we used that  $r < \frac{\pi}{2b}$ . The rest of the proof is similar.  $\square$

Before stating our main estimates, let us introduce some notation and provide some auxiliary results.

**Lemma 2.10.** (i) For every  $\alpha > 0$  and  $s > 0$ , the resolvent operator  $\mathbf{G}_{s,\alpha} := (\alpha - \Delta)^{-s}$  on  $H = L^2(M, \text{vol}_g)$  is an integral operator with kernel given by

$$G_{s,\alpha}(x, y) := \frac{1}{\Gamma(s)} \int_0^\infty e^{-\alpha t} t^{s-1} p_t(x, y) dt .$$

Since  $\langle P_t u \rangle_g = \langle u \rangle_g$  for all  $u$ , the heat operator  $P_t = e^{t\Delta}$  also acts on the grounded  $L^2$ -space  $\dot{H} = \{u \in L^2(M, \text{vol}_g) : \langle u \rangle_g = 0\}$ , and so do the resolvent operators  $\mathbf{G}_{s,\alpha}$ .

(ii) Restricted to  $\dot{H}$ , the resolvent operator

$$\mathring{\mathbf{G}}_{s,\alpha} = (\alpha - \Delta)^{-s} \Big|_{\dot{H}}$$

is a compact, symmetric operator for every  $\alpha > -\lambda_1$  and  $s > 0$ . It admits a symmetric integral kernel

$$\mathring{G}_{s,\alpha}(x, y) := \frac{1}{\Gamma(s)} \int_0^\infty e^{-\alpha t} t^{s-1} \mathring{p}_t(x, y) dt ,$$

defined in terms of the grounded heat kernel  $\mathring{p}_t(x, y) := p_t(x, y) - \text{vol}(M)^{-1}$ .



(iii) By compactness of  $M$ , the operator  $-\Delta$  has discrete spectrum  $(\lambda_j)_{j \in \mathbb{N}_0}$ , counted with multiplicity, and the corresponding eigenfunctions  $(\chi_j)_{j \in \mathbb{N}_0}$  form an orthonormal basis for  $L^2(M, \text{vol})$ . In terms of these spectral data, the symmetric grounded resolvent kernel is given as

$$\mathring{G}_{s,\alpha}(x,y) = \sum_{j=1}^{\infty} \frac{\chi_j(x)\chi_j(y)}{(\alpha + \lambda_j)^s}. \quad (29)$$

(iv) For every  $s > n/2$  and  $\alpha > 0$  there exists  $C$  such that for all  $x, y \in M$

$$|\mathring{G}_{s,\alpha}(x,y)| \leq C, \quad (30)$$

and for every  $s < n/2$  and  $\alpha > 0$  there exists  $C$  such that for all  $x, y \in M$

$$G_{s,\alpha}(x,y) \leq \frac{C}{d(x,y)^{n-2s}}. \quad (31)$$

*Proof.* All of (i)–(iii) but (29) are proven by Strichartz [Str83, §4]. Regarding (29), by the Spectral Theorem, for all  $u \in H$  and  $\alpha > 0$  (or  $u \in \mathring{H}$  and  $\alpha > -\lambda_1$ ),

$$(\alpha - \Delta)^s u = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\alpha t} t^{s-1} P_t u dt = \sum_{j=0}^{\infty} \frac{\langle u | \chi_j \rangle_{L^2}}{(\alpha + \lambda_j)^s} \chi_j$$

with convergence of integral and sum in  $H$  (or in  $\mathring{H}$ , resp.). Thus (29) readily follows.

(iv) Estimate (30) is a consequence of [DKS20, Thm. 6.2]. In order to show (31) fix  $\varepsilon \ll 1$ ,  $x, y \in M$  with  $0 < r := d_g(x, y) < \varepsilon$ , and set

$$G_{s,\alpha}(x,y) = \underbrace{\int_0^\varepsilon e^{-\alpha^2 t} t^{s-1} p_t(x,y) dt}_{I_1} + \underbrace{\int_\varepsilon^\infty e^{-\alpha^2 t} t^{s-1} p_t(x,y) dt}_{I_2}.$$

As a consequence of the upper heat kernel estimate [DKS20, Eqn. (6.1)], there exists a constant  $C = C(g, s, \alpha, \varepsilon) > 0$  independent of  $x, y$ , and such that  $I_2 \leq C$  and

$$I_1 \leq C \int_0^\varepsilon e^{-\frac{r^2}{t}} t^{s-n/2-1} dt.$$

Combining these estimates together,

$$G_{s,\alpha}(x,y) \leq C \left( 1 + \int_0^\varepsilon e^{-\frac{r^2}{t}} t^{s-n/2-1} dt \right),$$

and the assertion now follows from the known asymptotic expansion of the Exponential Integral function

$$E_{s-n/2+1}\left(\frac{r^2}{\varepsilon}\right) := \int_0^\varepsilon e^{-\frac{r^2}{t}} t^{s-n/2-1} dt \asymp \Gamma(n/2 - s) r^{2s-n} \quad \text{as } r \rightarrow 0. \quad \square$$

*Remark 2.11.* For all  $s > 0$ , the operators  $G_{s,\alpha}$  and  $\mathring{G}_{s,\alpha}$  are powers of  $G_\alpha := G_{1,\alpha}$  and  $\mathring{G}_\alpha := \mathring{G}_{1,\alpha}$ , that is,

$$G_{s,\alpha} = (G_\alpha)^s, \quad \mathring{G}_{s,\alpha} = (\mathring{G}_\alpha)^s.$$

*Example 2.12.* Let  $(M, g)$  be the 2-dimensional round sphere  $\mathbb{S}^2$ . Then, according to [DKS20],

$$\mathring{G}_{1,0}(x,y) = -\frac{1}{4\pi} \left( 1 + 2 \log \sin \frac{d(x,y)}{2} \right).$$

**Proposition 2.13.** *Let  $(M, g)$  be a compact  $n$ -dimensional manifold and  $\alpha > -\lambda_1$ . Then, for all  $x$  and  $y \in M$ :*

$$\begin{aligned} \left| G_{n/2, \alpha}(x, y) - a_n \log \frac{1}{d(x, y)} \right| &\leq C_0; \\ \left| \check{G}_{n/2, \alpha}(x, y) - a_n \log \frac{1}{d(x, y)} \right| &\leq C_0; \end{aligned}$$

for some  $C_0 = C_0(g, \alpha) > 0$  and

$$a_n := \frac{2}{\Gamma(n/2) (4\pi)^{n/2}}. \quad (32)$$

*Proof.* For convenience we split the proof.

*Lower estimate for the ungrounded kernel.* Take  $\lambda_*$  as in Proposition 2.9 (i) and  $\alpha > \lambda_*$ . For the non-grounded resolvent kernel, the lower heat kernel estimate (25) yields, with  $x$  and  $y \in M$ , and  $r = d(x, y)$ ,

$$\begin{aligned} G_{n/2, \alpha}(x, y) &= \frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-\alpha t} p_t(x, y) t^{n/2-1} dt \\ &\geq \frac{1}{\Gamma(\frac{n}{2}) (4\pi)^{n/2}} \left( \frac{ar}{\sinh(ar)} \right)^{\frac{n-1}{2}} \int_0^\infty e^{-(\alpha+\lambda_*)t} e^{-\frac{r^2}{4t}} \frac{dt}{t}. \end{aligned}$$

By [DKS20], Eqn. (6.14) and the asymptotic formulas thereafter, for every  $\beta > 0$ :

$$\int_0^\infty e^{-\beta t} e^{-\frac{r^2}{4t}} \frac{dt}{t} = 4\pi \cdot G_{1, \beta}^{\mathbb{R}^2}(r) \geq 2 \log \frac{1}{r} - C_\beta.$$

Combining the two previous estimates yields

$$G_{n/2, \alpha}(x, y) - a_n \log \frac{1}{d(x, y)} > -C_{\alpha+\lambda_*}, \quad x, y \in M.$$

*Upper estimate for the ungrounded kernel with Dirichlet boundary conditions.* Consider the case  $\alpha > \lambda^*$ , with  $\lambda^*$  as in Lemma 2.9 (ii). We estimate the contribution of  $p_t^0$  as before, with  $x$  and  $y \in M$ , and  $r = d(x, y)$ :

$$\begin{aligned} G_{n/2, \alpha}^0(x, y) &:= \frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-\alpha t} p_t^0(x, y) t^{n/2-1} dt \\ &\leq \frac{1}{\Gamma(\frac{n}{2}) (4\pi)^{n/2}} \left( \frac{ar}{\sin(br)} \right)^{\frac{n-1}{2}} \int_0^\infty e^{(-\alpha+\lambda^*)t} e^{-\frac{r^2}{4t}} \frac{dt}{t}, \end{aligned}$$

and we can use the fact that, by [DKS20], *ibid.*:

$$\int_0^\infty e^{-\beta t} e^{-\frac{r^2}{4t}} \frac{dt}{t} = 4\pi \cdot G_{1, \beta}^{\mathbb{R}^2}(r) \leq 2 \log \frac{1}{r} + C_\beta, \quad \beta > 0.$$

The two estimates yield

$$G_{n/2, \alpha}^0(x, y) - a_n \log \frac{1}{d(x, y)} < C_{\alpha-\lambda_*}, \quad x, y \in M.$$

*Upper estimate for the ungrounded kernel.* We now estimate the remainder  $G_{n/2, \alpha} - G_{n/2, \alpha}^0$ . Choose  $0 < \beta < \alpha$ . For every  $n \geq 2$  and suitable  $C, C' > 0$ ,

$$\begin{aligned} 0 &\leq G_{n/2, \alpha}(x, y) - G_{n/2, \alpha}^0(x, y) := \frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-\alpha t} (p_t(x, y) - p_t^0(x, y)) t^{n/2-1} dt \\ &\leq C \int_0^\infty e^{-\beta t} (p_t(x, y) - p_t^0(x, y)) dt = C (G_{1, \beta} - G_{1, \beta}^0)(x, y) \\ &\leq C \sup_{z \in \partial B} G_{1, \beta}(x, z) \leq C'. \end{aligned}$$

Above, the second to last inequality follows from the maximum principle for local solutions to  $(-\Delta_g + \beta)u = 0$ , and the last inequality from the elliptic Harnack inequality for positive local solutions to  $(-\Delta_g + \beta)u = 0$ .

*Bounds for the grounded kernel.* The lower and upper bounds for the *grounded* resolvent kernel  $\mathring{G}_\alpha$  for  $\alpha > \lambda_*$  then follow from the previous bounds and the fact that  $\mathring{p}_t(x, y) = p_t(x, y) - \frac{1}{\text{vol}(M)}$  and

$$\frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-\alpha t} t^{n/2-1} dt = \alpha^{-n/2} .$$

*Bounds for all  $\alpha > -\lambda_1$ .* In order to show the desired estimates for  $G_\alpha$  in the whole range of  $\alpha > -\lambda_1$ , we use a perturbation argument based on the *resolvent identity*

$$\mathring{G}_\alpha = \mathring{G}_\beta + (\beta - \alpha) \mathring{G}_\beta \mathring{G}_\alpha, \quad (33)$$

valid for all  $\beta > -\lambda_1$  and employed below for  $\beta > \lambda_*$ . By iteration, it follows that

$$\mathring{G}_\alpha = \mathring{G}_\beta \left( \sum_{\ell=0}^{\infty} ((\beta - \alpha) \mathring{G}_\beta)^\ell \right) .$$

The series is absolutely converging in  $\text{Lin}(\mathring{H}, \mathring{H})$ , since

$$\|(\beta - \alpha) \mathring{G}_\beta\|_{\mathring{H}, \mathring{H}} \leq (\beta - \alpha) / (\beta + \lambda_1) < 1 .$$

Let  $\mathring{T} = (\beta - \alpha) \sum_{\ell=0}^{\infty} ((\beta - \alpha) \mathring{G}_\beta)^\ell$ . Then,

$$\mathring{G}_\alpha^{n/2} = \mathring{G}_\beta^{n/2} (\text{Id} + \mathring{G}_\beta \mathring{T})^{n/2} = \mathring{G}_\beta^{n/2} \left( 1 + \sum_{k=1}^{n/2} \binom{n/2}{k} \mathring{G}_\beta^k \mathring{T}^k \right) = \mathring{G}_\beta^{n/2} + \mathring{G}_\beta^{n/2+1} \tilde{\mathring{T}},$$

where  $\tilde{\mathring{T}} = \sum_{k=0}^{n/2-1} \binom{n/2}{k+1} \mathring{G}_\beta^k \mathring{T}^{k+1}$ . Consequently, since all operators involved commute with each other,

$$\mathring{G}_\alpha^{n/2} - \mathring{G}_\beta^{n/2} = \underbrace{\mathring{G}_\beta^{n/4+1/2}}_{\mathring{L}^2 \rightarrow \mathring{L}^\infty} \underbrace{\tilde{\mathring{T}}}_{\mathring{L}^2 \rightarrow \mathring{L}^2} \underbrace{\mathring{G}_\beta^{n/4+1/2}}_{\mathring{L}^1 \rightarrow \mathring{L}^2} .$$

Moreover,  $\mathring{G}_\beta^{n/4+1/2}$  is a bounded linear operator both from  $\mathring{L}^1$  to  $\mathring{L}^2$  and from  $\mathring{L}^2$  to  $\mathring{L}^\infty$ . Indeed, for  $u \in \mathring{L}^1$ ,

$$\begin{aligned} & \left\| \mathring{G}_\beta^{n/4+1/2} u \right\|_{L^2}^2 = \\ &= \int \left[ \int \mathring{G}_{(n+2)/4, \beta}(x, y) u(y) d\text{vol}(y) \int \mathring{G}_{(n+2)/4, \beta}(x, z) u(z) d\text{vol}(z) \right] d\text{vol}(x) \\ &= \iint \mathring{G}_{(n+2)/2, \beta}(y, z) u(y) u(z) d\text{vol}(y) d\text{vol}(z) \\ &\leq \sup_{y, z} \mathring{G}_{(n+2)/2, \beta}(y, z) \cdot \|u\|_{L^1}^2 , \end{aligned}$$

and for  $u \in \mathring{L}^2$ ,

$$\begin{aligned} \left\| \mathring{G}_\beta^{n/4+1/2} u \right\|_{L^\infty}^2 &= \sup_x \left( \int \mathring{G}_{(n+2)/4, \beta}(x, y) u(y) d\text{vol}(y) \right)^2 \\ &\leq \sup_x \int \mathring{G}_{(n+2)/4, \beta}^2(x, y) d\text{vol}(y) \cdot \int u^2(y) d\text{vol}(y) \\ &= \sup_x \mathring{G}_{(n+2)/2, \beta}(x, x) \cdot \|u\|_{L^2}^2 . \end{aligned}$$

Finiteness of both expressions is granted for  $\beta > 0$  by Lemma 2.10. Thus summarizing we obtain

$$\left\| \mathring{G}_\alpha^{n/2} - \mathring{G}_\beta^{n/2} \right\|_{\mathring{L}^1, \mathring{L}^\infty} < \infty .$$

Furthermore, consider  $(\mathring{G}_\alpha^{n/2} - \mathring{G}_\beta^{n/2}) \circ \pi_g: L^1 \rightarrow L^\infty$ , where as usual  $\pi_g: u \mapsto u - \langle u \rangle_g$ . Since  $\pi_g$  is the identity on  $\mathring{L}^1$ , by virtue of [DP40, Thm. 2.2.5], the operator  $(\mathring{G}_\alpha^{n/2} - \mathring{G}_\beta^{n/2}): \mathring{L}^1 \rightarrow L^\infty$  admits a bounded integral kernel. Therefore,  $\mathring{G}_\alpha^{n/2}$  admits an integral kernel with the same logarithmic divergence as  $\mathring{G}_\beta^{n/2}$ .  $\square$

For curiosity, we provide an estimate for the co-polyharmonic heat kernel which, in the case  $n = 2$ , reduces to the standard Gaussian estimate.

*Remark 2.14* ([tR97, Theorem 1.1]). Assume that the compact manifold  $(M, g)$  of even dimension  $n$  is a Lie group. Then, the co-polyharmonic heat semigroup  $e^{-tP_g}$  has an integral kernel ('co-polyharmonic heat kernel'), the modulus of which can be estimated by

$$|p_t(x, y)| \leq \frac{C_1}{t \wedge 1} \exp \left[ - \left( \frac{d(x, y)^n}{C_2 t} \right)^{\frac{1}{n-1}} \right] .$$

## 2.2 Estimates for co-polyharmonic Green kernels

**Lemma 2.15.** *For every admissible manifold  $(M, g)$ ,*

- (i) *the co-polyharmonic operator is a compact perturbation of the poly-Laplacian: for every  $\alpha > -\lambda_1$  there exists  $C_\alpha = C(\alpha, g) > 0$  such that the operator  $S_\alpha := P_g - (\alpha - \Delta_g)^{n/2}$  satisfies*

$$\langle S_\alpha u | u \rangle_{\mathring{H}} \leq C_\alpha \cdot \|(\alpha - \Delta)^{\frac{n-1}{4}} u\|_{\mathring{H}}^2, \quad \forall u \in \mathring{H}^{n/2}. \quad (34)$$

*In particular,  $\langle S_\alpha u | u \rangle_{\mathring{H}} \leq C_\alpha \|u\|_{\mathring{H}^{(n-1)/2}}^2$ .*

- (ii) *for every  $s > 0$ ,  $u \mapsto \|P_g^s u\|_{L^2}$  defines a Hilbert norm on  $\mathring{H}^{sn}$ , bi-Lipschitz equivalent to the  $\mathring{H}^{sn}$ -norm.*
- (iii) *for every  $r \in \mathbb{R}$ , the bounded operator  $P_g: \mathring{H}^{n+r} \rightarrow \mathring{H}^r$  has bounded inverse  $K_g: \mathring{H}^r \rightarrow \mathring{H}^{n+r}$ .*
- (iv) *for every  $s > 0$ ,  $f \mapsto \|(K_g)^s f\|_{L^2}$  defines a Hilbert norm on  $\mathring{H}^{-sn}$ , bi-Lipschitz equivalent to the  $\mathring{H}^{-sn}$ -norm.*
- (v)  *$(P_g, H^n)$  has discrete spectrum  $\text{spec}(P_g) = \{\nu_j\}_{j \in \mathbb{N}_0}$ , indexed with multiplicities, satisfying  $\nu_j \geq 0$  for all  $j$ , and  $\nu_0 = 0$  with multiplicity 1. The corresponding family of eigenfunctions  $(\psi_j)_{j \in \mathbb{N}_0}$  forms an orthonormal basis of  $H$ .*
- (vi) *The operator  $K_g: \mathring{H} \rightarrow \mathring{H}$  is nuclear. It admits a unique non-reabeled extension  $K_g: H \rightarrow \mathring{H}$ , vanishing on constants and satisfying  $K_g P_g = \pi_g$  on  $H$ . This extension is an integral operator on  $H$  with symmetric kernel*

$$K_g(x, y) := \sum_{j=1}^{\infty} \frac{\psi_j(x) \psi_j(y)}{\nu_j}, \quad x, y \in M, \quad (35)$$

*where the convergence of the series is understood in  $L^2 \otimes L^2$ .*

- (vii) *For  $\ell \in \mathbb{N}$ , define the operators  $K_{g,\ell}: H \rightarrow \mathring{H}$  by*

$$K_{g,\ell} u := \sum_{j=1}^{\ell} \frac{\langle \psi_j | u \rangle_{L^2}}{\nu_j} \psi_j .$$

*Then, for every  $u \in H = L^2$ , as  $\ell \rightarrow \infty$ ,*

$$K_{g,\ell} u \longrightarrow K_g u \quad \text{in } L^\infty . \quad (36)$$

(viii) The spectrum of  $\mathsf{P}$  satisfies the Weyl asymptotic. With  $N(\nu)$  the number of eigenvalues lower than  $\nu$ , we get  $N(\nu) = c\nu + O(\nu^{1-1/n})$  as  $\nu \rightarrow \infty$ . In particular, we find

$$\nu_j = cj + O(j^{1-1/n}), \quad j \rightarrow \infty. \quad (37)$$

*Proof.* (i) For every  $\alpha > -\lambda_1$ , the operator  $\mathsf{S}_\alpha$  is a linear differential operator of order  $\leq n-1$  with smooth (hence bounded) coefficients on  $M$ , and (34) readily follows.

(ii) It suffices to show the statement for  $s = 1/2$ . As a consequence of Theorem 1.3 (iv) and admissibility, the (strictly) positive operator  $(\mathsf{P}_g, \dot{H}^n)$  has positive self-adjoint square root  $(\sqrt{\mathsf{P}_g}, \dot{H}^{n/2})$ , and the latter defines a Hilbert norm on  $\dot{H}^{n/2}$ . Thus, the linear operator  $\iota := (-\Delta_g)^{-n/4} \sqrt{\mathsf{P}_g}: \dot{H}^{n/2} \rightarrow \dot{H}^{n/2}$  is well-defined, positive, and injective. Moreover,  $\iota$  is an isometry

$$\iota: \left( \dot{H}^{n/2}, \|\sqrt{\mathsf{P}_g} \cdot\|_{\dot{H}} \right) \longrightarrow \left( \dot{H}^{n/2}, \|\cdot\|_{\dot{H}^{n/2}} \right),$$

and in fact unitary, since  $\ker \iota = \{0\}$  by strict positivity of both  $\sqrt{\mathsf{P}_g}$  and  $(-\Delta_g)^{-n/4}$  on the appropriate spaces of grounded functions. As a consequence,  $\iota: \dot{H}^{n/2} \rightarrow \dot{H}^{n/2}$  is surjective, and thus bijective. It suffices to show it is also  $\dot{H}^{n/2}$ -bounded, in which case it has a  $\dot{H}^{n/2}$ -bounded inverse  $\iota^{-1}$  by the Bounded Inverse Theorem. The former fact follows if we show that  $\iota^*$  is  $\dot{H}^{n/2}$ -bounded. We have

$$\iota^* = (-\Delta_g)^{-n/4} \mathsf{P}_g (-\Delta_g)^{-n/4} = \text{Id}_{\dot{H}^{n/2}} + (-\Delta_g)^{-n/4} \mathsf{S}_0 (-\Delta_g)^{-n/4}.$$

By squaring the operators in (34) with  $\alpha = 0$ , the latter is a  $\dot{H}^{n/2}$ -bounded perturbation of the identity on  $\dot{H}^{n/2}$ , and the assertion follows.

(iii) It suffices to show the statement for  $r = 0$ . We show that  $\sqrt{\mathsf{P}_g}: \dot{H}^{n/2} \rightarrow \dot{H}$  is invertible with bounded inverse, say  $\sqrt{\mathsf{K}_g}$ , in which case the assertion follows setting  $\mathsf{K}_g := (\sqrt{\mathsf{K}_g})^2$ . As a consequence of the bijectivity of  $\iota$  in (ii), and since  $(-\Delta)^{n/2}: \dot{H}^{n/2} \rightarrow \dot{H}$  is surjective, the operator  $\sqrt{\mathsf{P}_g} = (-\Delta_g)^{n/2} \iota: \dot{H}^{n/2} \rightarrow \dot{H}$  is as well surjective, and thus bijective. Its inverse  $\sqrt{\mathsf{K}_g} := \iota^{-1} (-\Delta_g)^{-n/2}: \dot{H} \rightarrow \dot{H}^{n/2}$  is a bounded operator, since so are  $\iota^{-1}: \dot{H}^{n/2} \rightarrow \dot{H}^{n/2}$ , by (ii), and  $(-\Delta_g)^{-n/2}: \dot{H} \rightarrow \dot{H}^{n/2}$ .

(iv) is well-posed by (iii). It follows from (ii) by a standard duality argument.

(v) Since  $\dot{H}^n$  embeds compactly into  $\dot{H}$  by the Rellich–Kondrashov Theorem, the operator  $\mathsf{K}_g: \dot{H} \rightarrow \dot{H}$  is compact, being the composition of the bounded operator  $\mathsf{K}_g: \dot{H} \rightarrow \dot{H}^n$  with the compact Sobolev embedding. The spectral properties follow from the (strict) positivity of  $(\mathsf{P}_g, \dot{H}^n)$  on  $\dot{H}$  and the  $\dot{H}$ -compactness of  $\mathsf{K}_g$ . The assertion on eigenfunctions holds by the Spectral Theorem for unbounded self-adjoint operators.

(vi) In order to show that the operator  $\mathsf{K}_g: \dot{H} \rightarrow \dot{H}$  is trace-class, it suffices to show that  $\sqrt{\mathsf{K}_g}$  is Hilbert–Schmidt. This latter fact holds since  $\sqrt{\mathsf{K}_g} = \iota^{-1} (-\Delta_g)^{n/2}$  by (iii),  $\iota^{-1}: \dot{H}^{n/2} \rightarrow \dot{H}^{n/2}$  is bounded and  $(-\Delta_g)^{-n/2}: \dot{H}^{n/2} \rightarrow \dot{H}$  is Hilbert–Schmidt. Since  $\mathsf{K}_g: \dot{H} \rightarrow \dot{H}$  is trace-class, the kernel’s representation in (35) on  $\dot{H}$  follows from the spectral characterization of trace-class operators. The extension of  $\mathsf{K}_g$  to  $H$  is then defined as the operator on  $H$  with integral kernel  $K_g$ . The equality  $\mathsf{K}_g \mathsf{P}_g = \pi_g$  is readily verified, since  $\langle u \rangle_g = \langle u | \psi_0 \rangle_H$  and the series in the definition of  $\mathsf{K}_g$  starts at  $j = 1$ .

(vii) By the norm equivalence stated in (ii),

$$\|\mathsf{K}_g u - \mathsf{K}_{g,\ell} u\|_{\dot{H}^n}^2 \simeq \|\mathsf{P}_g(\mathsf{K}_g u - \mathsf{K}_{g,\ell} u)\|_{L^2}^2 = \sum_{j=\ell+1}^{\infty} \langle \psi_j | u \rangle_{L^2}^2 \longrightarrow 0$$

as  $\ell \rightarrow \infty$  for every  $u \in L^2$ . Hence, by Sobolev embedding,  $\mathsf{K}_{g,\ell} u \rightarrow \mathsf{K}_g u$  in  $L^\infty$ .

(viii) Hörmander’s Weyl law [H668] for positive pseudo-differential operators.  $\square$

**Definition 2.16.** Given any admissible manifold  $(M, g)$ , we define the quadratic form  $\mathfrak{p} = \mathfrak{p}_g$  on  $H^{n/2}$  associated to  $\mathsf{P}_g$  by

$$\mathfrak{p}(u, v) = \int \sqrt{\mathsf{P}_g} u \sqrt{\mathsf{P}_g} v \, d \text{vol}_g, \quad \forall u, v \in H^{n/2}. \quad (38)$$

Moreover, we always implicitly extend the operator  $K_g$  to  $H$  by setting  $K_g c = 0$  for all constant  $c$ . It is the pseudo-inverse of  $P_g$  on  $H$  in the sense that:

$$K_g P_g = P_g K_g = \pi_g .$$

We call  $K$  the co-polyharmonic Green operator. It has the integral kernel  $K_g$  given in (35), and we call  $K_g$  the co-polyharmonic Green kernel.

*Remark 2.17.* (a) Elliptic regularity theory implies that off the diagonal of  $M \times M$ , the function  $(x, y) \mapsto K_g(x, y)$  is  $C^\infty$ .

(b) The symmetry of the integral kernel  $K_g$  implies that

$$\int_M K_g(x, y) d\text{vol}_g(y) = 0 , \quad x \in M . \quad (39)$$

Indeed,  $K_g f \in \mathring{H}$  implies  $\int [\int K_g(x, y) d\text{vol}(x)] f(y) d\text{vol}(y) = 0$  for all  $f \in \mathring{H}$  which in turn implies that  $\int K_g(x, y) d\text{vol}(x)$  is constant in  $y$ . By symmetry, this constant must vanish.

**Theorem 2.18.** *For every admissible manifold  $(M, g)$ , the co-polyharmonic Green kernel  $K_g$  satisfies*

$$\left| K_g(x, y) - a_n \log \frac{1}{d_g(x, y)} \right| \leq C_0 \quad (40)$$

for some  $C_0 = C_0(g)$  and  $a_n$  as in (32).

*Proof.* By the second resolvent identity for the operators  $K_g, \mathring{G}_{n/2}: \mathring{H} \rightarrow \mathring{H}$ ,

$$K_g - \mathring{G}_{n/2} = K_g S_0 \mathring{G}_{n/2} = K_g S_0 \mathring{G}_{\frac{n-1}{4}} \mathring{G}_{\frac{n+1}{4}} \quad (41)$$

with  $S_0 = P - (-\Delta_g)^{n/2}$  as in Lemma 2.15 (i). Similarly to the proof of Proposition 2.13, the operators  $\mathring{G}_{\frac{n+1}{4}}: \mathring{L}^1 \rightarrow \mathring{H}$  and  $\mathring{G}_{\frac{n-1}{4}}: \mathring{H} \rightarrow \mathring{H}^{-\frac{n-1}{2}}$  are bounded. By Theorem 1.3 (iii),  $S_0$  is a differential operator of order at most  $n-1$  with smooth (hence bounded) coefficients. As a consequence,  $S_0: \mathring{H}^{-\frac{n-1}{2}} \rightarrow \mathring{H}^{-\frac{n-1}{2}}$  is a bounded operator. Furthermore, choosing  $r = -\frac{n-1}{2}$  in Lemma 2.15 (ii), the operator  $K_g: \mathring{H}^{-\frac{n-1}{2}} \rightarrow \mathring{H}^{-\frac{n-1}{2}}$  is bounded.

Combining the previous assertions with (41) shows that  $K_g - \mathring{G}_{n/2}: \mathring{L}^1 \rightarrow \mathring{H}^{-\frac{n+1}{2}}$  is bounded, thus  $K_g - \mathring{G}_{n/2}: \mathring{L}^1 \rightarrow L^\infty$  is bounded as well, by continuity of the Sobolev–Morrey embedding. Finally, by [DP40, Thm. 2.2.5], the latter operator admits a bounded integral kernel, and the conclusion follows from Proposition 2.13.  $\square$

The previous theorem has also been derived (with completely different arguments) in [Ndi07, Lemma 2.1], a reference which we had not been aware of before publishing a first draft of this paper.

**Proposition 2.19.** *Assume that  $(M, g)$  is admissible and that  $g' := e^{2\varphi} g$  for some  $\varphi \in C^\infty(M)$ . Then, the co-polyharmonic Green operator  $K_{g'}$  is given by*

$$K_{g'} u = (\pi_{g'} \circ K_g)(e^{n\varphi} \pi_{g'}(u)) , \quad u \in L^2 . \quad (42)$$

and the co-polyharmonic Green kernel  $K_{g'}$  by

$$K_{g'}(x, y) = K_g(x, y) - \frac{1}{2} \bar{\phi}(x) - \frac{1}{2} \bar{\phi}(y) \quad (43)$$

with  $\bar{\phi} \in \mathcal{D}$  defined by

$$\bar{\phi} := \frac{2}{\text{vol}_{g'}(M)} \int K_g(\cdot, z) d\text{vol}_{g'}(z) - \frac{1}{\text{vol}_{g'}(M)^2} \iint K_g(z, w) d\text{vol}_{g'}(z) d\text{vol}_{g'}(w) .$$

*Proof.* Let  $K_{g'}$  be the integral kernel defined by the right-hand side of (43). Obviously,  $K_{g'}$  is symmetric. Furthermore, by (40)

$$\begin{aligned} |K_{g'}(x, y) - K_g(x, y)| &\leq \frac{3}{\text{vol}_{g'}(M)} \sup_w \int |K(z, w)| d\text{vol}_{g'}(z) \\ &\leq 3C \text{vol}_{g'}(M) + \frac{3a_n}{\text{vol}_{g'}(M)} \sup_w \int \left| \log \frac{1}{d_g(z, w)} \right| d\text{vol}_{g'}(z) < \infty, \end{aligned} \quad (44)$$

and, since  $e^{\inf \varphi} d_g \leq d_{g'} \leq e^{\sup \varphi} d_g$ ,

$$\left| \log \frac{1}{d_g(x, y)} - \log \frac{1}{d_{g'}(x, y)} \right| \leq C_{\varphi, g}. \quad (45)$$

Thus the kernel  $K_{g'}$  satisfies (40) with  $K_{g'}$  in place of  $K_g$  and  $g'$  in place of  $g$  for some constant  $C_0(g')$ .

Moreover, straightforward calculation yields the identity (42) for the integral operator  $\mathbf{K}_{g'}$  associated with the kernel  $K_{g'}$ . It remains to prove that the operator  $\mathbf{K}_{g'}$  is the inverse of  $\mathbf{P}_{g'}$ . To see this, recall that we have  $\mathbf{P}_{g'} = e^{-n\phi} \mathbf{P}_g$ . Thus for all  $u \in \dot{H}^{n/2}(\text{vol}_{g'})$ ,

$$\mathbf{K}_{g'} \mathbf{P}_{g'} u = \mathbf{K}_g \mathbf{P}_g u - \langle \mathbf{K}_g \mathbf{P}_g u \rangle_{g'} = u - \langle u \rangle_g - \langle u - \langle u \rangle_g \rangle_{g'} = u.$$

Consequently we have  $\mathbf{K}_{g'} \mathbf{P}_{g'} u = u = \mathbf{P}_{g'} \mathbf{K}_{g'} u$  and the claim follows by uniqueness of the inverse.  $\square$

*Remark 2.20.* The transformation formula (43) for the co-polyharmonic Green kernels can be re-phrased as follows. Given  $\varphi \in C^\infty(M)$ , let  $\varphi_0 := \varphi - c$  with  $c$  chosen such that  $\int e^{n\varphi_0} d\text{vol} = 1$ . Then,

$$k_{e^{2\varphi}g}(x, y) = k_g(x, y) - \mathbf{K}_g(e^{n\varphi_0})(x) - \mathbf{K}_g(e^{n\varphi_0})(y) + \langle \mathbf{K}_g(e^{n\varphi_0}), e^{n\varphi_0} \rangle_{L^2(\text{vol}_g)}. \quad (46)$$

### 3 The co-polyharmonic Gaussian field

In what follows, we consider an admissible manifold  $(M, g)$  of even dimension  $n$ . We make use of the normalized operator  $\mathbf{k}_g := \frac{1}{a_n} \mathbf{K}_g$  with  $a_n$  from (32); its associated integral kernel is

$$k_g(x, y) := \frac{1}{a_n} K_g(x, y), \quad \forall x, y \in M.$$

By construction,  $k_g$  is a symmetric integral kernel which annihilates constants and by (40) has precise logarithmic divergence

$$\left| k_g(x, y) - \log \frac{1}{d_g(x, y)} \right| \leq C, \quad \forall x, y \in M. \quad (47)$$

We define the bilinear form  $\mathfrak{k}_g$  on  $L^2(M, \text{vol}_g)$ :

$$\mathfrak{k}_g(u, v) := \langle \mathbf{k}_g u | v \rangle_{L^2} = \iint u(x) k_g(x, y) v(y) d\text{vol}_g(x) d\text{vol}_g(y). \quad (48)$$

Observe that  $\mathfrak{k}_g(u + C, v + C') = \mathfrak{k}_g(u, v)$  for  $u$  and  $v \in L^2(M, \text{vol}_g)$ , and  $C$  and  $C' \in \mathbb{R}$ . According to (42), we also have that, for every  $g' = e^{2\varphi}g$  with  $\varphi \in C^\infty(M)$ :

$$\mathfrak{k}_{g'}(u, v) = \mathfrak{k}_g(e^{n\varphi} \pi_{g'}(u), e^{n\varphi} \pi_{g'}(v)) \quad \forall u, v \in L^2. \quad (49)$$



### 3.1 Existence and uniqueness, equivalent characterizations

We define the co-polyharmonic Gaussian field on  $(M, g)$  in a similar fashion as we did for fractional Gaussian fields in [DKS20]. The key role is played by a probability measure on  $\mathfrak{D}'$ , heuristically characterized as

$$d\nu(h) = \frac{1}{Z_g} \exp\left(-\frac{a_n}{2} \mathfrak{p}_g(h, h)\right) dh \quad (50)$$

where  $dh$  stands for the (non-existing) uniform distribution on  $\mathfrak{D}'$  and  $Z_g$  denotes some normalization constant.

**Theorem 3.1.** *For every admissible manifold  $(M, g)$  there exists a unique probability measure  $\nu$  on  $\mathfrak{D}'$ , called law of the co-polyharmonic Gaussian field and denoted by  $\text{CGF}^{(M, g)}$ , that satisfies*

$$\int_{\mathfrak{D}'} e^{i\langle h | u \rangle} d\nu(h) = \exp\left[-\frac{1}{2} \mathfrak{k}_g(u, u)\right] \quad \forall u \in \mathfrak{D}. \quad (51)$$

Equivalently,  $\text{CGF}^{(M, g)}$  can be characterized as the unique centered Gaussian probability measure  $\nu$  on  $\mathfrak{D}'$  that satisfies

$$\int_{\mathfrak{D}'} \langle h | u \rangle^2 d\nu(h) = \mathfrak{k}_g(u, u) \quad \forall u \in \mathfrak{D}. \quad (52)$$

*Proof.* Set  $\chi(u) := \exp\left[-\frac{1}{2} \mathfrak{k}_g(u, u)\right]$ . It satisfies  $\chi(0) = 1$ . Moreover, since  $M$  is admissible,  $\mathfrak{k}_g$  is a semi-definite inner product on  $\mathfrak{D}$ , thus, by, e.g., [LSSW16, Prop. 2.4],  $\chi$  is totally positive definite. By Lemma 2.15(ii),  $u \mapsto \sqrt{\mathfrak{k}_g(u, u)}$  is continuous with respect to the  $H^{-n/2}$ -norm on  $\mathfrak{D}$ . Since  $\mathfrak{D}$  embeds continuously into  $H^{-s}$  for every  $s \in \mathbb{R}$ , the functional  $\chi$  is continuous on  $\mathfrak{D}$ . The claim follows by the Bochner–Minlos Theorem [VTC87, §IV.4.3, Thm. 4.3, p. 410].  $\square$

**Definition 3.2.** *A measurable map  $h^\bullet : \Omega \rightarrow \mathfrak{D}'$ ,  $\omega \mapsto h^\omega$ , defined on some probability space  $(\Omega, \mathfrak{F}, \mathbf{P})$ , is called co-polyharmonic Gaussian field on  $(M, g)$  if it is distributed according to  $\text{CGF}^{(M, g)}$ .*

Here and henceforth, we use the notation  $h^\bullet$  if we want to emphasize the dependency on some underlying ‘random’ parameter  $\omega$ . Often, however, we simply write  $h$  instead, and then do not distinguish in notation between  $h$  distributed according to  $d\text{CGF}(h)$  and  $h^\omega$  distributed according to  $d\mathbf{P}(\omega)$ .

*Remark 3.3.* (a) In view of (52), the mapping  $\langle h | \cdot \rangle : \mathfrak{D} \rightarrow L^2(\nu)$  can be extended to a linear isometry  $\dot{H}^{-n/2} \rightarrow L^2(\nu)$ .

(b) Occasionally, with slight abuse of notation, we assume that  $\mathbf{P}$  is a probability measure on  $\Omega = \mathfrak{D}'$  and we regard  $h \mapsto \langle h | u \rangle$  for  $u \in \mathfrak{D}$  as a family of centered Gaussian random variables.

(c) Our definition implies that a co-polyharmonic Gaussian field  $h$  is *grounded*, in the sense that  $\langle h | c \rangle = 0$  for all constant  $c$ .

*Remark 3.4.* In view of Weyl’s asymptotic for eigenvalues of  $\mathbf{P}_g$  in Lemma 2.15(viii), we can argue as in [She07, Proposition 2.1] and show that we can realize the co-polyharmonic Gaussian field via the abstract Wiener space approach of [Gro67]. In that case the Cameron–Martin space is  $\dot{H}^{n/2}$  and the abstract Wiener space can be chosen to be  $\dot{H}^{-\varepsilon}$  for any  $\varepsilon > 0$ . See also Theorem 3.9.

*Remark 3.5.* A  $\mathfrak{D}'$ -valued centered Gaussian random field  $h$  is a co-polyharmonic Gaussian field on  $(M, g)$  if and only if  $\xi := \sqrt{a_n \mathbf{P}_g} h$  is a *grounded white noise* on  $(M, g)$ , i.e., a  $\mathfrak{D}'$ -valued centered Gaussian random distribution with covariance

$$\mathbf{E}[\langle \xi | u \rangle \langle \xi | v \rangle] = \langle \pi_g u | \pi_g v \rangle_{\dot{L}^2} \quad \forall u, v \in \mathfrak{D}. \quad (53)$$

Vice versa, given any grounded white noise  $\xi$  on  $(M, g)$ , then  $h := \sqrt{k_g} \xi$  is a co-polyharmonic Gaussian field on  $(M, g)$ .

The heuristic characterization (50) of the measure  $\text{CGF}^{(M,g)}$  manifests itself in various important properties. As every Gaussian measure,  $\text{CGF}^{(M,g)}$  satisfies a *large deviation principle* whose rate function is given by the Cameron–Martin norm [Aze80, Chap. II, Prop. 1.5 and Thm. 1.6]. In our case, this yields

**Proposition 3.6.** *For every co-polyharmonic field  $h$ , and for every Borel set  $A \subset \mathfrak{D}'$ :*

$$\begin{aligned} - \inf_{u \in A^0} a_n \mathfrak{p}_g(u) &\leq \liminf_{\beta \rightarrow 0} 2\beta^2 \mathbf{P}[\beta h \in A] \\ &\leq \limsup_{\beta \rightarrow 0} 2\beta^2 \mathbf{P}[\beta h \in A] \leq - \inf_{u \in \bar{A}} a_n \mathfrak{p}_g(u). \end{aligned}$$

Here  $A^0$  and  $\bar{A}$  respectively denote the interior and the closure of  $A$  in the topology of  $\mathfrak{D}'$ , and  $\mathfrak{p}$  is defined in (38) and we set  $\mathfrak{p}(u) = \infty$  if  $u \notin H^{n/2}$ .

Next we recall the celebrated *change of variable formula of Girsanov type*, also known as Cameron–Martin theorem, see, for instance [Jan97, Theorem 14.1].

**Proposition 3.7.** *If  $\varphi \in \mathfrak{D}$  and  $h \sim \text{CGF}^{(M,g)}$ , then  $h + \pi_g(\varphi) \text{vol}_g$  is distributed according to*

$$\exp \left( a_n \langle h | \mathbf{P}_g \varphi \rangle - \frac{a_n}{2} \mathfrak{p}_g(\varphi, \varphi) \right) d\text{CGF}^{(M,g)}(h).$$

*Remark 3.8.* Many of our subsequent results rely on the seminal work of J.-P. Kahane [Kah85] on Gaussian multiplicative chaos. His results apply to Gaussian random fields  $\tilde{h}$  on a metric space  $(M, d)$  with covariance kernel  $\tilde{k}$  with a logarithmic divergence:  $|\tilde{k}(x, y) + \log d(x, y)| \leq C$ . In addition to non-negative definiteness, he assumes that  $\tilde{k}$  is non-negative. Of course, this is not satisfied by our kernel  $k_g$ . However, as we are going to explain now, it imposes no serious obstacle to applying his results in our setting.

Given the kernel  $k_g$  as defined above, observe that it is smooth outside the diagonal and positive in the neighborhood of the diagonal. Define a new kernel by

$$\bar{k}(x, y) := k_g(x, y) + C \geq 0$$

with  $C := -\min_{x, y \in M} k_g(x, y) < \infty$ . By construction  $\bar{k}$  is non-negative. Furthermore, it is also non-negative definite since it is the covariance kernel for the Gaussian field

$$\bar{h} := h + \sqrt{C} \xi \cdot \text{vol}_g$$

where  $h$  denotes the co-polyharmonic Gaussian field associated with  $k_g$ , and  $\xi$  denotes a standard Gaussian variable independent of  $h$ .

## 3.2 Approximations

As anticipated, our goal is to construct the random measure  $d\mu(x) = e^{h(x)} d\text{vol}_g(x)$ . Due to the non-smooth nature of  $h$  this requires approximating  $h$  by smooth fields (and properly renormalizing). Co-polyharmonic Gaussian Fields may be approximated in various ways, the random measure obtained being essentially independent on the choice of the approximation [Sha16]. Here, we present a number of different approximations: through their expansion in terms of eigenfunctions of the co-polyharmonic operator  $\mathbf{P}_g$ ; by convolution with (smooth or non-smooth) functions; by a discretization procedure.

Let us first discuss the eigenfunctions approximation. As before, we denote by  $(\psi_j)_{j \in \mathbb{N}_0}$  the complete  $L^2$ -orthonormal system consisting of eigenfunctions of  $\mathbf{P}_g$ , each with corresponding eigenvalue  $\nu_j$ . In addition, we consider a sequence  $(\xi_j)_{j \in \mathbb{N}_1}$  of independent and identically distributed standard Gaussian variables. For each  $\ell \in \mathbb{N}_0$ , we define the random test function  $h_\ell$  by

$$h_\ell(x) := \sum_{j=1}^{\ell} \frac{\psi_j(x) \xi_j}{\sqrt{a_n \nu_j}}, \quad x \in M. \quad (54)$$

The covariance of the random field  $h_\ell$  is given by:

$$k_\ell(x, y) := \mathbf{E} \left[ h_\ell(x) h_\ell(y) \right] = \sum_{j=1}^{\ell} \frac{\psi_j(x) \psi_j(y)}{a_n \nu_j}, \quad x, y \in M. \quad (55)$$

Our next result establishes that the random field  $h_\ell$  converges to the random field  $h$ .

**Proposition 3.9.** *Let  $(M, g)$  be admissible and  $(h_\ell)_{\ell \in \mathbb{N}}$  defined as above. Then:*

- (i) *for all  $\varepsilon > 0$ , the field  $h_\ell \text{vol}_g$ , regarded as a random element of  $\dot{H}^{-\varepsilon}$ , converges as  $\ell \rightarrow \infty$  to a co-polyharmonic Gaussian field  $h$  in  $L^2(\mathbf{P})$  and  $\mathbf{P}$ -a.s. In particular,  $h \in \dot{H}^{-\varepsilon}$   $\mathbf{P}$ -a.s.;*
- (ii) *for every  $u \in \dot{H}^{-n/2}$ , the sequence  $(\langle h_\ell | u \rangle)_{\ell \in \mathbb{N}}$  is a centered,  $L^2$ -bounded martingale on  $(\Omega, \mathfrak{F}, \mathbf{P})$  converging to  $\langle h | u \rangle$   $\mathbf{P}$ -a.s. and in  $L^2(\mathbf{P})$  as  $\ell \rightarrow \infty$ . Furthermore, we have that  $\mathbf{P}$ -a.s., cf. Remark 3.3 (a),*

$$\langle h | u \rangle = \sum_{j=1}^{\infty} \frac{\langle u | \psi_j \rangle}{\sqrt{a_n \nu_j}} \xi_j.$$

*Proof.* The proof follows from the abstract construction of [Gro67] and Remark 3.4. For completeness, we outline a simple proof in our setting.

(i) Let  $\ell$  and  $p \in \mathbb{N}$ , and  $\varepsilon > 0$ . According to Lemma 2.15 (iv), we have that,  $\mathbf{P}$ -almost surely

$$\left\| \sum_{j=\ell+1}^p \frac{\psi_j \xi_j}{\sqrt{\nu_j}} \right\|_{\dot{H}^{-\varepsilon}}^2 \simeq \sum_{j=\ell+1}^p \frac{\xi_j^2}{\nu_j^{1+2\varepsilon/n}} \simeq \sum_{j=\ell+1}^p \frac{\xi_j^2}{j^{1+2\varepsilon/n}}.$$

The sum on the right-hand side is a generalized chi-square random variable with variance  $\sum_{j=\ell+1}^p j^{-2\varepsilon/n-1}$ . It converges as  $p \rightarrow \infty$  if and only if  $\varepsilon > 0$ . This shows that the series

$$h^\bullet := \frac{1}{a_n} \sum_{j=1}^{\infty} \frac{\psi_j \xi_j^\bullet}{\sqrt{\nu_j}}, \quad (56)$$

exists  $\mathbf{P}$ -almost surely in  $\dot{H}^{-\varepsilon}$ . The proof of the convergence in  $L^2(\mathbf{P})$  is carried out in the same way. Since  $h$  is an  $L^2(\mathbf{P})$ -limit of Gaussian fields, it is itself Gaussian. For  $u, v \in \mathfrak{D}$  its covariance is given by

$$\mathbf{E}[\langle h | u \rangle \langle h | v \rangle] = \frac{1}{a_n} \sum_{j=1}^{\infty} \frac{\langle \psi_j | u \rangle_{L^2} \langle \psi_j | v \rangle_{L^2}}{\nu_j} = \mathfrak{k}(u, v). \quad (57)$$

(ii) For all  $u \in \dot{H}^{-n/2}$ , the sequence  $(\langle u | h_\ell \rangle)_{\ell \in \mathbb{N}}$  is a martingale as a sum of independent and identically distributed random variables. Moreover, by orthogonality, for all  $\ell \in \mathbb{N}$ ,

$$\mathbf{E}[\langle u | h_\ell \rangle^2] \leq \mathbf{E}[\langle h | u \rangle^2] = \sum_{j=1}^{\infty} \frac{\langle u | \psi_j \rangle^2}{a_n \nu_j} = \mathfrak{k}(u, u) \leq C \|u\|_{\dot{H}^{-n/2}}^2 < \infty.$$

We used Lemma 2.15 (iv) for the second inequality. Thus, the martingale is  $L^2(\mathbf{P})$ -bounded. The convergence follows from Doob's Martingale Convergence Theorem.  $\square$

The previous result allows us to construct a co-polyharmonic Gaussian field on every probability space that supports a sequence of independent and identically distributed standard normal variables. It is also important to know that an approximation  $h_\ell \rightarrow h$  as in the previous proposition holds for every co-polyharmonic Gaussian field, independently of the construction of the latter.

*Remark 3.10.* Given any co-polyharmonic Gaussian field  $h$ , and the sequence of eigenfunctions  $(\psi_j)_{j \in \mathbb{N}_0}$  as above, define a sequence  $(\xi_j)_{j \in \mathbb{N}}$  of independent and identically distributed standard normal variables by setting  $\xi_j := \langle h | \psi_j \rangle$  for all  $j \in \mathbb{N}$ , and a sequence of Gaussian random fields  $(h_\ell)_{\ell \in \mathbb{N}}$  by

$$h_\ell : \Omega \longrightarrow L^2(M, \text{vol}_g) , \quad h_\ell^\omega(x) := \sum_{j=1}^{\ell} \frac{\psi_j(x)}{\sqrt{a_n} \nu_j} \langle h^\omega | \psi_j \rangle . \quad (58)$$

Then, for every  $u \in \mathfrak{D}$ , as  $\ell \rightarrow \infty$ ,

$$\langle h_\ell | u \rangle \longmapsto \langle h | u \rangle \quad \mathbf{P}\text{-a.s. and in } L^2(\mathbf{P}) .$$

Now, let us consider more general approximations. The previous eigenfunction approximation will appear as a particular case.

**Proposition 3.11.** *For each  $\ell \in \mathbb{N}$  let  $q_\ell \in L^2(\text{vol}_g \otimes \text{vol}_g)$  be such that  $q_\ell u \rightarrow u$  in  $L^2$  for all  $u \in L^2$ , where*

$$\mathbf{q}_\ell u(x) := \langle q_\ell(x, \cdot) | u \rangle_{L^2} .$$

(i) *Then, for every  $\ell \in \mathbb{N}$ , the field of functions  $h_\ell$  on  $M$  defined by*

$$h_\ell(y) = (\mathbf{q}_\ell^* h)(y) := \langle h | q_\ell(\cdot, y) \rangle \quad (59)$$

*is a centered Gaussian field with covariance function*

$$k_\ell(x, y) = ((\mathbf{q}_\ell \otimes \mathbf{q}_\ell)k)(x, y) := \iint k(x', y') q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(y') d\text{vol}_g(x') . \quad (60)$$

(ii) *As  $\ell \rightarrow \infty$ , for every  $u \in L^2$ ,*

$$\langle h_\ell | u \rangle \longmapsto \langle h | u \rangle \quad \mathbf{P}\text{-a.s. and in } L^2(\mathbf{P}) . \quad (61)$$

*Proof.* (i) is obvious. To see (ii), observe that  $\langle h_\ell | u \rangle_{L^2} = \langle h | \mathbf{q}_\ell u \rangle$  and thus by (52) and Lemma 2.15 for every  $u \in L^2$ ,

$$\begin{aligned} \mathbf{E} \left[ \left| \langle h | u \rangle - \langle \mathbf{q}_\ell | u \rangle_{L^2} \right|^2 \right] &= \mathbf{E} \left[ \left| \langle h | u - \mathbf{q}_\ell u \rangle \right|^2 \right] \simeq \|u - \mathbf{q}_\ell u\|_{\dot{H}^{-n/2}}^2 \\ &\leq \|u - \mathbf{q}_\ell u\|_{L^2}^2 \xrightarrow{\ell \rightarrow \infty} 0 . \quad \square \end{aligned}$$

*Example 3.12. (i) Probability kernels.* Let  $\{q_\ell(x, \cdot) \text{vol}_g : \ell \in \mathbb{N}, x \in M\}$  be a family of probability measures on  $M$  with  $q_\ell \in L^\infty(\text{vol}_g \otimes \text{vol}_g)$  non-negative, and such that  $q_\ell(x, \cdot) \text{vol}_g$  converges weakly to  $\delta_x$  as  $\ell \rightarrow \infty$  for each  $x \in M$ . Then  $\mathbf{q}_\ell u \rightarrow u$  in  $L^2$  as  $\ell \rightarrow \infty$  for all  $u \in L^2$ .

Particular cases of (i) are (ii) and (iii) below.

(ii) *Discretization.* Let  $(\mathfrak{P}_\ell)_{\ell \in \mathbb{N}}$  be a family of Borel partitions of  $M$  with  $\sup\{\text{diam}_g(A) : A \in \mathfrak{P}_\ell\} \rightarrow 0$  as  $\ell \rightarrow \infty$ . For  $\ell \in \mathbb{N}$  put

$$q_\ell(x, y) := \sum_{A \in \mathfrak{P}_\ell} \frac{1}{\text{vol}_g(A)} \mathbf{1}_A(x) \mathbf{1}_A(y) .$$

In other words, for given  $x \in M$  we have that  $q_\ell(x, \cdot) = \frac{1}{\text{vol}_g(A)} \mathbf{1}_A$  with the unique  $A \in \mathfrak{P}_\ell$  which contains  $x$ . Letting  $h_\ell$  be defined as in Proposition 3.11 then yields

$$h_\ell(x) = \frac{1}{\text{vol}_g(A)} \langle h | \mathbf{1}_A \rangle \quad A \in \mathfrak{P}_\ell , x \in A .$$

This is a centered Gaussian random field  $(h_\ell(x))_{x \in M}$  with covariance function

$$k_\ell(x, y) = \frac{1}{\text{vol}_g(A_\ell^x) \text{vol}_g(A_\ell^y)} \int_{A_\ell^x} \int_{A_\ell^y} k(x', y') d\text{vol}_g(x') d\text{vol}_g(y') ,$$

where  $A_\ell^x$  is the unique element of  $\mathfrak{P}_\ell$  containing  $x$ .

(iii) *Heat kernel approximation.* Let  $q_\ell(x, y) := p_{1/\ell}(x, y)$  be defined in terms of the heat kernel on  $M$ . Then  $\mathbf{q}_\ell u \rightarrow u$  in  $L^2$  and thus in particular (61) holds for all  $u \in L^2$ . Even more, (61) holds for all  $u \in \dot{H}^{-n/2}$ .

(iv) *Eigenfunctions approximation.* In terms of the eigenfunctions for the co-polyharmonic operator  $\mathbf{P}_g$  we define

$$q_\ell(x, y) := \sum_{j=0}^{\ell} \psi_j(x) \psi_j(y) .$$

In other words,  $\mathbf{q}_\ell : L^2 \rightarrow L^2$  is the projection onto the linear span of the first  $1 + \ell$  eigenfunctions. Then  $\mathbf{q}_\ell u \rightarrow u$  in  $L^2$  as  $\ell \rightarrow \infty$  for all  $u \in L^2$ .

*Proof.* (i) Since  $\mathcal{C}_b(M)$  is dense in  $L^2(X)$  and since by Jensen's inequality  $\|\mathbf{q}_\ell u - \mathbf{q}_\ell v\|_{L^2} \leq \|u - v\|_{L^2}$ , it suffices to prove that  $\mathbf{q}_\ell u \rightarrow u$  in  $L^2$  as  $\ell \rightarrow \infty$  for  $u \in \mathcal{C}_b(M)$ . To see the latter, observe that  $\mathbf{q}_\ell u(x) \rightarrow u(x)$  for each  $x$  by weak convergence of  $q_\ell(x, \cdot) \text{vol}_g$  to  $\delta_x$ , and that  $\|\mathbf{q}_\ell u\|_{L^\infty} \leq \|u\|_{L^\infty} < \infty$ .

(ii) is straightforward.

(iii) If  $q_\ell = p_{1/\ell}$  and  $u \in \dot{H}^{-n/2}$  we have with  $v := \mathring{\mathbf{G}}^{n/4} u \in L^2$ ,

$$\|u - \mathbf{q}_\ell u\|_{\dot{H}^{-n/2}} \lesssim \|v - \mathbf{q}_\ell v\|_{L^2} \xrightarrow{\ell \rightarrow \infty} 0 .$$

(iv) Readily follows from the fact that  $(\psi_j)_{j \in \mathbb{N}_0}$  is a complete  $L^2$ -orthonormal system, Lemma 2.15 (v).  $\square$

### 3.3 Conformal quasi-invariance

**Theorem 3.13.** *Consider an admissible Riemannian manifold  $(M, g)$  and  $g' = e^{2\varphi} g$  with  $\varphi \in \mathcal{C}^\infty(M)$ . If  $h$  is distributed according to  $\text{CGF}^{(M, g)}$  then*

$$h' := \pi_{g'}^*(e^{n\varphi} h) = e^{n\varphi} h - \frac{\langle h | e^{n\varphi} \rangle}{\text{vol}_{g'}(M)} \text{vol}_{g'} \quad (62)$$

is distributed according to  $\text{CGF}^{(M, g')}$ . Here,  $\pi_{g'}^*$  denotes the dual for the grounding operator  $\pi_{g'}$ , and the distribution  $\pi_{g'}^*(e^{n\varphi} h) \in \mathfrak{D}'$  is defined through its action  $u \mapsto \langle h | e^{n\varphi} \pi_{g'} u \rangle$  on  $\mathfrak{D}$ .

*Proof.* Recall that  $\mathbf{E}[\langle h | u \rangle \cdot \langle h | v \rangle] = \mathfrak{k}_g(u, v)$  for all  $u, v \in \mathfrak{D}$ . Thus, with  $h'$  as defined above and for  $u, v \in \mathfrak{D}$ ,

$$\begin{aligned} \mathbf{E}[\langle h' | u \rangle \cdot \langle h' | v \rangle] &= \mathbf{E}[\langle h | e^{n\varphi} \pi_{g'} u \rangle \cdot \langle h | e^{n\varphi} \pi_{g'} v \rangle] = \mathfrak{k}_g(e^{n\varphi} \pi_{g'} u, e^{n\varphi} \pi_{g'} v) \\ &= \mathfrak{k}_{g'}(u, v) \end{aligned}$$

according to (49).  $\square$

The conformal quasi-invariance of the CGF indeed holds true in a more general form. Assume that  $(M, g)$  and  $(M', g')$  are conformally equivalent with diffeomorphism  $\Phi$  and conformal weight  $e^{2\varphi}$  such that  $\Phi^* g' = e^{2\varphi} g$ . Furthermore assume that  $h$  is distributed according to  $\text{CGF}^{(M, g)}$  and  $h'$  is distributed according to  $\text{CGF}^{(M', g')}$ . Then,

$$h' \stackrel{(d)}{=} \Phi_* \left( \pi_{g'}^*(e^{n\varphi} h) \right) . \quad (63)$$

**Corollary 3.14.** *On each class of conformally equivalent admissible  $n$ -dimensional compact Riemannian manifolds,  $\text{CGF}^{(M, g)}$  defines a conformally quasi-invariant random field.*

*Remark 3.15.* (i) In the above transformation formulas, as mostly in this paper, we regard a co-polyharmonic field  $h \sim \text{CGF}^{(M, g)}$  as a map  $h: \Omega \rightarrow \mathfrak{D}'$ . As already noted in Remark 3.4, we also can regard it as a map  $h: \Omega \rightarrow H^{-s}(M)$  for any  $s > 0$ . Since the latter depends on the choice of the representative  $g \in [g]$ , one better denotes it by  $h_g$ .

Such a formalism in particular is used in [GRV19]. This allows one to get rid of the multiplicative correction in (62) so that

$$h'_{g'} = h_g - \frac{1}{\text{vol}_{g'}(M)} \langle h_g | \mathbf{1} \rangle_{H^{-s}(M, g'), H^s(M, g')} .$$

(ii) To get rid of the additive correction term in (62), one can consider the ‘random variable’  $h + a \text{vol}_g$ , called *ungrounded co-polyharmonic Gaussian field*, where  $h$  is distributed according to  $\text{CGF}^{M, g}$  and where  $a$  is a constant informally distributed according to the Lebesgue measure on the line (the latter not being a probability measure).

More formally, given any admissible manifold  $(M, g)$ , the distribution of the corresponding co-polyharmonic Gaussian field is a probability measure  $\nu_g$  on the space of distributions  $\mathfrak{D}'$  on  $M$ . To override the influence of additive constants, we consider the (non-finite) measure  $\widehat{\nu}_g$  on  $\mathfrak{D}'$  defined as the image measure of  $\nu_g \otimes \mathfrak{L}^1$  under the map

$$(h, a) \mapsto h + a \text{vol}_g .$$

**Definition 3.16.** *The measure  $\widehat{\nu}_g$  is called law of the ungrounded co-polyharmonic Gaussian field and denoted by  $\widehat{\text{CGF}}^{M, g}$ .*

We write  $h \sim \widehat{\text{CGF}}^{M, g}$  to indicate that a measurable map  $h : \Omega \rightarrow \mathfrak{D}'$ , defined on some measure space  $(\Omega, \mathfrak{F}, \mathbf{m})$ , is distributed according to  $\widehat{\text{CGF}}^{M, g}$ , i.e.,  $h_* \mathbf{m} = \widehat{\text{CGF}}^{M, g}$ .

The conformal quasi-invariance of the probability measures  $\text{CGF}^{M, g}$  leads to an analogous but simpler quasi-invariance of the measures  $\widehat{\text{CGF}}^{M, g}$ .

**Proposition 3.17.** *If  $h \sim \widehat{\text{CGF}}^{M, g}$  and  $h' \sim \widehat{\text{CGF}}^{M, g'}$  with  $g' = e^{2\varphi} g$  then*

$$h' \stackrel{(d)}{=} e^{n\varphi} h .$$

*Proof.* Set  $\langle h \rangle_{g'} := \frac{\langle h | e^{n\varphi} \rangle}{\text{vol}_{g'}(M)}$ . Then, for all  $u \in \mathfrak{D}$ , by translation invariance of 1-dimensional Lebesgue measure,

$$\begin{aligned} \int_{\mathfrak{D}'} \langle h' | u \rangle^2 d\widehat{\nu}_{g'}(h') &= \int_{\mathfrak{D}'} \int_{\mathbb{R}} \langle h' + a \text{vol}_{g'} | u \rangle^2 da d\nu_{g'}(h') \\ &= \int_{\mathfrak{D}'} \int_{\mathbb{R}} \langle h - \langle h \rangle_{g'} \text{vol}_g + a \text{vol}_g | e^{n\varphi} u \rangle^2 da d\nu_g(h) \\ &= \int_{\mathfrak{D}'} \int_{\mathbb{R}} \langle h + a \text{vol}_g | e^{n\varphi} u \rangle^2 da d\nu_g(h) \\ &= \int_{\mathfrak{D}'} \langle e^{n\varphi} h | u \rangle^2 d\widehat{\nu}_g(h) . \end{aligned} \quad \square$$

Also the change of variable formula of Girsanov type of Proposition 3.7 takes on a simpler form: the projection onto the subspace of grounded functions is no longer needed.

**Corollary 3.18.** *If  $\varphi \in \mathfrak{D}$  and  $h \sim \widehat{\text{CGF}}^{(M, g)}$ , then  $h + \varphi \text{vol}_g$  is distributed according to*

$$\exp \left( a_n \langle h | P_g \varphi \rangle - \frac{a_n}{2} \mathfrak{p}_g(\varphi, \varphi) \right) d\widehat{\text{CGF}}^{(M, g)}(h) .$$

## 4 Liouville Quantum Gravity measure

Fix an admissible manifold  $(M, g)$  and a co-polyharmonic Gaussian field  $h : \Omega \rightarrow \mathfrak{D}'$ . Our naive goal is to study the ‘random geometry’  $(M, g_h)$  obtained by the random conformal transformation,

$$g_h = e^{2h} g ,$$

and in particular to study the associated ‘random volume measure’ given as

$$d\text{vol}_{g_h}(x) = e^{nh(x)} d\text{vol}_g(x) . \quad (64)$$

It easily can be seen that — due to the singular nature of the noise  $h$  — all approximating sequences of this measure diverge as long as no additional renormalization is built in.

A more tractable goal is to study (for suitable  $\gamma \in \mathbb{R}$ ) the random measure  $\mu^h$  formally given as

$$d\mu^h(x) = e^{\gamma h(x) - \frac{\gamma^2}{2} \mathbf{E}[h(x)^2]} d\text{vol}_g(x). \quad (65)$$

Since  $h$  is not a function but only a distribution, both (64) and (65) are ill-defined. However, replacing  $h$  by its finite-dimensional noise approximation  $h_\ell$  as constructed in Proposition 3.10, leads to a sequence  $(\mu^{h_\ell})_\ell$  of random measures on  $M$  which, as  $\ell \rightarrow \infty$ , almost surely, converges to a random measure  $\mu^h$  on  $M$ , the *Liouville Quantum Gravity measure* on the  $n$ -dimensional manifold  $M$ . Let  $\mathcal{M}_b(M)$  denote the set of finite positive Borel measures on  $M$ . We equip it with the Borel  $\sigma$ -algebra associated with its usual *weak topology*.

## 4.1 Gaussian multiplicative chaos

In the following Theorem, we construct the Gaussian multiplicative chaos  $\mu_{g,\gamma}^h$  associated to a co-polyharmonic Gaussian field  $h$  on  $(M, g)$ . For the sake of simplicity, we drop the subscripts  $\gamma$  and  $g$  from the notation whenever its specification is not relevant to the discussion. In view of Theorem 3.9, we can look at the co-polyharmonic Gaussian field  $h$  as a random element of  $H^{-\varepsilon}$  for some  $\varepsilon > 0$ .

**Theorem 4.1.** *Let an admissible manifold  $(M, g)$  and a real number  $\gamma$  with  $|\gamma| < \sqrt{2n}$  be given. Then, there exists a measurable map*

$$\mu: \mathring{H}^{-\varepsilon}(M) \rightarrow \mathcal{M}_b(M), \quad h \mapsto \mu^h,$$

with the following properties:

(i) for  $\mathbf{P}$ -a.e.  $h$  and every  $\varphi \in \mathring{H}^{n/2}(M)$ ,

$$\mu^{h+\varphi} = e^{\gamma \varphi} \mu^h. \quad (66)$$

(ii) for all Borel measurable  $f: \mathring{H}^{-\varepsilon}(M) \times M \rightarrow [0, \infty]$ , we have that

$$\mathbf{E} \int f(h, x) d\mu^h(x) = \mathbf{E} \int f(h + \gamma k(x, \cdot), x) d\text{vol}_g(x). \quad (67)$$

(iii) for all  $p \in (-\infty, \frac{2n}{\gamma^2})$ ,

$$\mathbf{E}[\mu^h(M)^p] < \infty.$$

*Remark 4.2.* (67) implies that  $\mathbf{E}[\mu^h] = \text{vol}_g$ .

**Definition 4.3.** *The random measure  $\mu^h = \mu_{g,\gamma}^h$  is called the plain Liouville Quantum Gravity measure on  $(M, g)$ .*

*Proof.* The result follows from general results regarding the theory of Gaussian multiplicative chaos by Kahane [Kah85] and Shamov [Sha16]. Shamov [Sha16] gives an axiomatic definition of Gaussian multiplicative chaos and shows that the limit measure is in fact independent of the choice of approximating sequence. In the language of [Sha16], our result follows from the existence of a *sub-critical Gaussian multiplicative chaos over the Gaussian field*  $h: \mathring{H}^{-n/2} \rightarrow L^2(\nu)$  and the operator  $k: \mathring{H}^{-n/2} \rightarrow \mathring{H}^{n/2} \subset L^0(\text{vol}_g)$ . In this case  $h$  is, almost surely, seen as a (non-continuous) linear form over  $\mathring{H}^{-n/2}$ . Properties (i) and (ii) being respectively [Sha16, Dfn. 11 (3)] and [Sha16, Thm. 4]. The moments estimates (iii) can be found in [Kah85, Thm. 4] for  $p > 0$  and [RV14, Thm. 2.12] for  $p < 0$ .

The existence of the Gaussian multiplicative chaos  $\mu$  — or, more precisely, the existence of the random variables  $\int_M u d\mu$  as a limit of uniformly integrable martingales — follows from an argument of [Kah85, Thm. 4 Variant 1]. This argument is stated in the slightly more restrictive setting of positive kernels. This restriction, however, does not harm in our case. Indeed, passing from  $h$  to  $\hat{h} := h + C\xi \text{vol}_g$  with some standard normal variable  $\xi$



independent of  $h$  will change  $k$  into  $\hat{k} + C^2$  which is eventually (for sufficiently large  $C$ ) a positive kernel. The corresponding random measures are then related to each other according to

$$\hat{\mu}_g^h = \exp\left(\gamma C\xi - \frac{\gamma^2}{2}C^2\right)\mu_g^h. \quad \square$$

*Remark 4.4.* Regarding uniform integrability and the existence of Liouville Quantum Gravity measure, the work [Ber17] provides an alternative approach based on the study of thick points of the underlying Gaussian fields.

## 4.2 Approximations

Let us recall the content of [Sha16, Thm. 25] specified to our setting.

**Lemma 4.5.** *Let  $q_\ell \in L^2(\text{vol}_g \otimes \text{vol}_g)$  be a family of kernels as in Proposition 3.11 and let  $(h_\ell(y))_{y \in M}$  be Gaussian fields as in (59) with covariance kernel  $k_\ell$  as in (60). Further set*

$$d\mu^{h_\ell}(x) := \exp\left(\gamma h_\ell(x) - \frac{\gamma^2}{2}k_\ell(x, x)\right) d\text{vol}_g(x). \quad (68)$$

Assume that

- (i) The family  $(\mu^{h_\ell}(M))_{\ell \in \mathbb{N}}$  is uniformly integrable;
- (ii) For all  $u \in \dot{H}^{n/2}$ ,  $q_\ell u \rightarrow u$  in  $L^0(\text{vol}_g)$ ;
- (iii)  $k_\ell \rightarrow k$  in  $L^0(\text{vol}_g \otimes \text{vol}_g)$ .

Then,  $\mu^{h_\ell} \rightarrow \mu^h$  weakly as Borel measures on  $M$  in  $\mathbf{P}$ -probability as  $\ell \rightarrow \infty$ . Even more, for every  $u \in L^1(\text{vol}_g)$ ,

$$\int_M u d\mu^{h_\ell} \rightarrow \int_M u d\mu^h \quad \text{in } L^1(\mathbf{P}) \quad \text{as } \ell \rightarrow \infty. \quad (69)$$

*Remark 4.6* (Transformation of Shamov's results to our setting). The basic objects in [Sha16] are generalized  $H$ -valued functions and generalized  $H$ -valued random fields. To apply these results to our setting, the Hilbert space  $(H, \langle \cdot | \cdot \rangle_H)$  there should be chosen as the space  $\dot{H}^{n/2}$  equipped with the scalar product  $a_n \mathfrak{p}(\cdot, \cdot) = a_n \langle \cdot | \mathbf{P}(\cdot) \rangle_{L^2}$ . Attention has to be paid to the fact that the pairing  $\langle \cdot | \cdot \rangle$  in [Sha16] is an extension of the scalar product in  $H$  whereas in our paper it is the natural pairing of distributions and test functions, extending the scalar product in  $L^2$ . To distinguish between the two, in the subsequent discussion a subscript  $H$  will always refer to the pairing in [Sha16].

The *generalized Gaussian field*  $(X, Y)$  in Shamov's notation is in our case the pair  $(h, \gamma k)$ . In particular,  $X := h$  is a *standard Gaussian random vector in  $H$*  and  $Y := \gamma k$  is a *generalized  $H$ -valued function*. Indeed, for all  $u \in H$ ,

$$\mathbf{E}\left[\langle X | u \rangle_H^2\right] = a_n^2 \mathbf{E}\left[\langle h | \mathbf{P}u \rangle^2\right] = a_n^2 \langle \mathbf{P}u | \mathbf{k} \mathbf{P}u \rangle_{L^2} = a_n \langle \mathbf{P}u | u \rangle_{L^2} = \|u\|_H^2.$$

Moreover,

$$\langle Y(x) | u \rangle_H = a_n \langle \gamma k(x, \cdot) | \mathbf{P}u \rangle_{L^2} = \gamma u(x)$$

and thus  $\|Y(x)\|_H^2 = \gamma^2 k(x, x)$ . In particular,  $X : u \mapsto \langle X | u \rangle_H = a_n \langle h | \mathbf{P}u \rangle$  and  $Y : u \mapsto \langle Y | u \rangle_H = \gamma u$  can be regarded as operators  $X : H \rightarrow L^2(\mathbf{P})$  and  $Y : H \rightarrow L^2(\text{vol}_g)$ . For  $q_\ell$  as in Lemma 4.5, we define the generalized  $H$ -valued function  $Y_\ell$  by

$$\begin{aligned} \langle Y_\ell(x) | u \rangle_H &:= \gamma q_\ell u(x) := \gamma \langle q_\ell(x, \cdot) | u \rangle_{L^2} \\ &= a_n \gamma \langle \mathbf{k} q_\ell(x, \cdot) | \mathbf{P}u \rangle_{L^2} = \gamma \langle \mathbf{k} q_\ell(x, \cdot) | u \rangle_H. \end{aligned}$$

Thus

$$\begin{aligned} \|Y_\ell(x)\|_H^2 &= \gamma^2 \|\mathbf{k} q_\ell(x, \cdot)\|_H^2 = \gamma^2 \langle q_\ell(x, \cdot) | \mathbf{k} q_\ell(x, \cdot) \rangle_{L^2} \\ &= \gamma^2 \iint q_\ell(x, y) k(y, z) q_\ell(x, z) d\text{vol}_g(y) d\text{vol}_g(z) = \gamma^2 k_\ell(x, x). \end{aligned}$$

The subcritical GMC over the Gaussian field  $(X, Y_\ell)$  then is given ([Sha16, Example 12]) by (68).

In practice, the criteria (ii) and (iii) of the previous lemma are easy to verify. The remaining challenge is the verification of (i).

**Lemma 4.7.** *Assume that for every  $\vartheta > 1$  there exists  $C \geq 0$ ,  $\ell_\vartheta \in \mathbb{N}$ , and a non-decreasing sequence  $(c_\ell)_{\ell \in \mathbb{N}}$  such that for all  $\ell \geq \ell_\vartheta$  and all  $x, y \in M$ ,*

$$k_\ell(x, y) \leq \vartheta \log \left( \frac{1}{d(x, y)} \wedge c_\ell \right) + C. \quad (70)$$

Then for every  $\gamma \in (0, \sqrt{2n})$ , the family  $(\mu^{\hbar_\ell}(M))_{\ell \in \mathbb{N}}$  as in (68) is uniformly integrable.

*Proof.* This follows from Kahane's comparison lemma [Kah85], cf. [Sha16, Thm.s 27, 28].  $\square$

In the rest of this section, let  $q_\ell$  be a family of probability kernels as in Example 3.12 (i). The lemmas above allow us to obtain the following crucial approximation results.

**Theorem 4.8.** *Let the kernels  $q_\ell$  be given in terms of a compactly supported, non-increasing function  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  as*

$$q_\ell(x, y) := \frac{1}{N_\ell(x)} \eta(\ell d(x, y)), \quad N_\ell(x) := \int_M \eta(\ell d(x, y)) d\text{vol}_g(y).$$

Then with  $\mu^{\hbar_\ell}$  defined as in (68),

$$\mu^{\hbar_\ell} \rightarrow \mu^{\hbar} \quad \text{as } \ell \rightarrow \infty$$

in the sense made precise in (69).

*Remark 4.9.* The assertion of the previous theorem holds as well for

$$q_\ell(x, y) := \frac{1}{N_\ell^*} \eta(\ell d(x, y))$$

with the 'Euclidean normalization'  $N_\ell^* := \ell^{-n} \int_{\mathbb{R}^n} \eta(|y|) d\mathcal{L}^n(y)$  in the place of the 'Riemannian normalization'  $N_\ell(x)$ .

*Proof.* Assume that  $\eta$  is supported in  $[0, R]$ . The verification of the criteria (ii) and (iii) in Lemma 4.5 is straightforward: (ii) was proven in Example 3.12 (i). (iii) follows from the fact that  $k(x, \cdot)$  is bounded and continuous outside of any  $\varepsilon$ -neighborhood of  $x$ , that  $q_\ell(x, \cdot)$  is supported in an  $R/\ell$ -neighborhood of  $y$ , and that  $q_\ell(x, \cdot) d\text{vol}_g \rightarrow \delta_x$  as  $\ell \rightarrow \infty$ . Thus for every  $x, y$  with  $d(x, y) \geq 2\varepsilon$  and every  $\ell \geq R/\varepsilon$ ,

$$k_\ell(x, y) = \int_{B_\varepsilon(y)} \int_{B_\varepsilon(x)} k_g(x', y') \rho_\ell(x', x) \rho_\ell(y', y) d\text{vol}_g(x') d\text{vol}_g(y') \xrightarrow{\ell \rightarrow \infty} k(x, y).$$

Our verification of the criterion (i) in Lemma 4.5 is based on Lemma 4.7, the verification of which will in turn be based on the following auxiliary results.

*Claim 4.10.* For all  $x, y \in M$  with  $d(x, y) \geq 3R/\ell$ ,

$$\iint \log \frac{1}{d(x', y')} q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(x') d\text{vol}_g(y') \leq \log \frac{1}{d(x, y)} + \log 3.$$

*Proof.* Combining the assumption  $d(x, y) \geq 3R/\ell$  and the facts that  $d(x, x') \leq R/\ell$  for all  $x'$  in the support of  $q_\ell(x, \cdot)$  and  $d(y, y') \leq R/\ell$  for all  $y'$  in the support of  $q_\ell(y, \cdot)$ , yields

$$d(x', y') \geq d(x, y) - d(x, x') - d(y, y') \geq d(x, y) - 2R/\ell \geq \frac{1}{3}d(x, y).$$

Thus, the claim readily follows.  $\square$

*Claim 4.11.* For every  $\vartheta > 1$  there exist  $\ell_\vartheta \in \mathbb{N}$  such that

$$\int \log \frac{1}{d(x, z)} \rho_\ell(y, z) d\text{vol}_g(z) \leq \frac{\vartheta}{N_\ell^*} \int_{\mathbb{R}^n} \log \frac{1}{|x' - z|} \eta(\ell |y' - z|) d\mathcal{L}^n(z) + \vartheta \log \vartheta$$

for all  $\ell \geq \ell_\vartheta$ , all  $x, y \in M$  with  $d(x, y) < 4R/\ell$ , and all  $x', y' \in \mathbb{R}^n$  with  $d(x, y) = |x' - y'|$ .

*Proof.* Denote by  $\text{inj}_g(M) > 0$  the injectivity radius of  $(M, g)$ . For every  $y \in M$  set  $y' := 0 \in \mathbb{R}^n$  and use the exponential map  $\exp_y : \mathbb{R}^n \rightarrow M$  to identify  $\varepsilon$ -neighborhoods of  $y \in M$  with  $\varepsilon$ -neighborhoods of  $y' \in \mathbb{R}^n$  for all  $\varepsilon \in (0, \text{inj}_g(M))$ . Since  $M$  is compact and smooth, for every  $\vartheta > 1$  there exists  $\varepsilon_\vartheta \in (0, \text{inj}_g(M))$  so small that  $\exp_y$  deforms both distances and volume elements in  $\varepsilon$ -neighborhoods of  $y$  by a factor less than  $\vartheta$ , for every  $y \in M$  and every  $\varepsilon \in (0, \varepsilon_\vartheta)$ . Choose  $\ell_\vartheta$  so that  $4R/\ell_\vartheta < \varepsilon_\vartheta$ . Thus,

$$\begin{aligned} \int \log \frac{1}{d(x, z)} q_\ell(y, z) d\text{vol}_g(z) &\leq \frac{\vartheta}{N_\ell^*} \int_{\mathbb{R}^n} \log \frac{\vartheta}{|x' - z|} \eta(\ell |y' - z|) d\mathcal{L}^n(z) \\ &= \frac{\vartheta}{N_\ell^*} \int_{\mathbb{R}^n} \log \frac{1}{|x' - z|} \eta(\ell |y' - z|) d\mathcal{L}^n(z) + \vartheta \log \vartheta. \quad \square \end{aligned}$$

*Claim 4.12.* For all  $x, y \in \mathbb{R}^n$ ,

$$\int_{\mathbb{R}^n} \log \frac{1}{|x - z|} \eta(\ell |y - z|) d\mathcal{L}^n(z) \leq \int_{\mathbb{R}^n} \log \frac{1}{|z|} \eta(\ell |z|) d\mathcal{L}^n(z).$$

*Proof.* Without restriction  $y = 0$  and  $\ell = 1$ . For  $r \geq 0$ , consider

$$\phi(r) := \int_{\mathbb{R}^n} \log \frac{1}{|rx - z|} \eta(|z|) d\mathcal{L}^n(z).$$

Then

$$\begin{aligned} \phi'(r) &= \int_{\mathbb{R}^n} \frac{\langle rx - z, x \rangle}{|rx - z|^2} \eta(|z|) d\mathcal{L}^n(z) = \int_{\mathbb{R}^n} \frac{\langle z, x \rangle}{|z|^2} \eta(|z - rx|) d\mathcal{L}^n(z) \\ &= \int_{\{z: \langle z, x \rangle \geq 0\}} \frac{\langle z, x \rangle}{|z|^2} \left( \eta(|z - rx|) - \eta(|z + rx|) \right) d\mathcal{L}^n(z) \leq 0 \end{aligned}$$

since  $t \mapsto \eta(t)$  is non-increasing. □

*Claim 4.13.* There exists  $C^* \geq 0$  such that, for all  $\ell \in \mathbb{N}$ ,

$$\frac{1}{N_\ell^*} \int_{\mathbb{R}^n} \log \frac{1}{|z|} \eta(\ell |z|) d\mathcal{L}^n(z) \leq \log \ell + C^*.$$

*Proof.* Straightforward with  $C^* := \frac{1}{N_1^*} \int_{\mathbb{R}^n} \log \frac{1}{|z|} \eta(|z|) d\mathcal{L}^n(z)$ . □

Now let us conclude the *proof of Theorem 4.8*. Fix  $\vartheta > 1$ , and choose  $\ell_\vartheta$  as in the proof of Claim 4.11. It remains to verify the estimate (70). For  $x, y$  with  $d(x, y) \geq 3R/\ell$ , this is derived in Claim 4.10. For  $x, y$  with  $d(x, y) < 3R/\ell$  (hence  $d(x, y) < \varepsilon_\vartheta$ ), the Claims 4.11, 4.12, 4.13 yield

$$\int \log \frac{1}{d(x', y')} q_\ell(y, y') d\text{vol}_g(y') \leq \vartheta (\log \ell + C^*) + \vartheta \log \vartheta$$

for every  $x'$  in the support of  $q_\ell(x, \cdot)$ , and thus

$$\iint \log \frac{1}{d(x', y')} q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(x') d\text{vol}_g(y') \leq \vartheta (\log \ell + C^*) + \vartheta \log \vartheta.$$

This proves the estimate (70) with  $c_\ell := \ell$  and  $C := C^* \vartheta^2 \log \vartheta$ , and the proof of the theorem is herewith complete. □

The previous results in particular applies to the kernel  $q_\ell(x, y) := \frac{1}{\text{vol}_g(B_{1/\ell}(x))} \mathbf{1}_{B_{1/\ell}(x)}(y)$ . Similar arguments apply to discretization kernels.

**Theorem 4.14.** *Let  $(\mathfrak{P}_\ell)_{\ell \in \mathbb{N}}$  be a family of partitions of  $M$  with  $d_\ell := \sup\{\text{diam}(A) : A \in \mathfrak{P}_\ell\} \rightarrow 0$  as  $\ell \rightarrow \infty$ , see Example 3.12 (ii), and  $\inf\{\text{vol}_g(A)/d_\ell^n : A \in \mathfrak{P}_\ell, \ell \in \mathbb{N}\} > 0$ . Let*

$$q_\ell := \sum_{A \in \mathfrak{P}_\ell} \frac{1}{\text{vol}_g(A)} \mathbf{1}_A \otimes \mathbf{1}_A .$$

Then, with  $\mu^{h_\ell}$  defined as in (68),

$$\mu^{h_\ell} \rightarrow \mu^h \quad \text{as } \ell \rightarrow \infty$$

in the sense made precise in (69).

*Proof.* Again the argumentation will be based on Lemma 4.5. The verification of the criteria (ii) and (iii) there is again straightforward. Criterion (i) will be verified as before by means of Lemma 4.7. To verify (70), assume without restriction that  $(\mathfrak{P}_\ell)_{\ell \in \mathbb{N}}$  is given with

$$\text{diam}(A) \leq d_\ell, \quad \text{vol}_g(A) \geq v_\ell \geq V d_\ell^n$$

for all  $A \in \mathfrak{P}_\ell$ ,  $\ell \in \mathbb{N}$  and for some constant  $V > 0$  independent of  $\ell$ . Then, for  $x, y \in M$  with  $d(x, y) > 3d_\ell$ , we obtain as in Claim 4.10 that

$$\iint \log \frac{1}{d(x', y')} q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(x') d\text{vol}_g(y') \leq \log \frac{1}{d(x, y)} + \log 3 .$$

Furthermore, for every  $\vartheta > 1$ , every sufficiently large  $\ell$ , and for all  $x, y \in M$  with  $d(x, y) \leq 3d_\ell$ , we have that

$$\begin{aligned} & \iint \log \frac{1}{d(x', y')} q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(x') d\text{vol}_g(y') \\ & \leq \sup_{x' \in A_x} \frac{1}{\text{vol}_g(A_y)} \int_{A_y} \log \frac{1}{d(x', y')} d\text{vol}_g(y') \\ & \leq \sup_{x' \in \mathbb{R}^n} \sup_{\substack{A \subset \mathbb{R}^n \\ \mathcal{L}^n(A) \leq v_\ell}} \frac{\vartheta}{\mathcal{L}^n(A)} \int_A \log \frac{1}{|x' - y'|} d\mathcal{L}^n(y') \end{aligned}$$

by comparison of Riemannian and Euclidean distances and volumes. Since  $|x' - y'|$  is translation invariant, we may dispense with the supremum over  $x'$  and assume instead that  $x' = 0 \in \mathbb{R}^n$ . Furthermore,

$$\begin{aligned} \sup_{\substack{A \subset \mathbb{R}^n \\ \mathcal{L}^n(A) \leq v_\ell}} \frac{1}{\mathcal{L}^n(A)} \int_A \log \frac{1}{|x' - y'|} d\mathcal{L}^n(y') &= \sup_{v \leq v_\ell} \sup_{\substack{A \subset \mathbb{R}^n \\ \mathcal{L}^n(A) = v}} \frac{1}{v} \int_{\mathbb{R}^n} \mathbf{1}_A(y') \log \frac{1}{|y'|} d\mathcal{L}^n(y') \\ &\leq \sup_{v \leq v_\ell} \frac{1}{v} \int_{B_r(0)} \log \frac{1}{|y'|} d\mathcal{L}^n(y') \end{aligned}$$

by Hardy–Littlewood inequality and spherical symmetry of  $-\log |y'|$ , where  $r = r(v)$  is so that  $\mathcal{L}^n(B_r(0)) = v$ . Furthermore, since  $\ell \mapsto v_\ell$  is monotone decreasing to 0, we may choose  $\ell$  additionally so large that  $v_\ell \leq 1$ . For all such  $\ell$ , since  $r(v) \leq r(v_\ell) \leq 1$  and  $-\log |y'| \geq 1$  on  $B_r(0)$ , the function  $v \mapsto \frac{1}{v} \int_{B_r(v)(0)} \log \frac{1}{|y'|} d\mathcal{L}^n(y')$  is increasing for  $v \in [0, v_\ell)$ . We have therefore that, for every  $\vartheta > 1$ , every sufficiently large  $\ell$ , every  $x, y \in M$  with  $d(x, y) \leq 3d_\ell$ ,

$$\iint \log \frac{1}{d(x', y')} q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(x') d\text{vol}_g(y') \leq \frac{1}{v_\ell} \int_{B_r(0)} \log \frac{1}{|y'|} d\mathcal{L}^n(y')$$

with  $r > 0$  such that  $\mathcal{L}^n(B_r(0)) = v_\ell$ . For such  $r$  we may compute

$$\begin{aligned} \frac{1}{\mathcal{L}^n(B_r(0))} \int_{B_r(0)} \log \frac{1}{|y'|} d\mathcal{L}^n(y') &= \frac{n}{r^n} \int_0^r \log \frac{1}{s} s^{n-1} ds = \frac{1}{n r^n} \int_0^{r^n} \log \frac{1}{t} dt \\ &= \frac{1}{n r^n} r^n (1 - \log r^n) = \frac{1}{n} + \log \frac{1}{r} . \end{aligned}$$

That is, for  $d(x, y) \leq 3d_\ell$  with sufficiently large  $\ell$ ,

$$\iint \log \frac{1}{d(x', y')} q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(x') d\text{vol}_g(y') \leq \vartheta \left( \frac{1}{n} + \log \frac{1}{r} \right) \leq C + \vartheta \log \frac{1}{3d_\ell}$$

since  $r = (v_\ell/c_n)^{1/n} \geq (V/c_n)^{1/n} d_\ell$  with  $c_n = \mathcal{L}^n(B_1(0))$ . Thus, summarizing, for all  $x, y \in M$  and all sufficiently large  $\ell$ ,

$$\iint \log \frac{1}{d(x', y')} q_\ell(x, x') q_\ell(y, y') d\text{vol}_g(x') d\text{vol}_g(y') \leq C + \vartheta \log \frac{1}{d(x, y) \vee 3d_\ell} . \quad \square$$

The previous theorems do *not* apply to the kernels

$$q_\ell := \sum_{j=1}^{\ell} \psi_j \otimes \psi_j$$

for the eigenspace projections. These kernels are not nonnegative and not supported on small balls, even for large  $\ell$ . Nevertheless,  $\mu^h$  can also be obtained via eigenfunctions approximation according to our next result.

**Theorem 4.15.** *Consider the eigenfunctions approximation  $(h_\ell)_{\ell \in \mathbb{N}}$  given in (54) with covariance kernel  $(k_\ell)_{\ell \in \mathbb{N}}$  as in (55). Let  $(\mu_\ell)_{\ell \in \mathbb{N}}$  be as in (68). Then, for all Borel  $B \subset M$ ,*

$$\mathbf{E}[\mu^h(B) \mid \xi_1, \dots, \xi_\ell] = \mu^{h_\ell}(B) .$$

*In particular,  $(\mu^{h_\ell}(B))_{\ell \in \mathbb{N}}$  is a uniformly integrable martingale.*

*Proof.* Fix a Borel  $B \subset M$ . Since  $h_\ell$  is almost surely smooth, in view of (66) we find that

$$\mu^h(B) = F(h_\ell, h - h_\ell) ,$$

where for  $\Phi \in \mathfrak{D}$  and  $u \in \mathring{H}^{-\varepsilon}$  for some  $\varepsilon > 0$  we write

$$F(\Phi, u) = \int_B e^{\gamma\Phi(x)} d\mu^u(x) .$$

Again by (66), we see that

$$G(\Phi) := \mathbf{E}F(\Phi, h - h_\ell) = \mathbf{E} \int_B e^{\gamma\Phi(x)} e^{-\gamma h_\ell(x)} d\mu^h(x) .$$

Applying (67) and using that  $h_\ell(x)$  is Gaussian, we thus have that

$$G(\Phi) = \mathbf{E} \int_B e^{\gamma\Phi(x)} e^{-\gamma h_\ell(x) - \gamma^2 k_\ell(x, x)} d\text{vol}_g(x) = \int_B e^{\gamma\Phi(x)} e^{-\frac{\gamma^2}{2} k_\ell(x, x)} d\text{vol}_g(x) .$$

Since  $h_\ell$  and  $h - h_\ell$  are independent and  $h_\ell$  is measurable with respect to  $u_1, \dots, u_\ell$ , we have that

$$\mathbf{E}[\mu^h(B) \mid u_1, \dots, u_\ell] = \mathbf{E}[F(h_\ell, h - h_\ell) \mid u_1, \dots, u_\ell] = G(h_\ell) = \mu^{h_\ell}(B) . \quad \square$$

### 4.3 Conformal quasi-invariance

**Theorem 4.16.** *Assume that the Riemannian manifold  $(M, g)$  is admissible and that  $g' = e^{2\varphi}g$  with  $\varphi \in C^\infty(M)$ . Set  $v' := \text{vol}_{g'}(M)$ , and define a centered Gaussian random variable  $\xi$  and a function  $\bar{\varphi} \in C^\infty(M)$  by*

$$\xi := \langle h \rangle_{g'} := \frac{1}{v'} \langle h | e^{n\varphi} \rangle, \quad \bar{\varphi} := \frac{2}{v'} \mathbf{k}_g(e^{n\varphi}) - \frac{1}{v'^2} \mathbf{k}_g(e^{n\varphi}, e^{n\varphi}). \quad (71)$$

For  $\gamma \in (-\sqrt{2n}, \sqrt{2n})$ , let  $\mu_g^h$  and  $\mu_{g'}^{h'}$  denote the Liouville Quantum Gravity measures on  $(M, g)$  and  $(M, g')$ , resp., with  $h \sim \text{CGF}^{M,g}$  and  $h' \sim \text{CGF}^{M,g'}$ . Then,

$$\mu_{g'}^{h'} \stackrel{(d)}{=} e^{-\gamma\xi + \frac{\gamma^2}{2}\bar{\varphi} + n\varphi} \mu_g^h. \quad (72)$$

*Remark 4.17.* Our formulation of the Liouville quantum measure is slightly different from the one usually considered in dimension 2 (see, for instance [DS11, Proposition 1.1]). See Section 4.5 for more details.

*Proof.* Let  $h \sim \text{CGF}^{M,g}$ . For  $\ell \in \mathbb{N}$ , let  $h_\ell$  be the Gaussian random field defined by (54), and define the random fields

$$h'_\ell := h_\ell - \langle h_\ell \rangle_{g'} \in \mathfrak{D}, \quad h' := e^{n\varphi} h - \langle h \rangle_{g'} \text{vol}_{g'} \in \mathfrak{D}'. \quad (73)$$

Here, we regard the  $h'_\ell$  and  $h_\ell$  as *random functions*. If we regarded them as *random distributions*, then they would transform in the same way as  $h'$  and  $h$  do.

*Random fields.* The convergence  $h_\ell \rightarrow h$  for  $\ell \rightarrow \infty$  as stated in Prop. 3.9 implies an analogous convergence  $h'_\ell \rightarrow h'$  in  $\mathfrak{D}'$ . More precisely, for every  $u \in \mathfrak{D}$ ,

$$\lim_{\ell \rightarrow \infty} \langle h_\ell \text{vol}_{g'} | u \rangle = \lim_{\ell \rightarrow \infty} \langle h_\ell \text{vol}_g | e^{n\varphi} u \rangle = \langle h | e^{n\varphi} u \rangle = \langle e^{n\varphi} h | u \rangle,$$

as well as  $\lim_{\ell \rightarrow \infty} \langle h_\ell \rangle_{g'} = \langle h \rangle_{g'}$ , the convergences being **P**-a.s. and in  $L^2(\mathbf{P})$ , and thus

$$\lim_{\ell \rightarrow \infty} \langle h'_\ell \text{vol}_{g'} | u \rangle = \langle h' | u \rangle, \quad \mathbf{P}\text{-a.s. and in } L^2(\mathbf{P}). \quad (74)$$

Let us set  $k'_\ell(x, y) := \mathbf{E}[h'_\ell(x) h'_\ell(y)]$  and let us denote by  $\mathbf{k}'_\ell$  the corresponding integral operator on  $L^2(\text{vol}_{g'})$ , namely

$$(\mathbf{k}'_\ell u)(x) := \int k'_\ell(x, y) u(y) d\text{vol}_{g'}(y), \quad u \in L^2(\text{vol}_{g'}).$$

Then,

$$\begin{aligned} \lim_{\ell} \iint u(x) k'_\ell(x, y) v(y) d\text{vol}_{g'}^{\otimes 2}(x, y) &= \lim_{\ell} \mathbf{E}[\langle h'_\ell \text{vol}_{g'} | u \rangle \langle h'_\ell \text{vol}_{g'} | v \rangle] \\ &= \mathbf{E}[\langle h' | u \rangle \langle h' | v \rangle] \\ &= \iint u(x) k'(x, y) v(y) d\text{vol}_{g'}^{\otimes 2}(x, y), \end{aligned}$$

where the first equality holds by definition of  $k'_\ell$ , the second equality holds by (74), and the third equality holds since  $h' \sim \text{CGF}^{M,g'}$  by Theorem 3.13. In particular, we have the following convergences

$$\lim_{\ell} \sqrt{\mathbf{k}'_\ell} u = \sqrt{\mathbf{k}'} u, \quad \forall u \in L^2(\text{vol}_{g'}), \quad (75)$$

$$\lim_{\ell} k'_\ell = k', \quad \text{a.e. on } M \times M. \quad (76)$$

*Random measures.* Now, let us set, for all  $\ell \in \mathbb{N}$ :

$$\mu_g^{h_\ell} := e^{\gamma h_\ell - \frac{\gamma^2}{2} \mathbf{E}[h_\ell^2]} \text{vol}_g, \quad \text{resp.} \quad \mu_{g'}^{h'_\ell} := e^{\gamma h'_\ell - \frac{\gamma^2}{2} \mathbf{E}[(h'_\ell)^2]} \text{vol}_{g'}.$$

On the one hand, by Theorem 4.1 we have that

$$\lim_{\ell} \int u d\mu_g^{h_\ell} = \int u d\mu_g^h, \quad u \in \mathcal{C}(M), \quad (77)$$

in  $L^1(\mathbf{P})$ . On the other hand, similarly to the proof of Theorem 4.1, the martingale  $\{\mu_{g'}^{h'_\ell}(M) : \ell \in \mathbb{N}\}$  is uniformly integrable. Together with (75) and (76), this verifies the assumptions in [Sha16, Thm. 25], hence

$$\lim_{\ell} \int u d\mu_{g'}^{h'_\ell} = \int u d\mu_{g'}^{h'}, \quad u \in \mathcal{C}_b(M), \quad (78)$$

in  $L^1(\mathbf{P})$ .

*Radon–Nikodym derivative.* Similarly to Theorem 2.19, we can compute  $k'_\ell$  explicitly. For short write  $\mathbf{m}_{g'} = \text{vol}_{g'}/v'$ . Then, we have

$$\begin{aligned} k'_\ell(x, y) &:= \mathbf{E} \left[ h'_\ell(x) h'_\ell(y) \right] = \mathbf{E} \left[ (h_\ell(x) - \langle h_\ell \rangle_{g'}) (h_\ell(y) - \langle h_\ell \rangle_{g'}) \right] \\ &= k_\ell(x, y) + \iint k_\ell(w, z) d\mathbf{m}_{g'}^{\otimes 2}(w, z) \\ &\quad - \int k_\ell(x, z) d\mathbf{m}_{g'}(z) - \int k_\ell(y, w) d\mathbf{m}_{g'}(w) \\ &= k_\ell(x, y) - \frac{1}{2} \bar{\varphi}_\ell(x) - \frac{1}{2} \bar{\varphi}_\ell(y), \end{aligned}$$

where we have set

$$\begin{aligned} \bar{\varphi}_\ell(\cdot) &:= 2 \int k_\ell(\cdot, z) d\mathbf{m}_{g'}(z) - \iint k_\ell(w, z) d\mathbf{m}_{g'}^{\otimes 2}(w, z) \\ &= \frac{2}{v'} \mathbf{k}_\ell(e^{n\varphi}) - \frac{1}{v'^2} \mathbf{k}_\ell(e^{n\varphi}, e^{n\varphi}). \end{aligned}$$

Thus in particular,

$$k'_\ell(x, x) - k_\ell(x, x) = \bar{\varphi}_\ell(x).$$

Furthermore, set  $\xi_\ell := \langle h_\ell \rangle_{g'} = \langle h_\ell | e^{n\bar{\varphi}} \rangle$ . Then, almost surely:

$$\begin{aligned} \log \frac{d\mu_g^{h_\ell}}{d\mu_{g'}^{h'_\ell}}(x) &= \gamma h_\ell(x) - \frac{\gamma^2}{2} \mathbf{E} \left[ h_\ell(x)^2 \right] - \gamma h'_\ell(x) + \frac{\gamma^2}{2} \mathbf{E} \left[ h'_\ell(x)^2 \right] - n\varphi(x) \\ &= \gamma \langle h_\ell \rangle_{g'} + \frac{\gamma^2}{2} \mathbf{E} \left[ h'_\ell(x)^2 - h_\ell(x)^2 \right] - n\varphi(x) \\ &= \gamma \xi_\ell + \frac{\gamma^2}{2} (k'_\ell(x, x) - k_\ell(x, x)) - n\varphi(x) \\ &= \gamma \xi_\ell - \frac{\gamma^2}{2} \bar{\varphi}_\ell(x) - n\varphi(x), \end{aligned}$$

and thus for every  $u \in \mathcal{C}_b(M)$ ,

$$\int_M u(x) d\mu_{g'}^{h'_\ell}(x) = \int_M e^{-\gamma \xi_\ell + \frac{\gamma^2}{2} \bar{\varphi}_\ell(x) + n\varphi(x)} u(x) d\mu_g^{h_\ell}(x). \quad (79)$$

*Convergence.* As  $\ell \rightarrow \infty$ , by (74) applied with  $u = e^{n\bar{\varphi}}$ , we have that  $\xi_\ell \rightarrow \xi$ ,  $\mathbf{P}$ -a.s. Moreover,  $\bar{\varphi}_\ell \rightarrow \bar{\varphi}$  in  $L^\infty(M, \text{vol}_g)$  according to Lemma 2.15 (vii). Together with the representation formula (79) and the convergence obtained in (77) and (78), this implies



that, in  $L^1(\mathbf{P})$ ,

$$\begin{aligned}
\int_M u(x) d\mu_{g'}^h(x) &= \lim_{\ell \rightarrow \infty} \int_M u(x) d\mu_{g'}^{h_\ell}(x) \\
&= \lim_{\ell \rightarrow \infty} \int_M e^{-\gamma \xi_\ell + \frac{\gamma^2}{2} \bar{\varphi}_\ell(x) + n\varphi(x)} u(x) d\mu_g^{h_\ell}(x) \\
&= \lim_{\ell \rightarrow \infty} \int_M e^{-\gamma \xi + \frac{\gamma^2}{2} \bar{\varphi}(x) + n\varphi(x)} u(x) d\mu_g^{h_\ell}(x) \\
&= \int_M e^{-\gamma \xi + \frac{\gamma^2}{2} \bar{\varphi}(x) + n\varphi(x)} u(x) d\mu_g^h(x).
\end{aligned}$$

This proves the claim.  $\square$

**Corollary 4.18.** *Assume that  $(M, g)$  and  $(M', g')$  are admissible and conformally equivalent with diffeomorphism  $\Phi$  and conformal weight  $e^{2\varphi}$ . Let  $h$  and  $h'$  denote the co-polyharmonic random fields, and  $\mu_g^h$  and  $\mu_{g'}^{h'}$  the corresponding Liouville Quantum Gravity measures on  $(M, g)$  and  $(M', g')$ , resp. Then,*

$$\mu_{g'}^{h'} \stackrel{(d)}{=} \Phi_* \left( e^{-\gamma \xi + \frac{\gamma^2}{2} \bar{\varphi} + n\varphi} \mu_g^h \right) \quad (80)$$

with  $\xi$  and  $\bar{\varphi}$  as above.

As for the co-polyharmonic Gaussian field, the conformal quasi-invariance simplifies whenever we consider Liouville quantum measures constructed from ungrounded fields.

**Corollary 4.19.** *Assume that  $h \sim \widehat{\text{CGF}}^{M,g}$  and  $h' \sim \widehat{\text{CGF}}^{M,g'}$ , then*

$$\mu_{g'}^{h'} \stackrel{(d)}{=} e^{n\varphi + \frac{\gamma^2}{2} \bar{\varphi}} \mu_g^h.$$

Even more, for all measurable  $F: \mathfrak{D}' \times \mathcal{M}_b(M) \rightarrow \mathbb{R}_+$ :

$$\int F(h, \mu_{g'}^h) d\widehat{\text{CGF}}_{g'}(h) = \int F(e^{n\varphi} h, e^{n\varphi + \frac{\gamma^2}{2} \bar{\varphi}} \mu_g^h) d\widehat{\text{CGF}}_g(h).$$

*Proof.* Expanding the definition of  $\widehat{\text{CGF}}$  and using Theorems 3.13 and 4.16, we find

$$\begin{aligned}
\int F(h, \mu_{g'}^h) d\widehat{\text{CGF}}_{g'}(h) &= \int F(h + a \text{vol}_{g'}, e^{\gamma a} \mu_{g'}^h) da d\text{CGF}_{g'}(h) \\
&= \int F(e^{n\varphi}(h - \xi \text{vol}_g + a \text{vol}_g), e^{\gamma(a-\xi)} e^{\frac{\gamma^2}{2} \bar{\varphi} + n\varphi} \mu_g^h) da d\text{CGF}_g(h).
\end{aligned}$$

We conclude by the translation invariance of the Lebesgue measure.  $\square$

## 4.4 Support properties

Since a typical realization of the Liouville Quantum Gravity measure  $\mu_g^h$  is singular with respect to the volume measure of  $M$ , it gives positive mass to certain sets  $E \subset M$  of vanishing volume measure. However, it does not give mass to sets of vanishing  $H^s$ -capacity (for sufficiently large  $s$ ), a classical scale of ‘smallness of sets’ involving Green kernels and thus well-suited for our purpose.

**Definition 4.20.** *For  $s > 0$ , the  $H^s$ -capacity (aka Bessel capacity) of a set  $E \subset M$  is*

$$\text{cap}_s(E) := \inf \left\{ \|f\|_{L^2}^2 : \mathbf{G}_{s/2,1} f \geq 1 \text{ vol}_g\text{-a.e. on } E, f \geq 0 \right\}. \quad (81)$$

A set with vanishing  $H^s$ -capacity, also has vanishing  $H^r$ -capacity for every  $r \in (0, s)$ . We call a set  $E$  such that  $\text{cap}_s(E) = 0$ , a  $\text{cap}_s$ -zero or a  $\text{cap}_s$ -polar set.

**Theorem 4.21.** Consider the co-polyharmonic Gaussian field  $h \sim \text{CGF}^{M,g}$  and the associated Liouville Quantum Gravity measure  $\mu_g^h$  on  $(M, g)$  with  $|\gamma|^2 < 2n$ . Then, for a.e.  $h$  and every  $s > \gamma^2/4$ , the measure  $\mu_g^h$  does not charge sets of vanishing  $H^s$ -capacity. That is,

$$\text{cap}_s(E) = 0 \implies \mu_g^h(E) = 0 \quad \forall \text{ Borel } E \subset M .$$

For applications of this result in the remainder of this paper, two choices of  $s$  are relevant,  $s = n/2$  and  $s = 1$ .

**Corollary 4.22.** Consider  $h$  and  $\mu_g^h$  as above.

- If  $|\gamma| < \sqrt{2n}$ , then  $\mathbf{P}$ -almost surely  $\mu_g^h$  does not charge sets of vanishing  $H^{n/2}$ -capacity.
- If  $|\gamma| < 2$ , then  $\mathbf{P}$ -almost surely  $\mu_g^h$  does not charge sets of vanishing  $H^1$ -capacity.

In the particular case  $n = 2$ , both assertions coincide. In general, none of the two assertions is an immediate consequence of the other one.

Our proof of the theorem relies on results on Bessel capacities and on a celebrated estimate for the volume of balls by J.-P. Kahane for random measures defined in terms of covariance kernels with logarithmic divergence.

Concerning capacities, we adapt to manifolds results in [Zie89] that do not follow from [DK07]. Denote by  $\mathcal{M}_b(M)$  the space of non-negative finite Borel measures on  $M$ . For  $\mu \in \mathcal{M}_b(M)$  we set

$$\mathbb{G}_{s,\alpha}\mu(x) = \int G_{s,\alpha}(x, y) d\mu(y), \quad s, \alpha > 0 ,$$

and, for a measurable set  $E \subset M$ :

$$b_s(E) := \sup \left\{ \mu(E) : \mu \in \mathcal{M}_b(M), \|\mathbb{G}_{s/2,1}(\mathbf{1}_E \mu)\|_{L^2} \leq 1 \right\}. \quad (82)$$

*Remark 4.23.* The Bessel capacities as defined above are ‘order 1 capacities’ in the sense of Dirichlet forms. The corresponding ‘order 0 capacities’ would be defined by replacing the operator  $\mathbb{G}_{s,1}$  with its grounded version  $\mathring{\mathbb{G}}_s$ . As a consequence of the compactness of  $M$ , these capacities define the same class of cap-zero subsets of  $M$ .

**Lemma 4.24.** Let  $s > 0$ . The following assertions hold true:

- (i)  $\text{cap}_s$  is a regular Choquet capacity;
- (ii) for every Suslin set  $E \subset M$ ,

$$b_s(E)^2 = \text{cap}_s(E) ;$$

- (iii) if  $\mu \in \mathcal{M}_b(M)$  satisfies  $\|\mathbb{G}_{s/2,1}\mu\|_{L^2} < \infty$ , then  $\mu$  does not charge  $\text{cap}_s$ -zero sets;
- (iv) any function in  $H^s$  is pointwise determined (and finite) up to a  $\text{cap}_s$ -zero set;
- (v) if  $(u_k)_k \subset H^s$  and  $u \in H^s$  satisfy  $\lim_k |u_k - u|_{H^s} = 0$ , then there exists a subsequence  $(u_{k_j})_j \subset H^s$  so that  $u = \lim_j u_{k_j}$  pointwise up to a  $\text{cap}_s$ -zero set.

*Proof.* Since assertions (i) and (ii) above are set-theoretical in nature, their proof is adapted *verbatim* from [Zie89]. In particular, (i) is concluded as in [Zie89, Cor. 2.6.9], and (ii) as in [Zie89, Thm. 2.6.12]. These adaptations hold provided we substitute the operator  $g_\alpha^*$  in [Zie89],  $\alpha = s/2$ , with  $\mathbb{G}_{s/2,1}$ , and noting that  $G_{s/2,1}(x, \cdot)$  is continuous away from  $x$ .

In order to show (iii), let  $E \subset M$  be  $\text{cap}_s$ -polar. By standard facts on Choquet capacities,  $E$  can be covered by countably many Suslin  $\text{cap}_s$ -polar sets. Thus, we may assume with no loss of generality that  $E$  be additionally Suslin. By (ii) and definition (82) of  $b_s$ , we then have

$$\|\mathbb{G}_{s/2,1}\mu\|_{L^2} \cdot \text{cap}_s(E)^{1/2} \geq \|\mathbb{G}_{s/2,1}(\mathbf{1}_E \mu)\|_{L^2} \cdot b_s(E) \geq \mu E ,$$

which concludes the proof by assumption on  $E$ .

(iv) By definition,  $u$  is in  $H^s$  if and only if there exists  $v \in L^2$  such that  $u = \mathbf{G}_{s/2,1}v$ , and  $\|u\|_{H^s} = \|v\|_{L^2}$ . By density of  $\mathfrak{D}$  in both  $L^2$  and  $H^s$ , it suffices to show that, whenever  $u_n \rightarrow u$   $\text{vol}_g$ -a.e. and in  $L^2$ , then  $\mathbf{G}_{s/2,1}u_n \rightarrow \mathbf{G}_{s/2,1}u$   $\text{cap}_s$ -q.e.. This latter fact holds as in [Zie89, Lem. 2.6.4], with identical proof.

(v) Firstly, let us show that

$$\text{cap}_s(\{|f| > a\}) \leq \|f\|_{H^s}^2 / a^2, \quad f \in H^s, a > 0. \quad (83)$$

Indeed,

$$\mathbf{G}_{s/2,1}(1 - \Delta_g)^{s/2} |f| / a = |f| / a \geq 1 \quad \text{on } \{|f| > a\},$$

hence, by definition of  $\text{cap}_s$  we have that

$$\text{cap}_s(\{|f| > a\}) \leq \left\| (1 - \Delta_g)^{s/2} |f| / a \right\|_{L^2}^2 = \| |f| / a \|_{H^s}^2 = \|f\|_{H^s}^2 / a^2.$$

Now, let  $(u_{k_j})_j \subset (u_k)_k$  be so that  $\|u - u_{k_j}\|_{H^s}^2 \leq 2^{-3j}$ , and set  $A_j := \{|u - u_{k_j}| > 2^{-j}\}$ . By (83),

$$\text{cap}_s(A_j) \leq 2^{2j} \|u - u_{k_j}\|_{H^s}^2 \leq 2^{-j}.$$

Set  $A := \bigcap_{\ell=1}^{\infty} \bigcup_{j=\ell}^{\infty} A_j$ . If  $x \notin A$ , it is readily seen that  $\lim_j |u(x) - u_{k_j}(x)| = 0$  by definition of the sets  $A_j$ . Thus, it suffices to show that  $\text{cap}_s(A) = 0$ . Since  $\text{cap}_s$  is a Choquet capacity, it is increasing and (countably) subadditive, and we have that

$$\text{cap}_s(A) \leq \text{cap}_s\left(\bigcup_{j=\ell}^{\infty} A_j\right) \leq \sum_{j=\ell}^{\infty} \text{cap}_s(A_j) \leq \sum_{j=\ell}^{\infty} 2^{-j} = 2^{-\ell+1}, \quad \ell \in \mathbb{N}.$$

Since  $\ell$  was arbitrary, the conclusion follows letting  $\ell \rightarrow \infty$ .  $\square$

For the above-mentioned, celebrated estimate for the volume of balls by J.-P. Kahane — concerning the so-called  $R_\alpha^+$  classes —, we refer to the survey [RV14] by R. Rhodes and V. Vargas. Note that  $|k_g(x, y) + \log d(x, y)| \leq C$  and recall Remark 3.8 concerning positivity of  $k_g$ .

**Lemma 4.25** ([RV14, Thm. 2.6]). *Take  $\alpha \in (0, n)$  and  $\gamma^2/2 \leq \alpha$ . Consider a copolyharmonic Gaussian field  $h \sim \text{CGF}^{M,g}$  and the associated Liouville quantum gravity measure  $\mu_g^h$  on  $(M, g)$ . Then, almost surely, for all  $\varepsilon > 0$  there exists  $\delta > 0$ ,  $C < \infty$ , and a compact set  $M_\varepsilon \subset M$  such that  $\mu_g^h(M \setminus M_\varepsilon) < \varepsilon$  and*

$$\mu_g^h(B_r(x) \cap M_\varepsilon) \leq Cr^{\alpha - \gamma^2/2 + \delta}, \quad \forall r > 0, \forall x \in M. \quad (84)$$

*Proof of Theorem 4.21.* For a.e.  $h$  the following holds true. Let numbers  $\gamma, s \in \mathbb{R}$  with  $\gamma^2 < 4s \leq 2n$  be given as well as a Borel set  $E \subset M$  with  $\mu_g^h(E) > 0$ . Applying Lemma 4.25 with  $\alpha := n + \gamma^2/2 - 2s \in [\gamma^2/2, n)$  and  $\varepsilon := \frac{1}{2}\mu_g^h(E) > 0$  yields the existence of  $\delta > 0$ ,  $C < \infty$ , and a compact set  $M_\varepsilon \subset M$  such that  $\mu_g^h(M \setminus M_\varepsilon) < \varepsilon$  and (84) holds. Set  $\mu_\varepsilon := \mathbf{1}_{M_\varepsilon} \mu_g^h$ . Then,  $\mu_\varepsilon(E) \geq \varepsilon > 0$ . Furthermore, with  $f(r) := r^{2s-n}$  and  $R := \text{diam}(M)$ , uniformly in  $y$ ,

$$\begin{aligned} \int_M f(d(x, y)) d\mu_\varepsilon(x) &= - \int_M \int_0^R \mathbf{1}_{\{r > d(x, y)\}} f'(r) dr d\mu_\varepsilon(x) \\ &= - \int_0^R \mu_\varepsilon(B_r(y)) f'(r) dr \\ &\leq (n - 2s) \int_0^R r^{\alpha - \gamma^2/2 + \delta} r^{2s-n} \frac{dr}{r} \\ &= (n - 2s) \int_0^R r^\delta \frac{dr}{r} \leq C' < \infty. \end{aligned}$$

Hence, according to Lemma 2.10,

$$0 \leq G_{s,1}\mu_\varepsilon(y) \leq C' . \quad (85)$$

Thanks to the convolution property of the kernels  $G_{r,1}$  for  $r > 0$  [DKS20, Lem. 2.3(ii)], the uniform estimate (85), and the fact that  $\mu_g^h$  is a finite measure, we find, with  $\mu' := \mathbf{1}_E\mu_\varepsilon$ :

$$\begin{aligned} \|G_{s/2,1}(\mu')\|_{L^2}^2 &= \iiint G_{s/2,1}(x,y) d\mu'(y) G_{s/2,1}(x,z) d\mu'(z) d\text{vol}_g(x) \\ &= \iint G_{s,1}(y,z) d\mu'(y) d\mu'(z) \\ &\leq C' \cdot \mu'(M) =: C'' < \infty . \end{aligned}$$

Hence, by the very definition of  $b_s$ ,

$$b_s(E) \geq \frac{\mu'(E)}{\|G_{s/2,1}(\mu')\|_{L^2}} \geq \frac{\varepsilon}{\sqrt{C''}} > 0 ,$$

and thus in turn  $\text{cap}_s(E) > 0$  according to Lemma 4.24.  $\square$

## 4.5 Refined and adjusted Liouville Quantum Gravity measures

Recall that  $k_g$  is equal up to multiplicative normalization to the kernel  $K_g$  of the inverse of the co-polyharmonic operator  $P_g$ . Now we propose a further additive normalization in terms of the function

$$r_g(x) = \limsup_{y \rightarrow x} \left[ k_g(x,y) - \log \frac{1}{d_g(x,y)} \right], \quad \forall x \in M .$$

This function has an important quasi-invariance property under conformal changes.

**Lemma 4.26.** *Let  $\varphi$  smooth and  $g' = e^{2\varphi}g$ . Then, with the notation of Theorem 4.16.*

$$r_{g'} - r_g = -\bar{\varphi} + \varphi .$$

*Proof.* By Proposition 2.19, for  $x \neq y \in M$ :

$$\begin{aligned} &\left[ k_{g'}(x,y) + \log d_{g'}(x,y) \right] - \left[ k_g(x,y) + \log d_g(x,y) \right] \\ &= -\frac{1}{2}\bar{\varphi}(x) - \frac{1}{2}\bar{\varphi}(y) + \log d_{g'}(x,y) - \log d_g(x,y) . \end{aligned}$$

Thus the claim is obtained immediately by letting  $y \rightarrow x$ , and noting that

$$\frac{d_{g'}(x,y)}{d_g(x,y)} \longrightarrow e^{\varphi(x)} \quad \text{as } y \longrightarrow x . \quad \square$$

To proceed, let us assume for the sake of discussion that the function  $r_g$  is smooth. This is known to be true in the case  $n = 2$ ; in arbitrary even dimension, according to [Ndi07, Lem. 2.1], the function  $r_g$  is at least  $\mathcal{C}^2$ . With respect to the smooth function  $r_g$ , we define the *refined co-polyharmonic kernel* by

$$\tilde{k}_g(x,y) := k_g(x,y) - \frac{1}{2}r_g(x) - \frac{1}{2}r_g(y) + c_g ,$$

where  $c_g := \langle r_g \rangle_g + \frac{a_n}{4}\mathfrak{p}_g(r_g, r_g)$ . With  $\tilde{k}_g$  it shares the estimate (47), and in addition it satisfies

$$\limsup_{y \rightarrow x} \left[ \tilde{k}_g(x,y) - \log \frac{1}{d_g(x,y)} \right] = c_g , \quad x \in M . \quad (86)$$

**Proposition 4.27.** *The kernel  $\tilde{k}_g$  is the covariance kernel associated with the refined co-polyharmonic field given by*

$$\tilde{h} := h - \frac{a_n}{2} \langle h | \mathbb{P}_g r_g \rangle \text{vol}_g ,$$

where  $h$  denotes the co-polyharmonic field as considered before.

*Proof.* Straightforward calculations yield for  $u, v \in \mathfrak{D}$ ,

$$\begin{aligned} & \iint \tilde{k}_g(x, y) u(x) v(y) d\text{vol}_g(x) d\text{vol}_g(y) \\ &= \mathbf{E}[\langle \tilde{h} | u \rangle \cdot \langle \tilde{h} | v \rangle] \\ &= \iint k_g(x, y) u(x) v(y) d\text{vol}_g(x) d\text{vol}_g(y) \\ &\quad - \frac{a_n}{2} \int u d\text{vol}_g \cdot \mathfrak{k}_g(\mathbb{P}_g r_g, v) - \frac{a_n}{2} \int v d\text{vol}_g \cdot \mathfrak{k}_g(\mathbb{P}_g r_g, u) \\ &\quad + \frac{a_n^2}{4} \int u d\text{vol}_g \cdot \int v d\text{vol}_g \cdot \mathfrak{k}_g(\mathbb{P}_g r_g, \mathbb{P}_g r_g) . \quad \square \end{aligned}$$

As for their plain equivalent, the kernel  $\tilde{k}_g$  and the field  $\tilde{h}$  enjoy quasi-invariance properties under conformal changes.

**Proposition 4.28.** *Let  $\tilde{k}_{g'}$  and  $\tilde{h}'$  denote the refined kernel and refined field associated with the metric  $g' = e^{2\varphi}g$  for some  $\varphi \in \mathfrak{D}$ . Then,*

$$\tilde{k}_{g'}(x, y) = \tilde{k}_g + \frac{1}{2}\varphi(x) - \frac{1}{2}\varphi(y) + c_{g'} - c_g , \quad (87)$$

and

$$\tilde{h}' \stackrel{(d)}{=} e^{n\varphi} \tilde{h} - \frac{a_n}{2} \langle \tilde{h} | \mathbb{P}_g \varphi \rangle \text{vol}_g . \quad (88)$$

*Proof.* Immediate consequences of Lemma 4.26, Proposition 2.19, and Theorem 4.16.  $\square$

The Gaussian multiplicative chaos on  $(M, g)$  associated with the refined field  $\tilde{h}$  is given in terms of the plain Liouville measure:

$$\tilde{\mu}_{g, \gamma}^h = \exp\left(\frac{\gamma^2}{2}(r_g - c_g) - \frac{\gamma a_n}{2} \langle h | \mathbb{P}_g r_g \rangle\right) \mu_{g, \gamma}^h . \quad (89)$$

Passing to the law of the grounded field obtained by convolution with the one-dimensional Lebesgue measure, as in Remark 3.15 (ii), we can easily control the correction terms that do not depend on  $x$ . Thus, following the established procedure in the two-dimensional case, we now leave them aside and take into account only the  $x$ -dependent correction term  $\frac{\gamma^2}{2}r_g$ . This allows to cover the general situation, where we no longer assume that  $r_g$  is smooth.

**Definition 4.29.** *We define the adjusted Liouville Quantum Gravity measure by*

$$\bar{\mu}_g^h = \exp\left(\frac{\gamma^2}{2} r_g\right) \mu_g^h .$$

*Remark 4.30.* In dimension 2, this approach corresponds with the one used for instance in [DS11, DKRV16, GRV19], as well as with the one of [Cer19] on higher dimensional spheres. In these works, they obtain directly the adjusted Liouville measure by regularizing  $h$  via convolution and by normalizing  $e^{\gamma h_\varepsilon(x)} \text{vol}(dx)$  by some explicit power of  $\varepsilon$ .

**Theorem 4.31.** *Let  $\varphi$  smooth,  $g' = e^{2\varphi}g$ , and  $h$  and  $h'$  co-polyharmonic Gaussian fields with respect to  $g$  and  $g'$ . Then,*

$$\bar{\mu}_{g'}^{h'} = \exp\left[-\gamma\xi + \left(n + \frac{\gamma^2}{2}\right)\varphi\right] \bar{\mu}_g^h ,$$

where  $\xi = \langle h \rangle_{g'} = \frac{1}{\text{vol}_{g'}(M)} \langle h | e^{n\varphi} \rangle$ .

*Remark 4.32.* The adjusted LQG measure  $\bar{\mu}_g^h$  shares the same support properties as formulated for the plain LQG measure  $\mu_g^h$  in Theorem 4.21.

## 5 Applications and outlook

### 5.1 Random Dirichlet form and Liouville Brownian motion

For sufficiently small  $|\gamma|$ , the Liouville Quantum Gravity measure  $\mu_g^h$  does not charge sets of  $H^1$ -capacity zero. Hence, a random Brownian motion can easily be constructed through time change of the standard Brownian motion  $((B_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in M})$ .

**Theorem 5.1.** *Let  $(M, g)$  be admissible, let  $h \sim \text{CGF}^{M, g}$  denote the co-polyharmonic Gaussian field and  $\mu_g^h$  the associated Liouville Quantum Gravity measure with  $|\gamma| < 2$ . Then, for  $\mathbf{P}$ -a.e.  $h$ ,*

(i) *A regular strongly local Dirichlet form on  $L^2(M, \mu_g^h)$  is given by*

$$\mathcal{E}_\gamma^h(f, f) := \int_M |\nabla f|^2 d\text{vol}_g, \quad \mathcal{D}(\mathcal{E}_\gamma^h) := \left\{ f \in H^1(M) : \tilde{f} \in L^2(M, \mu_g^h) \right\} \quad (90)$$

where  $\tilde{f}$  denotes the quasi-continuous modification of  $f \in H^1(M)$ .

(ii) *The associated reversible continuous Markov process  $((X_t^h)_{t \geq 0}, (\mathbb{P}_x^h)_{x \in M})$ , called Liouville Brownian motion on  $(M, g)$ , is obtained by time change of the standard Brownian motion on  $(M, g)$ . Namely, let  $(A_t^h)_{t \geq 0}$  be the additive functional whose Revuz measure is given by  $\mu_g^h$ , then*

$$\mathbb{P}_x^h := \mathbb{P}_x, \quad X_t^h := B_{\tau_t^h}, \quad \tau_t^h := \inf\{s \geq 0 : A_s^h > t\}.$$

(iii) *Moreover, for every bounded probability density  $\rho$  on  $M$ , the additive functional  $(A_t^h)_{t \geq 0}$  is  $\mathbb{P}_\rho$ -a.s. given by*

$$A_t^h = \lim_{\ell \rightarrow \infty} \int_0^t \exp\left(\gamma h_\ell(B_s) - \frac{\gamma^2}{2} k_\ell(B_s, B_s)\right) ds, \quad (91)$$

with  $h_\ell$  and  $k_\ell$  as in (54) and (55).

*Remark 5.2.* Recall that the additive functional  $A^h$  associated with the measure  $\mu_g^h$  is the process characterized by

$$\mathbb{E}_x \left[ \int_0^t u(B_s) dA_s^h \right] = \int_0^t \int u(y) p_s(x, y) d\mu_g^h(y) ds, \quad u \in \mathfrak{B}_b, \quad t \geq 0, \quad (92)$$

where  $\mathfrak{B}_b$  denotes the space of real-valued bounded Borel functions on  $M$ . For further information on additive functionals, see [FOT11].

*Proof.* (i) and (ii) hold using standard argument in the theory of Dirichlet forms. Indeed, Corollary 4.22 and the compactness of  $M$  imply that for  $\mathbf{P}$ -a.e.  $h$ , the measure  $\mu_g^h$  is a Revuz measure of finite energy integral. For details see [GRV14, Thm. 1.7], where this argument is carried out in the case when  $M$  is the unit disk.

(iii) Fix  $t > 0$  and  $\rho$  a probability measure with bounded density on  $M$ . We consider the occupation measure

$$dL_t(x) = \int_0^t d\delta_{B_s}(x) ds.$$

Observe that for  $\alpha > 0$ ,

$$\begin{aligned} \mathbb{E}_\rho \left[ \iint \frac{dL_t(y) dL_t(z)}{d(y, z)^\alpha} \right] &= \int_0^t \int_0^t \mathbb{E}_\rho d(B_r, B_s)^{-\alpha} dr ds \\ &= 2 \int_0^t \int_s^t \iiint d(y, z)^{-\alpha} p_s(x, y) p_{r-s}(y, z) d\text{vol}_g(y) d\text{vol}_g(z) d\rho(x) dr ds \\ &\leq C t \sup_\ell \int_0^t \iiint d(y, z)^{-\alpha} p_s(y, z) d\text{vol}_g(y) d\text{vol}_g(z) ds \\ &\leq C t e^t \iint d(y, z)^{-\alpha} G_{1,1}(y, z) d\text{vol}_g(y) d\text{vol}_g(z). \end{aligned}$$

According to the estimate for the 1-Green kernel  $G_{1,1}$ , the latter integral is finite for all  $\alpha < 2$ . This means that,  $\mathbb{P}_\rho$ -almost surely,  $L_t$  satisfies [Kah85, Eqn. (39)] for all  $\alpha < 2$ . Thus  $L_t$  is,  $\mathbb{P}_\rho$ -almost surely, in the class  $M_{\alpha^+}^+$  for all  $\alpha < 2$ . Arguing as in the proof of Theorem 4.1, we find that, for all  $\gamma^2 < 4$ , having fixed the randomness with respect to  $\mathbb{P}_\rho$ , there exists a random measure  $\nu_t^h$  that is the Gaussian multiplicative chaos over  $(h, \gamma k)$  with respect to  $L_t$ .

Now, for all Borel sets  $A \subset M$  we set

$$\begin{aligned} \nu_t^{h_\ell}(A) &:= \int_A \exp\left(\gamma h_\ell(x) - \frac{\gamma^2}{2} k_\ell(x, x)\right) dL_t(x) \\ &= \int_0^t 1_A(B_s) \exp\left(\gamma h_\ell(B_s) - \frac{\gamma^2}{2} k_\ell(B_s, B_s)\right) ds. \end{aligned}$$

Since we choose  $(h_\ell)_\ell$  as in (54), the family  $(\nu_t^{h_\ell})_\ell$  is a  $\mathbf{P}$ -martingale for every fixed  $t \geq 0$ , similarly to Theorem 4.15. The fact that  $\nu_t^{h_\ell} \rightarrow \nu_t^h$  follows from the same uniform integrability argument for martingales as in Theorem 4.15.

For all  $t > 0$ , set  $A_t^h := \nu_t^h(M)$  and  $A_t^{h_\ell} := \nu_t^{h_\ell}(M)$  for each  $\ell \in \mathbb{N}$ . It is clear that  $t \mapsto A_t^{h_\ell}$  is the positive continuous additive functional associated to  $\mu^{h_\ell}$  by the Revuz correspondence, that is (cf. (92)),

$$\mathbb{E}_\rho \left[ \int_0^t u(B_s) dA_s^{h_\ell} \right] = \int_0^t \int \left[ \int p_s(x, y) u(y) d\mu_g^{h_\ell}(y) \right] d\rho(x) ds, \quad u \in \mathfrak{B}_b, t \geq 0. \quad (93)$$

Now let  $\tilde{A}_t^h$  denote the positive continuous additive functional associated with  $\mu^h$ . Then applying (69) twice — to  $\mu_g^{h_\ell} \rightarrow \mu_g^h$  and to  $\nu_t^{h_\ell} \rightarrow \nu_t^h$  — we obtain that in  $\mathbf{P}$ -probability:

$$\begin{aligned} \mathbb{E}_\rho \left[ \int_0^t u(B_s) dA_s^h \right] &= \mathbb{E}_\rho \left[ \int_M u d\nu_t^h \right] \\ &= \lim_{\ell \rightarrow \infty} \mathbb{E}_\rho \left[ \int_M u d\nu_t^{h_\ell} \right] = \lim_{\ell \rightarrow \infty} \mathbb{E}_\rho \left[ \int_0^t u(B_s) dA_s^{h_\ell} \right] \\ &= \lim_{\ell \rightarrow \infty} \int_0^t \int \left[ \int p_s(x, y) u(y) d\mu_g^{h_\ell}(y) \right] d\rho(x) ds \\ &= \int_0^t \int \left[ \int p_s(x, y) u(y) d\mu_g^h(y) \right] d\rho(x) ds = \mathbb{E}_\rho \left[ \int_0^t u(B_s) d\tilde{A}_s^h \right]. \end{aligned}$$

This shows that  $A^h = \tilde{A}^h$  a.s. w.r.t.  $\mathbf{P} \otimes \mathbb{P}_\rho$  and concludes the proof.  $\square$

*Remark 5.3.* The intrinsic distance associated to the Dirichlet form (90) vanishes identically. This can be easily verified, exactly as in [GRV14, Prop. 3.1].

*Remark 5.4.* The previous constructions work equally well with the adjusted Liouville measure  $\bar{\mu}_g^h$  (or with the refined Liouville measure  $\tilde{\mu}_g^h$ ) in the place of the plain Liouville measure  $\mu_g^h$ . For a.e.  $h$ , the resulting process, the adjusted (or refined, resp.) Brownian motion, can be regarded as the plain Brownian motion with drift.

*Remark 5.5.* In the case  $n = 2$ , Liouville Brownian motion shares an important quasi-invariance property under conformal transformations. In higher dimensions, no such — or similar — conformal quasi-invariance property holds true. Indeed, the generator of the Brownian motion, the Laplace–Beltrami operator, is quasi-invariant under conformal transformations if and only if  $n = 2$ .

For the 2-dimensional counterparts of the previous theorem, see [Ber15] and [GRV14, GRV16].

## 5.2 Random Paneitz and random GJMS operators

In higher dimensions, from the perspective of conformal quasi-invariance, the natural random operators to study are random perturbations of the co-polyharmonic operators  $\mathbf{P}_g$ . To simplify notation, we henceforth write  $\mathbf{P}$  and  $\text{vol}$  rather than  $\mathbf{P}_g$  and  $\text{vol}_g$ .

**Theorem 5.6.** *Let  $(M, g)$  be admissible, let  $h \sim \text{CGF}^{M, g}$  denote the co-polyharmonic Gaussian field and  $\mu_g^h$  the associated plain Liouville Quantum Gravity measure with  $|\gamma| < \sqrt{2n}$ . Then, for  $\mathbf{P}$ -a.e.  $h$ ,*

$$\mathfrak{p}^h(u, v) := \int_M \sqrt{\mathbb{P}}u \sqrt{\mathbb{P}}v \, d\text{vol}_g, \quad u, v \in \mathcal{D}(\mathfrak{p}^h) := H^{n/2} \cap L^2(M, \mu_g^h),$$

*is a well-defined non-negative closed symmetric bilinear form on  $L^2(M, \mu_g^h)$ .*

*Proof.* Since  $\mu_g^h$  does not charge  $\text{cap}_{n/2}$ -polar sets by Theorem 4.21, and since every  $f \in H^{n/2}$  is  $\text{cap}_{n/2}$ -q.e. finite by Proposition 4.24(iv), every  $f \in H^{n/2}$  admits a  $\mu_g^h$ -a.e. finite representative (possibly depending on  $h$ ). Thus,  $\mathfrak{p}^h$  is well-defined on  $H^{n/2} \cap L^0(\mu_g^h)$ . In order to show that  $\mathfrak{p}^h$  is finite on  $\mathcal{D}(\mathfrak{p}^h)$ , let  $u = \mathbb{G}_{n/4, 1}u'$ , resp.  $v = \mathbb{G}_{n/4, 1}v' \in H^{n/2}$ , with  $u', v' \in L^2$ , and note that

$$\begin{aligned} \mathfrak{p}^h(u, v) &= \langle \mathbb{G}_{n/4, 1}u' \mid \mathbb{P} \mathbb{G}_{n/4, 1}v' \rangle_{L^2} = \langle u' \mid \mathbb{G}_{n/4, 1} \mathbb{P} \mathbb{G}_{n/4, 1}v' \rangle_{L^2} \\ &\leq \|u'\|_{L^2} \|v'\|_{L^2} \|\mathbb{G}_{n/4, 1} \mathbb{P} \mathbb{G}_{n/4, 1}\|_{L^2 \rightarrow L^2} < \infty \end{aligned}$$

by admissibility of  $M$ .

In order to show closedness it suffices to show that  $\mathcal{D}(\mathfrak{p}^h)$  is complete in the graph-norm

$$\|u\|_{\mathcal{D}(\mathfrak{p}^h)} := \left( \mathfrak{p}^h(u) + \|u\|_{L^2(\mu_g^h)}^2 \right)^{1/2}, \quad u \in \mathcal{D}(\mathfrak{p}^h).$$

Since  $\mathfrak{p}^h$  vanishes on constant functions by Theorem 1.3(ii), it suffices to show that  $\mathcal{D}(\mathfrak{p}^h) := \dot{H}^{n/2} \cap L^2(\mu_g^h)$  is complete in the same norm. To this end, let  $(u_k)_k$  be  $\mathcal{D}(\mathfrak{p}^h)$ -Cauchy and note that it is in particular both  $L^2(\mu_g^h)$ - and  $\mathfrak{p}^h$ -Cauchy. In particular, there exists the  $L^2(\mu_g^h)$ -limit  $u$  of  $(u_k)_k$ , and, up to passing to a suitable non-relabeled subsequence, we may further assume with no loss of generality that  $\lim_k u_k = u$   $\mu_g^h$ -a.e.. Furthermore, by Lemma 2.15(ii),  $\mathfrak{p}^h$  defines a norm on  $\dot{H}^{n/2}$ , bi-Lipschitz equivalent to the standard norm of  $\dot{H}^{n/2}$ . As consequence,  $(u_k)_k$  is as well  $\dot{H}^{n/2}$ -Cauchy, and, by completeness of the latter, it admits an  $\dot{H}^{n/2}$ -limit  $u'$ . Up to passing to a suitable non-relabeled subsequence, by Proposition 4.24(v) we may further assume with no loss of generality that  $\lim_k u_k = u'$   $\text{cap}_{n/2}$ -q.e.. In particular, again since  $\mu_g^h$  does not charge  $\text{cap}_{n/2}$ -polar sets, we have that  $\lim_k u_k = u$   $\mu_g^h$ -a.e., i.e.  $u' = u$   $\mu_g^h$ -a.e., hence as elements of  $L^2(\mu_g^h)$ . It follows that  $L^2(\mu_g^h)$ - $\lim_k u_k = u$ , which concludes the proof of completeness.

Non-negativity is a consequence of the admissibility of  $M$ . Symmetry follows from that of  $\mathbb{P}$ , Theorem 1.3(iv).  $\square$

**Corollary 5.7.** *Let  $(M, g)$ ,  $h$ ,  $\gamma$ , and  $\mu_g^h$  be as above. Then, for  $\mathbf{P}$ -a.e.  $h$  there exists a unique nonnegative self-adjoint operator  $\mathbb{P}^h$  on  $L^2(M, \mu_g^h)$ , called random co-polyharmonic operator or random GJMS operators, defined by  $\mathcal{D}(\mathbb{P}^h) \subset \mathcal{D}(\mathfrak{p}^h)$  and*

$$\mathfrak{p}^h(u, v) = \int u \mathbb{P}^h v \, d\mu_g^h, \quad u \in \mathcal{D}(\mathbb{P}^h), \quad v \in \mathcal{D}(\mathfrak{p}^h).$$

In the case  $n = 4$ , the operators  $\mathbb{P}^h$  are also called *random Paneitz operators*.

**Corollary 5.8.** *With  $(M, g)$ ,  $h$ ,  $\gamma$ , and  $\mu_g^h$  as above, for a.e.  $h$  there exists a semigroup  $(e^{-t\mathbb{P}^h})_{t \geq 0}$  of bounded symmetric operators on  $L^2(M, \mu_g^h)$ , called random co-polyharmonic heat semigroup.*

**Proposition 5.9.** *The random co-polyharmonic heat flow  $(t, u) \mapsto e^{-t\mathbb{P}^h}u$  is the EDE-gradient flow for  $\frac{1}{2}\mathfrak{p}^h$  on  $L^2(M, \mu_g^h)$ .*

*Here ‘EDE’ stands for gradient flow in the sense of ‘energy-dissipation-equality’, see [AG13, Dfn. 3.4].*



*Proof.* The energy decays along the flow according to

$$\frac{d}{dt} \mathfrak{p}(u_t) = \left\langle \sqrt{\mathbb{P}^h} \frac{d}{dt} u_t \left| \sqrt{\mathbb{P}^h} u_t \right. \right\rangle_{L^2(\mu_g^h)} = - \langle \mathbb{P}^h u_t \mid \mathbb{P}^h u_t \rangle_{L^2(\mu_g^h)} = - \| \mathbb{P}^h u_t \|_{L^2(\mu_g^h)}^2$$

for  $u_t := e^{-t\mathbb{P}^h} u_0$ . Moreover for each differentiable curve  $v_t$  we have

$$\frac{d}{dt} \mathfrak{p}(v_t) = \frac{d}{dt} \int \sqrt{\mathbb{P}^h} v_t \sqrt{\mathbb{P}^h} v_t d\mu_g^h = \int \frac{d}{dt} v_t \mathbb{P}^h v_t d\mu_g^h = \left\langle \frac{d}{dt} v_t \left| \mathbb{P}^h v_t \right. \right\rangle_{L^2}. \quad (94)$$

Consequently  $\nabla \mathfrak{p}(v) = \mathbb{P}^h v$  which leads to

$$\frac{d}{dt} u_t = -\nabla \mathfrak{p}(u_t) \quad (95)$$

and thus the assertion.  $\square$

*Remark 5.10.* For a.e.  $h$  and every  $u \in L^2(M, \mu_g^h)$ , the solutions  $u_t := e^{-t\mathbb{P}^h} u$  are absolutely continuous with respect to  $\mu_g^h$  for all  $t > 0$ . It is plausible to conjecture that there exists a *random co-polyharmonic heat kernel*  $p_t^h$  such that

$$e^{-t\mathbb{P}^h} u(x) = \int_M p_t^h(x, y) u(y) d\mu_g^h(y) \quad \text{for a.e. } x \in M, \quad u \in L^2.$$

In the case  $n = 2$ , such a kernel exists, and it admits sub-Gaussian upper bounds (see [MRVZ16, AK15]),

$$p_t^h(x, y) \leq C_1 t^{-1} \log(t^{-1}) \exp \left( -C_2 \left( \frac{d(x, y)^\beta \wedge 1}{t} \right)^{\frac{1}{\beta-1}} \right), \quad t \in \left( \frac{1}{2}, 1 \right],$$

for any  $\beta > \frac{1}{2}(\gamma + 2)^2$  and constants  $C_i = C_i(\beta, \gamma, h, d(y, 0))$ .

Now let us address the conformal quasi-invariance of the random co-polyharmonic operators. For this purpose, of course, we have to emphasize all  $g$ -dependencies in the notation and thus write  $\mathbb{P}_g$  and  $\mathbb{P}_g^h$  rather than  $\mathbb{P}$  and  $\mathbb{P}^h$ . Furthermore, we fix  $\gamma$  throughout the sequel and write  $\mu_{g, \gamma}^h$  instead of  $\mu_{g, \gamma}^h$ .

Assume that the Riemannian manifold  $(M, g)$  is admissible and that  $|\gamma| < \sqrt{2n}$ . Let  $h \sim \text{CGF}^{M, g}$  denote the co-polyharmonic random field and  $\mu_g^h$  the corresponding plain Liouville Quantum Gravity measure on  $(M, g)$ .

Given any  $g' = e^{2\varphi} g$  with  $\varphi \in \mathcal{C}^\infty(M)$ , define (a version of) the Liouville Quantum Gravity measure on  $(M, g')$  according to Theorem 4.16 by

$$\mu_{g'}^{h'} := e^{F^h} \mu_g^h. \quad (96)$$

with  $v' = \text{vol}_{g'}(M)$  and

$$F^h := -\gamma \langle h \rangle_{g'} + \frac{\gamma^2}{2v'} \mathfrak{k}_g(e^{n\varphi}, e^{n\varphi}) - \left( \frac{\gamma}{v'} \right)^2 \mathfrak{k}_g(e^{n\varphi}) + n\varphi. \quad (97)$$

**Theorem 5.11.** *The random co-polyharmonic operator  $\mathbb{P}_g^h$  is conformally quasi-invariant: if  $g' = e^{2\varphi} g$  then*

$$\mathbb{P}_{g'}^{h'} \stackrel{(d)}{=} e^{-F^h} \mathbb{P}_g^h \quad (98)$$

with  $\bar{F}^h$  as above.

*Proof.* Recall from Theorem 4.16 that  $\mu_{g'}^h \stackrel{(d)}{=} e^{F^h} \mu_g^h$ . Thus by the conformal invariance of the bilinear form  $\mathfrak{p}_g$ ,

$$\int_M \mathbb{P}_g^h u v d\mu_g^h = \mathfrak{p}_g(u, v) = \mathfrak{p}_{g'}(u, v) = \int_M \mathbb{P}_{g'}^h u v d\mu_{g'}^h = \int_M e^{Z^h} \bar{\mathbb{P}}_{g'}^h u v d\mu_g^h$$

for all  $u$  and  $v$  in appropriate domains. Hence,  $\mathbb{P}_g^h u = e^{Z^h} \mathbb{P}_{g'}^h u$ . This proves the claim.  $\square$

*Remark 5.12.* The above construction can also be carried out with  $\bar{\mu}_g^h$  instead of  $\mu_g^h$  yielding the *adjusted random co-polyharmonic operator*  $\bar{P}_g^h$ . In that case, we get for the conformal quasi-invariance the following formula

$$\bar{P}_{g'}^h \stackrel{(d)}{=} e^{-\bar{F}^h} \bar{P}_g^h ,$$

where  $\bar{F}^h = \gamma \langle h | e^{n\bar{\varphi}} \rangle + (n + \gamma^2/2)\varphi$ .

### 5.3 The Polyakov–Liouville measure in higher dimensions

Our last objective in this paper is to propose a version of conformal field theory on compact manifolds of arbitrary even dimension, an approach based on Branson’s  $Q$ -curvature. We provide a rigorous meaning to the Polyakov–Liouville measure  $\nu_g^*$ , informally given as

$$\frac{1}{Z_g} \exp(-S_g(h)) dh$$

with the (non-existing) uniform distribution  $dh$  on the set of fields (thought as sections of some bundles over  $M$ ) and the action

$$S_g(h) := \int_M \left( \frac{a_n}{2} |\sqrt{P_g} h|^2 + \Theta Q_g h + \frac{\Theta^*}{\text{vol}_g(M)} h + m e^{\gamma h} \right) d\text{vol}_g , \quad (99)$$

where  $P_g$  is the co-polyharmonic operator,  $Q_g$  denotes Branson’s curvature,  $a_n$  is the constant from (32), and  $m, \Theta, \Theta^*, \gamma$  are parameters — subjected to some restrictions specified below, in particular,  $0 < |\gamma| < \sqrt{2n}$ .

#### 5.3.1 Heuristics and motivations

Before going into the details of our approach, let us briefly recall the longstanding challenge of conformal field theory and some recent breakthroughs in the two-dimensional case. Here (99) becomes the celebrated Polyakov–Liouville action

$$S_g(h) = \int_M \left( \frac{1}{4\pi} |\nabla h|^2 + \frac{\Theta}{2} R_g h + \frac{\Theta^*}{\text{vol}_g(M)} h + m e^{\gamma h} \right) d\text{vol}_g , \quad (100)$$

where  $R_g$  is the scalar curvature and  $m, \Theta, \Theta^*, \gamma$  are parameters. (Instead of  $m$  and  $\Theta$  mostly in the literature  $\bar{\mu}$  and  $Q$  are used. However, in this paper the latter symbols are already reserved for the Liouville Quantum Gravity measure and Branson’s curvature.) With the Polyakov–Liouville action, this ansatz for the measure  $\nu_g^*(dh) = \frac{1}{Z_g} e^{-S_g(h)} dh$  reflects the coupling of the gravitational field with a matter field. It can be regarded as quantization of the the classical Einstein–Hilbert action  $S_g^{EH}(h) = \frac{1}{2\kappa} \int_M (R_g - 2\Lambda) dx$  or, more precisely, of its coupling with a matter field

$$S_g^{EH}(h) = \int_M \left[ \frac{1}{2\kappa} (R_g - 2\Lambda) + \mathcal{L}_M \right] dx .$$

In the case  $n = 2$  and  $Q^* = 0$ , based on the concepts of Gaussian Free Fields and Liouville Quantum Gravity measures, the rigorous construction of such a Polyakov–Liouville measure  $\nu_g^*$  has been carried out recently in [DKRV16] for surfaces of genus 0, [DRV16] for surfaces of genus 1 (see also [HRV18] for the disk), and in [GRV19] for surfaces of higher genus. For related constructions, see [DMS21]. The approach of [GRV19] gives a rigorous meaning to

$$\nu_g^*(dh) = \frac{1}{Z_g} \exp \left[ - \int \left( \frac{\Theta}{2} R_g h + m e^{\gamma h} \right) d\text{vol}_g \right] \exp \left( - \frac{1}{4\pi} \|h\|_{H^1}^2 \right) dh ,$$

by interpreting

$$\hat{\nu}_g(dh) = \frac{1}{Z_g} \exp \left( - \frac{1}{4\pi} |\nabla h|^2 \right) dh ,$$

as an informal definition of the ungrounded Gaussian Free Field  $\widehat{\mathcal{D}}_g := \widehat{\text{GFF}}^{M,g}$ , and by setting

$$\nu_g^*(dh) := \exp\left(-\frac{\Theta}{2}\langle h | R_g \rangle - m \bar{\mu}_{g,\gamma}^h(M)\right) \widehat{\mathcal{D}}_g(dh),$$

where  $\bar{\mu}_{g,\gamma}^h$  denotes the adjusted Liouville Quantum Gravity measure on  $M$  with parameter  $\gamma \in (0, 2)$ , where  $\Theta = \frac{1}{2\pi}\left(\frac{\gamma}{2} + \frac{2}{\gamma}\right)$ .

This provides a complete solution to the above mentioned challenge in dimension 2. However, in dimension greater than 2 not much is known so far.

The relevance of the Polyakov-Liouville action is that it quantifies the conformal quasi-invariance of the functional determinant in dimension 2. Namely, we have that [OPS88, Eq. (1.17)]:

$$\log \det(-\Delta_{g'}) - \log \det(-\Delta_g) = -\frac{1}{12\pi} \int 2\varphi \text{scal}_g + |\nabla\varphi|_g^2 d\text{vol}_g.$$

Thus we can see the Polyakov-Liouville action as a potential accounting for the variation of the the functional determinant of the Laplacian coupled with the volume. In higher dimension, no such formula exists. However, it is established in low dimension and conjectured in higher dimension (see [BG08] and the references therein) that a physically relevant Polyakov formula for  $n > 2$  should involve the co-polyharmonic operators. Under our admissibility, it should take the form:

$$\log \det P_g - \log \det P_{g'} = \Theta \int \left[ \frac{1}{2}\varphi P_g \varphi + \varphi Q_g \right] d\text{vol}_g + \int F_{g'} d\text{vol}_{g'} - \int F_g d\text{vol}_g,$$

where  $\Theta$  is a constant and  $F$  is a local scalar invariant. In view of this formula, let us define a higher dimensional equivalent of the  $2d$  Polyakov-Liouville action:

$$S_g(h) = \Theta \int h Q_g d\text{vol}_g + \frac{\Theta^*}{\text{vol}_g(M)} \int h d\text{vol}_g + m \int e^{\gamma h} d\text{vol}_g + \frac{a_n}{2} \mathfrak{p}_g(h, h).$$

The remainder of this section is devoted to give a rigorous meaning to the measure

$$\nu_g^*(dh) = \exp(-S_g(h))dh.$$

As an ansatz, we regard the quantity  $\exp(-\frac{a_n}{2}\mathfrak{p}_g(h, h))dh$  as an informal definition of the law of the ungrounded co-polyharmonic field. With this interpretation, we regard  $\int e^{\gamma h} d\text{vol}_g$  as the volume of  $M$  with respect to the Liouville Quantum Gravity measure. Since the latter comes in two versions –the plain and the adjusted Liouville measure– we obtain two conformally quasi-invariant rigorous definitions of the above measure, denoted henceforth by  $\nu_g^*$  and  $\bar{\nu}_g^*$ .

Before going into further details, let us have a naive look on the transformation property of our action functional under conformal changes. Choose  $\Theta^* = 0$ . Then, by a direct computation, we have that for all  $\varphi$  smooth and all  $h \in H^{n/2}$ :

$$\begin{aligned} S_{e^{2\varphi}g}\left(h - \frac{n}{\gamma}\varphi\right) &= S_g(h) + \left(\Theta - a_n \frac{n}{\gamma}\right) \mathfrak{p}_g(h, \varphi) \\ &\quad + \left(\frac{a_n n^2}{2\gamma^2} - \Theta \frac{n}{\gamma}\right) \mathfrak{p}_g(\varphi, \varphi) - \Theta \frac{n}{\gamma} \int \varphi Q_g d\text{vol}_g, \end{aligned}$$

where we used that  $Q_{e^{2\varphi}g} = e^{-n\varphi}(Q_g + P_g \varphi)$ . In particular, when selecting the special value  $\Theta = a_n \frac{n}{\gamma}$  the above expression simplifies to

$$S_{e^{2\varphi}g}\left(h - \frac{n}{\gamma}\varphi\right) = S_g(h) - \frac{a_n n^2}{2\gamma^2} \left[ \mathfrak{p}_g(\varphi, \varphi) + 2 \int \varphi Q_g d\text{vol}_g \right].$$

Therefore, writing  $T$  for the shift by  $h \mapsto h + \frac{n}{\gamma}\varphi$ , we expect the following quasi-conformal invariance:

$$\log \frac{dT_* \nu_{e^{2\varphi}g}^*}{d\nu_g^*}(h) = \frac{a_n n^2}{2\gamma^2} \left( \mathfrak{p}_g(\varphi, \varphi) + 2 \int \varphi Q_g d\text{vol}_g \right).$$

However, due to the rough nature of the object involved in the exact definition of  $\nu_*$ , the quasi-conformal invariance arises at a different value of  $\Theta$  (and/or  $\Theta^*$ ). Indeed, the renormalization of the adjusted (or plain) Liouville Quantum Gravity measure  $\bar{\mu}_g^h$  (or  $\mu_g^h$ , resp.) produces in an additional term which corresponds to quasi-invariance under the shift

$$h \mapsto h + \left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)\varphi$$

(or  $h \mapsto h + \frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar{\varphi}$  for some function  $\bar{\varphi}$  given in terms of  $\varphi$ ). We derive rigorous statements below. For convenience, we treat the two procedures –in spite of their similarity– for the ‘plain’ and ‘adjusted’ cases separately.

*Remark 5.13.* The approach involving the adjusted measure  $\bar{\nu}^*$  is similar to (and inspired by) that of [GRV19] in the case  $n = 2$ . Results concerning the plain measure  $\nu^*$  seem to be new even in the two-dimensional case.

### 5.3.2 The plain Polyakov-Liouville measure

Let us address the challenge of giving a rigorous meaning to

$$d\nu_g^*(h) = \frac{1}{Z_g} \exp\left(-\int (\Theta Q_g h + \Theta^* \langle h \rangle_g + m e^{\gamma h}) d\text{vol}_g\right) \exp\left(-\frac{a_n}{2} \mathfrak{p}_g(h, h)\right) dh$$

on manifolds of arbitrary even dimension. Assume for the sequel that  $|\gamma| < \sqrt{2n}$ , and let

$$\nu_g := \text{CGF}^{M,g}$$

denote the law of the the co-polyharmonic field, a (rigorously defined) probability measure on  $\mathfrak{D}'$ .

Furthermore, let

$$\widehat{\nu}_g := \widehat{\text{CGF}}^{M,g} \tag{101}$$

denote the (infinite) measure on  $\mathfrak{D}'$  introduced in Proposition 3.17 as the distribution of the ungrounded co-polyharmonic field on  $(M, g)$ . As outlined in Section 3 the latter admits a heuristic characterization as

$$d\widehat{\nu}_g(h) = \frac{1}{Z_g} \exp\left(-\frac{a_n}{2} \mathfrak{p}_g(h, h)\right) dh$$

with a suitable constant  $Z_g$ .

Proceeding as in the two-dimensional case, in terms of this measure, we define the measure

$$d\nu_g^*(h) := \exp\left(-\Theta \langle h | Q_g \rangle - \Theta^* \langle h \rangle_g - m \mu_{g,\gamma}^h(M)\right) d\widehat{\nu}_g(h) \tag{102}$$

on  $\mathfrak{D}'$  with associated *partition function*

$$Z_g^* := \int_{\mathfrak{D}'} d\nu_g^*(h),$$

where  $\Theta, \Theta^*, m, \gamma \in \mathbb{R}$  are parameters with  $m > 0$ ,  $0 < |\gamma| < \sqrt{2n}$ , and where  $\mu_{g,\gamma}^h$  denotes the plain Liouville Quantum Gravity measure on the  $n$ -dimensional manifold  $M$ . Moreover,  $\langle h | Q_g \rangle$  denotes the pairing between the random field  $h$  and the (scalar valued)  $Q$ -curvature, and  $\langle h \rangle_g = \frac{1}{\text{vol}_g(M)} \langle h | \mathbf{1} \rangle$  denotes the pairing between  $h$  and the constant function  $\mathbf{1}$ , normalized by the volume of  $M$ . Set  $Q(M) := Q(M, g)$ .

**Theorem 5.14.** *Assume that  $0 < \gamma < \sqrt{2n}$  and  $\Theta Q(M) + \Theta^* < 0$ . Then,  $\nu_g^*$  is a finite measure.*

*Proof.*

$$\begin{aligned}
Z_g^* &= \int_{\mathfrak{D}'} \exp \left( -\Theta \langle h | Q_g \rangle - \Theta^* \langle h \rangle_g - m \mu_{g,\gamma}^h(M) \right) d\hat{\nu}_g(h) \\
&= \int_{\mathfrak{D}'} \int_{\mathbb{R}} \exp \left( -\Theta \langle h | Q_g \rangle - a(\Theta Q(M) + \Theta^*) - m e^{\gamma a} \mu_{g,\gamma}^h(M) \right) da d\nu_g(h) \\
&\stackrel{(a)}{=} \int_{\mathfrak{D}'} e^{-\Theta \langle h | Q_g \rangle} \int_0^\infty \left( \frac{t}{m \mu_{g,\gamma}^h(M)} \right)^{-\frac{\Theta Q(M) + \Theta^*}{\gamma}} e^{-t} \frac{dt}{\gamma t} d\nu_g(h) \\
&\stackrel{(b)}{=} \frac{1}{\gamma} \Gamma \left( -\frac{\Theta Q(M) + \Theta^*}{\gamma} \right) \cdot \int_{\mathfrak{D}'} e^{-\Theta \langle h | Q_g \rangle} (m \mu_{g,\gamma}^h(M))^{\frac{\Theta Q(M) + \Theta^*}{\gamma}} d\nu_g(h) .
\end{aligned}$$

Here (a) follows by change of variables  $a \mapsto t := m e^{\gamma a} \mu_{g,\gamma}^h(M)$ , and (b) by the very definition of Euler's  $\Gamma$  function. The final integral then can be estimated according to

$$\begin{aligned}
&\int_{\mathfrak{D}'} e^{-\Theta \langle h | Q_g \rangle} (m \mu_{g,\gamma}^h(M))^{\frac{\Theta Q(M) + \Theta^*}{\gamma}} d\nu_g(h) \\
&\leq \left( \int_{\mathfrak{D}'} e^{-2\Theta \langle h | Q_g \rangle} d\nu_g(h) \right)^{1/2} \cdot \left( \int_{\mathfrak{D}'} (m \mu_{g,\gamma}^h(M))^{\frac{2(\Theta Q(M) + \Theta^*)}{\gamma}} d\nu_g(h) \right)^{1/2} .
\end{aligned}$$

The finiteness of the first term on the right-hand side is obvious by the defining property of  $\nu_g$ :

$$\int_{\mathfrak{D}'} e^{-2\Theta \langle h | Q_g \rangle} d\nu_g(h) = e^{2\Theta^2 \mathfrak{t}_g(Q_g, Q_g)} .$$

The finiteness of  $\int_{\mathfrak{D}'} \mu_{g,\gamma}^h(M)^{\frac{2(\Theta Q(M) + \Theta^*)}{\gamma}} d\nu_g(h)$  for  $\frac{\Theta Q(M) + \Theta^*}{\gamma} < 0$  follows from Theorem 4.1 (iii).  $\square$

*Remark 5.15.* Assuming that  $\Theta$  is positive, the finiteness assumption  $\Theta Q(M) + \Theta^* < 0$  in the above theorem is equivalent to saying that the constant  $-\Theta^*/\Theta$  is larger than the total  $Q$ -curvature.

**Definition 5.16.** For every admissible manifold and every choice of parameters  $m$ ,  $\Theta$ ,  $\Theta^*$ ,  $\gamma$  as above, the plain Polyakov–Liouville measure

$$\nu_g^\sharp := \frac{1}{Z_g^*} \nu_g^*$$

is a well-defined probability measure on  $\mathfrak{D}'$ .

**Theorem 5.17.** Assume that  $0 < \gamma < \sqrt{2n}$ ,  $\Theta = a_n \frac{n}{\gamma}$ , and  $\Theta^* = \gamma$ . Then,  $\nu_g^*$  is conformally quasi-invariant modulo shift in the following sense:

$$\nu_{e^{2\varphi}g}^* = Z(g, \varphi) \cdot T_* \nu_g^* , \quad \varphi \in \mathfrak{D} , \quad (103)$$

where  $T_*$  denotes the push forward under the shift  $T : h \mapsto e^{n\varphi} (h - (\frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar{\varphi}) \text{vol}_g)$  on  $\mathfrak{D}'$  with  $\bar{\varphi}$  defined as in (71).

The conformal anomaly  $Z(g, \varphi)$  is given as

$$Z(g, \varphi) := \exp \left[ \Theta \int \left( \frac{n}{\gamma} \varphi + \frac{\gamma}{2} \bar{\varphi} \right) Q_g d\text{vol}_g + n \langle \varphi \rangle_{g'} + \frac{a_n n^2}{2 \gamma^2} \mathfrak{p}_g(\varphi, \varphi) \right] . \quad (104)$$

*Proof.* For the sake of brevity let us write

$$S_g(h) = \Theta \langle h | Q_g \rangle + \Theta^* \langle h \rangle_g + m \mu_{g,\gamma}^h(M) .$$

We also set  $\Phi := (\frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar{\varphi}) \in \mathfrak{D}$ . Let  $F : \mathfrak{D}' \rightarrow \mathbb{R}_+$  measurable. Then, by Girsanov Theorem (Corollary 3.18) for  $\hat{\nu}$ , we find that

$$\begin{aligned}
\int_{\mathfrak{D}'} F d\nu_{g'}^* &= \int F (h - \Phi \text{vol}_{g'}) \exp(-S_{g'}(h - \Phi \text{vol}_{g'})) \\
&\quad \cdot \exp \left( a_n \langle h | P_{g'} \Phi \rangle - \frac{a_n}{2} \mathfrak{p}_{g'}(\Phi, \Phi) \right) d\hat{\nu}_{g'}(h) .
\end{aligned}$$

By Corollary 4.19 and Theorem 4.1(i), we have that

$$\begin{aligned} \int_{\mathfrak{D}'} F d\nu_{g'}^* &= \int F(e^{n\varphi}(h - \Phi \text{vol}_g)) \exp(-\Theta \langle e^{n\varphi}(h - \Phi \text{vol}_g) | Q_{g'} \rangle - m\mu_{g,\gamma}^h(M)) \\ &\quad \cdot \exp\left(-\frac{\Theta^*}{\text{vol}_{g'}(M)} \langle e^{n\varphi}(h - \Phi \text{vol}_g) | \mathbf{1} \rangle\right) \\ &\quad \cdot \exp\left(a_n \langle e^{n\varphi} h | P_{g'} \Phi \rangle - \frac{a_n}{2} \mathfrak{p}_{g'}(\Phi, \Phi)\right) \hat{\nu}_g(dh) . \end{aligned}$$

Now recall that  $P_{g'} u = e^{-n\varphi} P_g u$ , that  $\mathfrak{p}_g$  is conformally invariant, and that, by Proposition 21,  $Q_{g'} = e^{-n\varphi}(Q_g + P_g \varphi)$ . Thus, we obtain

$$\begin{aligned} \int_{\mathfrak{D}'} F d\nu_{g'}^* &= \int F(e^{n\varphi}(h - \Phi \text{vol}_g)) \exp(-\Theta \langle h - \Phi \text{vol}_g | Q_g \rangle - m\mu_{g,\gamma}^h(M)) \\ &\quad \cdot \exp\left(-\frac{\Theta^*}{\text{vol}_{g'}(M)} \langle e^{n\varphi}(h - \Phi \text{vol}_g) | \mathbf{1} \rangle\right) \\ &\quad \cdot \exp\left(-\Theta \langle h - \Phi \text{vol}_g | P_g \varphi \rangle + a_n \langle h | P_g \Phi \rangle - \frac{a_n}{2} \mathfrak{p}_g(\Phi, \Phi)\right) d\hat{\nu}_g(h) . \end{aligned}$$

Since we have chosen  $\Theta = a_n \frac{n}{\gamma}$  some of the terms in the last line cancel out and we get:

$$\begin{aligned} \int_{\mathfrak{D}'} F d\nu_{g'}^* &= \int F(e^{n\varphi}(h - \Phi \text{vol}_g)) \exp(-\Theta \langle h - \Phi \text{vol}_g | Q_g \rangle - m\mu_{g,\gamma}^h(M)) \\ &\quad \exp\left(-\frac{\Theta^*}{\text{vol}_{g'}(M)} \langle e^{n\varphi}(h - \Phi \text{vol}_g) | \mathbf{1} \rangle\right) \\ &\quad \exp\left(a_n \frac{\gamma}{2} \langle h | P_g \bar{\varphi} \rangle + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) - \frac{a_n}{2} \frac{\gamma^2}{4} \mathfrak{p}_g(\bar{\varphi}, \bar{\varphi})\right) d\hat{\nu}_g(h) . \end{aligned}$$

Now by definition of  $\bar{\varphi}$ , we get

$$a_n P_g \bar{\varphi} = \frac{2}{\text{vol}_{g'}(M)} \pi_g(e^{n\varphi}) = \frac{2}{\text{vol}_{g'}(M)} e^{n\varphi} - \frac{2}{\text{vol}_g(M)} .$$

In particular, for every  $h \in \mathfrak{D}'$ ,

$$a_n \langle h | P_g \bar{\varphi} \rangle = 2 \langle h \rangle_{g'} - 2 \langle h \rangle_g . \quad (105)$$

As a consequence,

$$\begin{aligned} \int_{\mathfrak{D}'} F d\nu_{g'}^* &= \int F(e^{n\varphi}(h - \Phi \text{vol}_g)) \exp\left(-\Theta \langle h | Q_g \rangle + \Theta \int \Phi Q_g d\text{vol}_g - m\mu_{g,\gamma}^h(M)\right) \\ &\quad \exp\left(-\Theta^* \langle h \rangle_{g'} + \Theta^* \langle \Phi \rangle_{g'}\right) \\ &\quad \exp\left(\gamma(\langle h \rangle_{g'} - \langle h \rangle_g) + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) - \frac{a_n}{2} \frac{\gamma^2}{4} \mathfrak{p}_g(\bar{\varphi}, \bar{\varphi})\right) d\hat{\nu}_g(h) . \end{aligned}$$

Thus the choice of  $\Theta^* = \gamma$ , after cancellations and rearrangement, yields

$$\begin{aligned} \int_{\mathfrak{D}'} F d\nu_{g'}^* &= \int F(e^{n\varphi}(h - \Phi \text{vol}_g)) \exp\left(-\Theta \langle h | Q_g \rangle - \Theta^* \langle h \rangle_g - m\mu_{g,\gamma}^h(M)\right) d\hat{\nu}_g(h) \\ &\quad \cdot \exp\left(\int \Phi \left(\Theta Q_g + \Theta^* \frac{e^{n\varphi}}{\text{vol}_{g'}(M)}\right) d\text{vol}_g + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) - \frac{a_n}{2} \frac{\gamma^2}{4} \mathfrak{p}_g(\bar{\varphi}, \bar{\varphi})\right) . \end{aligned} \quad (106)$$

Again in light of (105) we further have that

$$\frac{a_n}{2} \mathfrak{p}_g(\bar{\varphi}, \bar{\varphi}) = \langle \bar{\varphi} \rangle_{g'} - \langle \bar{\varphi} \rangle_g .$$

Substituting the definitions of  $\Theta^* := \gamma$  and  $\Phi := (\frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar{\varphi})$  then yields

$$\begin{aligned} & \int \Phi \left( \Theta Q_g + \Theta^* \frac{e^{n\varphi}}{\text{vol}_{g'}(M)} \right) d\text{vol}_g + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) - \frac{a_n}{2} \frac{\gamma^2}{4} \mathfrak{p}_g(\bar{\varphi}, \bar{\varphi}) = \\ & = \Theta \int \Phi Q_g d\text{vol}_g + n \langle \varphi \rangle_{g'} + \frac{\gamma^2}{4} \left( \langle \bar{\varphi} \rangle_{g'} + \langle \bar{\varphi} \rangle_g \right) + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) . \end{aligned}$$

Finally, by definition (71) of  $\bar{\varphi}$ , and since  $\mathbf{k}_g(e^{n\varphi}) \in \mathring{L}^2(\text{vol}_g)$ ,

$$\begin{aligned} \langle \bar{\varphi} \rangle_{g'} + \langle \bar{\varphi} \rangle_g &= \frac{2}{\text{vol}_{g'}(M)^2} \int_M e^{n\varphi} \mathbf{k}_g(e^{n\varphi}) d\text{vol}_g - \frac{1}{\text{vol}_{g'}(M)^2} \mathfrak{k}_g(e^{n\varphi}, e^{n\varphi}) \\ &+ \frac{2}{\text{vol}_{g'}(M) \text{vol}_g(M)} \int_M \mathbf{k}_g(e^{n\varphi}) d\text{vol}_g - \frac{1}{\text{vol}_{g'}(M)^2} \mathfrak{k}_g(e^{n\varphi}, e^{n\varphi}) \\ &= \frac{2}{\text{vol}_{g'}(M)^2} \mathfrak{k}_g(e^{n\varphi}, e^{n\varphi}) - \frac{1}{\text{vol}_{g'}(M)^2} \mathfrak{k}_g(e^{n\varphi}, e^{n\varphi}) \\ &+ 0 - \frac{1}{\text{vol}_{g'}(M)^2} \mathfrak{k}_g(e^{n\varphi}, e^{n\varphi}) \\ &= 0 , \end{aligned}$$

and therefore

$$\begin{aligned} & \int \Phi \left( \Theta Q_g + \Theta^* \frac{e^{n\varphi}}{\text{vol}_{g'}(M)} \right) d\text{vol}_g + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) - \frac{a_n}{2} \frac{\gamma^2}{4} \mathfrak{p}_g(\bar{\varphi}, \bar{\varphi}) = \\ & = \Theta \int \Phi Q_g d\text{vol}_g + n \langle \varphi \rangle_{g'} + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) . \end{aligned} \quad (107)$$

Substituting (107) into (106), we finally have that

$$\begin{aligned} \int_{\mathfrak{D}'} F d\nu_{g'}^* &= \int F(e^{n\varphi}(h - \Phi \text{vol}_g)) \exp \left( -\Theta \langle h | Q_g \rangle - \Theta^* \langle h \rangle_g - m \mu_{g,\gamma}^h(M) \right) d\hat{\nu}_g(h) \\ &\cdot \exp \left( \Theta \int \Phi Q_g d\text{vol}_g + n \langle \varphi \rangle_{g'} + \frac{a_n}{2} \frac{n^2}{\gamma^2} \mathfrak{p}_g(\varphi, \varphi) \right) . \end{aligned}$$

This concludes the proof of the conformal quasi-invariance. To conclude for the expression of  $Z(g, \varphi)$  we take  $F = \mathbf{1}$ .  $\square$

**Corollary 5.18.** *Assume that  $\Theta = a_n \frac{n}{\gamma}$ ,  $\Theta^* = \gamma$ , and  $\gamma^2 < -n a_n Q(M)$ . Then,  $Z(g, \varphi) = \frac{Z_{g'}^*}{Z_g^*}$ , and  $\nu_g^\sharp$  is conformally invariant modulo shift:*

$$\nu_{e^{2\varphi}g}^\sharp = T_* \nu_g^\sharp \quad (108)$$

with  $T : h \mapsto e^{n\varphi}(h - (n\varphi/\gamma + \gamma\bar{\varphi}/2)\text{vol}_g)$ .

*Proof.* We have

$$\nu_{g'}^\sharp = \frac{\nu_{g'}^*}{Z_{g'}^*} = \frac{Z(g, \varphi)}{Z_g^*} \cdot T_* \nu_g^* = \frac{Z_g^*}{Z_{g'}^*} Z(g, \varphi) \cdot T_* \nu_g^\sharp = T_* \nu_g^\sharp . \quad \square$$

*Remark 5.19.* With the choices  $\Theta := \frac{n a_n}{\gamma}$  and  $\Theta^* := \gamma$  from above, the condition  $\Theta Q(M) + \Theta^* < 0$  reads as  $\frac{a_n n}{\gamma^2} Q(M) + 1 < 0$  or, in other words,

$$\gamma^2 < -n a_n Q(M) . \quad (109)$$

*Remark 5.20.* In view of Corollaries 3.14 and 4.18, the (quasi-)invariance assertion in the previous Theorem 5.17 and Corollary 5.18 also holds under the more general class of conformal transformations in the sense of Definition 1.1 (ii). In particular, the plain Polyakov–Liouville measure is invariant under isometric transformations  $\Phi : M \rightarrow M'$ .

### 5.3.3 The adjusted Polyakov–Liouville measures

As anticipated, our results for the plain Polyakov–Liouville measure can also be recasted in the setting of the adjusted Polyakov–Liouville measure. Let us set

$$d\bar{\nu}_g^*(h) = \exp(-\Theta \langle h | Q_g \rangle - m\bar{\mu}_{g,\gamma}^h(M)) d\widehat{\nu}_g(h) ,$$

which corresponds to the *adjusted Polyakov–Liouville measure*. The associated *partition function* is

$$\bar{Z}_g^* := \int_{\mathfrak{D}'} d\bar{\nu}_g^*(h) .$$

As for the plain measure, we have the following result.

**Theorem 5.21.** *Assume that  $0 < \gamma < \sqrt{2n}$  and  $\Theta Q(M) < 0$ . Then,  $\bar{\nu}_g^*$  is a finite measure.*

*Proof.* The proof is the same as in Theorem 5.14, simply remarking that Theorem 4.1 (iii) also applies for  $\bar{\mu}^h$  instead of  $\mu^h$ .  $\square$

**Theorem 5.22.** *Assume that  $0 < \gamma < \sqrt{2n}$  and that  $\Theta = a_n(\frac{n}{\gamma} + \frac{\gamma}{2})$ . Let  $\varphi$  be smooth and  $g' = e^{2\varphi}g$ . Then,  $\bar{\nu}^*$  is conformally quasi-invariant under the shift  $T: h \mapsto e^{n\varphi}(h - \Theta\varphi\text{vol}_g)$ , viz.*

$$\bar{\nu}_{g'}^* = \bar{Z}(g, \varphi) \cdot T_* \bar{\nu}_g^* ,$$

where

$$\bar{Z}(g, \varphi) = \exp\left(\frac{\Theta^2}{2a_n} \left[ \mathfrak{p}_g(\varphi, \varphi) + 2 \int \varphi Q_g d\text{vol}_g \right]\right) .$$

*Proof.* Let  $F: \mathfrak{D}' \rightarrow \mathbb{R}_+$  measurable. Write  $\Phi_g = (\frac{n}{\gamma} + \frac{\gamma}{2})\varphi\text{vol}_g \in \mathfrak{D}'$ . By Girsanov's theorem for  $\widehat{\nu}$  (Corollary 3.18), we have:

$$\begin{aligned} \int_{\mathfrak{D}'} F d\bar{\nu}_{g'}^* &= \int F(h - \Phi_{g'}) \exp\left(-\Theta \langle h - \Phi_{g'} | Q_{g'} \rangle - m\bar{\mu}_{g,\gamma}^{h-\Phi_{g'}}(M)\right) \\ &\quad \exp\left(a_n \left\langle h \left| P_{g'} \frac{\Theta}{a_n} \varphi \right. \right\rangle - \frac{a_n}{2} \mathfrak{p}_{g'} \left( \frac{\Theta}{a_n} \varphi, \frac{\Theta}{a_n} \varphi \right)\right) d\widehat{\nu}_{g'}(h) . \end{aligned}$$

In view of Theorems 3.17 and 4.31, we thus get

$$\begin{aligned} \int_{\mathfrak{D}'} F d\bar{\nu}_{g'}^* &= \int F(e^{n\varphi}(h - \Phi_g)) \exp\left(-\Theta \langle e^{n\varphi}(h - \Phi_g) | e^{-n\varphi}(Q_g + P_g \varphi) \rangle - m\bar{\mu}_{g,\gamma}^h(M)\right) \\ &\quad \exp\left(\Theta \langle e^{n\varphi}h | e^{-n\varphi} P_g \varphi \rangle - \frac{\Theta^2}{2a_n} \mathfrak{p}_g(\varphi, \varphi)\right) d\widehat{\nu}_g(h) . \end{aligned}$$

Expanding  $\langle h - \Phi_g | Q_g + P_g \varphi \rangle$  cancels out with some term on the second line and we obtain the announced result.  $\square$

**Corollary 5.23.** *Assume  $Q(M) < 0$  and set  $\bar{\nu}_g^\sharp := \frac{1}{Z_*} \bar{\nu}_g^*$ . Then, with  $\Theta$  and  $T$  as above,*

$$\bar{\nu}_{g'}^\sharp = T_* \bar{\nu}_g^\sharp .$$

### 5.3.4 Some examples

The above assertions impose two conditions on a given manifold  $(M, g)$ : positivity of the co-polyharmonic operator  $P_g$  and negativity of the total  $Q$ -curvature  $Q(M)$ .

Let us present some examples of such manifolds.

*Example 5.24* ( $n = 2$ ). Every compact Riemannian surface of genus  $\geq 2$  satisfies both of these conditions.

*Example 5.25* ( $n = 2, 6, 10, \dots$ ). Every compact hyperbolic Riemannian manifold of dimension  $n = 4\ell + 2$  for some  $\ell \in \mathbb{N}$  and with  $\lambda_1 > \frac{n(n-2)}{4}$  satisfies both of these conditions.



*Proof.* Combine Proposition 2.5 and Example 1.14. □

*Example 5.26* ( $n = 4$ ). Let  $M = M_1 \times M_2$  where  $M_1$  and  $M_2$  are compact Riemannian surfaces of constant curvature  $k_1$  and  $k_2$ , resp.

(i) Then,  $Q_g < 0$  if and only if

$$|k_1 + k_2| < \sqrt{3} \cdot |k_1 - k_2| .$$

(ii) Furthermore,  $P_g > 0$  on  $\mathring{H}$  if  $k_1 + k_2 \geq 0$ .

*Proof.* (i) According to Example 1.14 (ii),

$$Q_g = -k_1^2 - k_2^2 + \frac{2}{3}(k_1 + k_2)^2 = -\frac{1}{2}(k_1 - k_2)^2 + \frac{1}{6}(k_1 + k_2)^2 .$$

(ii) For  $i = 1, 2$ , denote by  $P_i = -\Delta_i$  the negative of the Laplacian on the manifold  $M_i$ . Then, by Proposition 1.5 (ii),

$$\begin{aligned} P_g &= (P_1 + P_2)^2 - 2k_1 P_1 - 2k_2 P_2 + \frac{4}{3}(k_1 + k_2)(P_1 + P_2) \\ &= P_1(P_1 - 2k_1) + P_2(P_2 - 2k_2) + 2P_1 P_2 + \frac{4}{3}(k_1 + k_2)(P_1 + P_2) \geq 0 \end{aligned}$$

according to the Lichnerowicz estimate  $P_i \geq 2k_i$  for  $i = 1, 2$  (which is valid independent of the sign of  $k_i$ ). Indeed,  $P_g$  is positive since the term  $2P_1 P_2$  is positive on the grounded  $L^2$ -space. □

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