A DISCOVERY TOUR IN RANDOM RIEMANNIAN GEOMETRY

By Lorenzo Dello Schiavo[†]
Eva Kopfer[‡]
Karl-Theodor Sturm[‡]

[†]*Institute of Science and Technology Austria*
[‡]*Institut für angewandte Mathematik*
Rheinische Friedrich-Wilhelms-Universität Bonn

October 26, 2022

Abstract. We study random perturbations of a Riemannian manifold (M,g) by means of so-called *Fractional Gaussian Fields*, which are defined intrinsically by the given manifold. The fields $h^\bullet:\omega\mapsto h^\omega$ will act on the manifold via the conformal transformation $\mathsf{g}\mapsto\mathsf{g}^\omega:=e^{2h^\omega}\,\mathsf{g}$. Our focus will be on the regular case with Hurst parameter $H>0$, the critical case $H=0$ being the celebrated Liouville geometry in two dimensions. We want to understand how basic geometric and functional-analytic quantities like: diameter, volume, heat kernel, Brownian motion, spectral bound, or spectral gap change under the influence of the noise. And if so, is it possible to quantify these dependencies in terms of key parameters of the noise? Another goal is to define and analyze in detail the Fractional Gaussian Fields on a general Riemannian manifold, a fascinating object of independent interest.

CONTENTS

1 Introduction . . . 2
  1.1 Random Riemannian Geometry . . . 2
  1.2 Fractional Gaussian Field (FGF) . . . 4
  1.3 Higher Order Green Kernel . . . 5
Acknowledgements . . . 6
2 The Riemannian Manifold . . . 6
  2.1 Higher Order Green Operators . . . 7
  2.2 The case of manifolds of bounded geometry . . . 7
    2.2.1 Bessel potential spaces . . . 7
    2.2.2 Standard Sobolev spaces . . . 8
    2.2.3 Manifolds of bounded geometry . . . 8
    2.2.4 Test functions . . . 9
    2.2.5 Heat-kernel estimates . . . 9
  2.3 The case of closed manifolds . . . 10
    2.3.1 Grounding . . . 11

[*]Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

[†]This author gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft through CRC 1060 as well as through SPP 2265, and by the Austrian Science Fund (FWF) grant F65 at Institute of Science and Technology Austria. He also acknowledges funding of his current position by the Austrian Science Fund (FWF) through the ESPRIT Programme (grant No. 208).

[‡]This author gratefully acknowledges funding by the Deutsche Forschungsgemeinschaft through the Hausdorff Center for Mathematics and through CRC 1060 as well as through SPP 2265.

Primary: 60G15, secondary: 58J65, 31C25.

Keywords and phrases: Fractional Gaussian Fields, Gaussian Free Field, random geometry, Liouville Quantum Gravity, Liouville Brownian Motion, spectral gap estimates.

2.3.2 Eigenfunction expansion . . . 14
2.3.3 Sobolev spaces on compact manifolds . . . 14
2.4 The noise distance . . . 15
3 The Fractional Gaussian Field . . . 16
3.1 Some characterizations . . . 16
3.2 Continuity of the FGF . . . 18
3.3 Series Expansions in the Compact Case . . . 20
3.4 The Grounded FGF . . . 22
3.5 Dudley's Estimate . . . 23
4 Random Riemannian Geometry . . . 24
4.1 Random Dirichlet Forms and Random Brownian Motions . . . 24
4.2 Random Brownian Motions in the $\mathcal{C}^1$-Case . . . 26
5 Geometric and Functional Inequalities for RRG's . . . 28
5.1 Volume, Length, and Distance . . . 29
5.2 Spectral Bound . . . 30
5.3 Spectral Gap . . . 31
6 Higher-Order Green Kernels — Asymptotics and Examples . . . 33
6.1 Green Kernel Asymptotics . . . 33
6.2 Supremum estimates . . . 36
6.3 Examples . . . 37
6.3.1 Euclidean space . . . 37
6.3.2 Torus . . . 38
6.3.3 Hyperbolic Space . . . 41
6.3.4 Sphere . . . 41
References . . . 43

1. Introduction.

1.1. *Random Riemannian Geometry.* Given a Riemannian manifold (M,g) and a Gaussian random field $h^\bullet:\Omega\to\mathcal{C}(\mathsf{M})$, $\omega\mapsto h^\omega$, we study random perturbations $(\mathsf{M},\mathsf{g}^\omega)$ of the given manifold with conformally changed metric tensors $\mathsf{g}^\omega := e^{2h^\omega}\mathsf{g}$. For this *Random Riemannian Geometry*

$$(\mathsf{M},\mathsf{g}^\bullet) \quad \text{with} \quad \mathsf{g}^\bullet := e^{2h^\bullet}\mathsf{g}$$

we want to understand how basic geometric and functional analytic quantities like diameter, volume, heat kernel, Brownian motion, or spectral gap change under the influence of the noise. If possible, we want to quantify these dependencies in terms of key parameters of the noise.

Our main interest in the sequel will be in the case $h^\bullet\notin\mathcal{C}^2(\mathsf{M})$ a.s., where standard Riemannian calculus is not directly applicable and where no classical curvature concepts are at our disposal. Our approach to geometry, spectral analysis, and stochastic calculus on the randomly perturbed Riemannian manifolds $(\mathsf{M},\mathsf{g}^\bullet)$ will be based on Dirichlet form techniques.

For convenience, we will assume throughout that the reference manifold (M,g) has bounded geometry.

THEOREM 1.1. *For every ω, a regular, strongly local Dirichlet form is given by*

$$\mathcal{E}^\omega(\varphi,\psi)=\frac{1}{2}\int_{\mathsf{M}}\langle\nabla\varphi\,|\,\nabla\psi\rangle_{\mathsf{g}}\ e^{(n-2)h^\omega}\,\mathrm{dvol}_{\mathsf{g}} \qquad on \quad L^2\big(\mathsf{M},e^{nh^\omega}\mathrm{vol}_{\mathsf{g}}\big)\ . \tag{1.1}$$

The associated *Laplace–Beltrami operator* $\big(\Delta^\omega,\mathcal{D}(\Delta^\omega)\big)$ on $(\mathsf{M},\mathsf{g}^\omega)$ is uniquely characterized by $\mathcal{D}(\Delta^\omega)\subset\mathcal{D}(\mathcal{E}^\omega)$ and $\mathcal{E}^\omega(\varphi,\psi)=-\frac{1}{2}\int(\Delta^\omega\varphi)\ \psi\, e^{nh^\omega}\mathrm{dvol}_{\mathsf{g}}$ for $\varphi\in\mathcal{D}(\Delta^\omega)$, $\psi\in\mathcal{D}(\mathcal{E}^\omega)$.

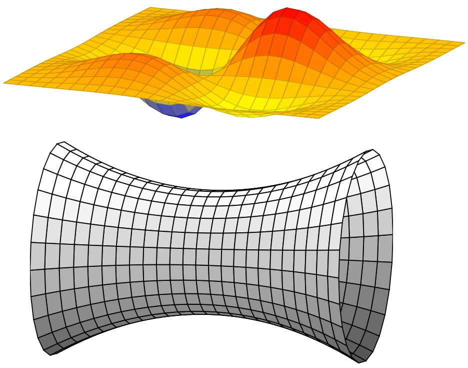


Fig 1: Gaussian random field over a toroid.

The associated *Riemannian metric* is given by

$$\mathsf{d}^\omega(x,y) := \inf\left\{\int_0^1 e^{h^\omega(\gamma_r)}\,|\dot\gamma_r|\,\mathrm{d}r : \ \gamma\in\mathcal{AC}\big([0,1];\mathsf{M}\big)\ ,\ \gamma_0=x\ ,\ \gamma_1=y\right\}\ ,$$

where $|\dot\gamma_r| := \sqrt{\mathsf{g}(\dot\gamma_r,\dot\gamma_r)}$ denotes the speed of an absolutely continuous curve γ_r.

Proposition 1.2. *The heat semigroup $\big(e^{t\Delta^\omega/2}\big)_{t>0}$ has an integral kernel $p^\omega_t(x,y)$ which is jointly locally Hölder continuous in t,x,y.*

The *Brownian motion on* $(\mathsf{M},\mathsf{g}^\omega)$, defined as the reversible, Markov diffusion process $\mathbf{B}^\omega$ associated with the heat semigroup $\big(e^{t\Delta^\omega/2}\big)_{t>0}$, allows for a more explicit construction if the conformal weight h^ω is differentiable.

Proposition 1.3. *If $h^\omega\in\mathcal{C}^1(\mathsf{M})$ then $\mathbf{B}^\omega$ is obtained from the Brownian motion $\mathbf{B}$ on (M,g) by the combination of time change with weight e^{2h^ω} and Girsanov transformation with weight $(n-2)h^\omega$.*

We will compare the random volume, random length, and random distance in the random Riemannian manifold $(\mathsf{M},\mathsf{g}^\bullet)$ with analogous quantities in deterministic geometries obtained by suitable conformal weights.

Proposition 1.4. *Put $\theta(x):=\mathbf{E}[h^\bullet(x)^2]\ \geq\ 0$ and $\bar{\mathsf{g}}^n := e^{n\,\theta}\mathsf{g}$, $\bar{\mathsf{g}}^1 := e^{\theta}\mathsf{g}$. Then for every measurable $A\subset\mathsf{M}$,*

$$\mathbf{E}[\mathrm{vol}_{\mathsf{g}^\bullet}(A)] = \mathrm{vol}_{\bar{\mathsf{g}}^n}(A) \geq \mathrm{vol}_{\mathsf{g}}(A)\ ,$$

and for every absolutely continuous curve $\gamma:[0,1]\to\mathsf{M}$,

$$\mathbf{E}[L_{\mathsf{g}^\bullet}(\gamma)] = L_{\bar{\mathsf{g}}^1}(\gamma) \geq L_{\mathsf{g}}(\gamma)\ .$$

Of particular interest is the rate of convergence to equilibrium for the random Brownian motion.

Theorem 1.5. *Assume that M is compact. Let λ^1 be the spectral gap of Δ, and for each ω, denote by λ_1^ω the spectral gap of Δ^ω. Then*

$$\mathbf{E}\Big[\big|\log\lambda_1^\bullet - \log\lambda_1\big|\Big] \leq \alpha\,\mathbf{E}\Big[\sup|h^\bullet|\Big] \tag{1.2}$$

with $\alpha := 2(n-1)$ if $n\geq 2$ and $\alpha:=2$ if $n=1$.

Let us emphasize that classical estimates for the spectral gap, based on Ricci curvature estimates, require that the metric tensor is of class $\mathcal{C}^2$, whereas our Theorem 1.5 — combined with Theorem 1.9 below — will apply whenever the random metric tensor is of class $\mathcal{C}^0$.

1.2. *Fractional Gaussian Field (FGF).* In our approach to Random Riemannian Geometry, we will restrict ourselves to the case where the random field $h^\bullet$ is a Fractional Gaussian Field, defined intrinsically by the given manifold. It is a fascinating object of independent interest.

Given a Riemannian manifold (M,g) of bounded geometry, for $m>0$ and $s\in\mathbb{R}$, we define the Sobolev spaces

$$H^s_m(\mathsf{M}) \coloneqq \left(m^2-\tfrac{1}{2}\Delta\right)^{-s/2}\left(L^2(\mathsf{M})\right) , \qquad \|u\|_{H^s_m} \coloneqq \left\| \left(m^2-\tfrac{1}{2}\Delta\right)^{s/2} u \right\|_{L^2} .$$

The scalar product $\langle u\,|\,v\rangle_{L^2}$ extends to a continuous bilinear pairing between $H^s_m(\mathsf{M})$ and $H^{-s}_m(\mathsf{M})$ as well as between $\mathscr{D}(\mathsf{M})$ and $\mathscr{D}'(\mathsf{M})$. It follows, that the functional $u\mapsto \exp\left(-\tfrac{1}{2}\|u\|^2_{H^{-s}_m}\right)$ is continuous on $\mathscr{D}(\mathsf{M})$, and is therefore the Fourier transform of a unique centered Gaussian field with variance $\|u\|^2_{H^{-s}_m}$ by Bochner–Minlos Theorem applied to the nuclear space $\mathscr{D}'(\mathsf{M})$.

THEOREM 1.6. *For every $s\in\mathbb{R}$ and $m>0$, there exists a unique centered Gaussian field $h^\bullet$ with*

$$\mathbf{E}\, e^{\mathrm{i}\,\langle u\,|\,h^\bullet\rangle} = e^{-\frac{1}{2}\|u\|^2_{H^{-s}_m}} , \qquad u\in\mathscr{D}(\mathsf{M}) , \tag{1.3}$$

called m-massive Fractional Gaussian Field on M of regularity s*, briefly* $\mathsf{FGF}^{\mathsf{M}}_{s,m}$.

For $s=0$ this is the white noise on M. Note that, if $h^\bullet$ is distributed according to $\mathsf{FGF}^{\mathsf{M}}_{s,m}$ on some compact M, then $\left(m^2-\tfrac{1}{2}\Delta\right)^{\frac{r-s}{2}} h^\bullet$ is distributed according to $\mathsf{FGF}^{\mathsf{M}}_{r,m}$.

THEOREM 1.7. *For $s>0$, the Fractional Gaussian Field $\mathsf{FGF}^{\mathsf{M}}_{s,m}$ is uniquely characterized as the centered Gaussian process $h^\bullet$ with covariance*

$$\mathrm{Cov}\left[\,\langle h^\bullet\,|\,\varphi\rangle\,,\langle h^\bullet\,|\,\psi\rangle\,\right] = \iint \varphi(x)\,G_{s,m}(x,y)\,\psi(y)\,\mathrm{dvol}^{\otimes 2}_{\mathsf{g}}(x,y) , \qquad \varphi,\psi\in\mathscr{D}\subset H^{-s}_m , \tag{1.4}$$

where $G_{s,m}(x,y) \coloneqq \frac{1}{\Gamma(s)}\int_0^\infty p_t(x,y)\,e^{-m^2t}\,t^{s-1}\,\mathrm{d}t$*. For $s>n/2$, this characterization simplifies to*

$$\mathbf{E}\left[h^\bullet(x)\,h^\bullet(y)\right] = G_{s,m}(x,y) , \qquad x,y\in\mathsf{M} . \tag{1.5}$$

Indeed, for $s>n/2$, the Fractional Gaussian Field $\mathsf{FGF}^{\mathsf{M}}_{s,m}$ is almost surely a continuous function. More precisely,

PROPOSITION 1.8. *Assume M is compact and let $h^\bullet\sim\mathsf{FGF}^{\mathsf{M}}_{s,m}$ with $s>n/2+k$, $k\in\mathbb{N}_0$. Then $h^\omega\in\mathcal{C}^k(\mathsf{M})$ for a.e. ω.*

A crucial role in our geometric estimates and functional inequalities for the Random Riemannian Geometry is played by estimates for the expected maximum of the random field.

THEOREM 1.9. *For every compact manifold M there exists a constant $C=C(\mathsf{M})$ such that for $h^\bullet\sim\mathsf{FGF}^{\mathsf{M}}_{s,m}$ with any $m>0$,*

$$\mathbf{E}\left[\sup_{x\in\mathsf{M}} h^\bullet(x)\right] \leq \begin{cases} C\cdot(\lambda_1/2)^{-s/2}, & s\geq\frac{n}{2}+1 , \\ C\cdot(s-n/2)^{-3/2}, & s\in\left(\frac{n}{2},\frac{n}{2}+1\right] . \end{cases}$$

If M is compact, then an analogous construction also works in the case $m = 0$ provided all function spaces H_m^{-s} are replaced by the subspaces $\mathring{H}_m^{-s}$ obtained via the *grounding* map $u \mapsto \mathring{u} := u - \frac{1}{\mathrm{vol}_{\mathsf{g}}(\mathsf{M})} \int u \mathrm{dvol}_{\mathsf{g}}$. The $\mathring{\mathsf{FGF}}^{\mathsf{M}}_{s,m}$ for $s = 1, m = 0$ is the celebrated *Gaussian Free Field* (GFF) on M.

In the compact case, the Fractional Gaussian Field also admits a quite instructive series representation.

THEOREM 1.10. *Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a complete orthonormal basis in L^2 consisting of eigenfunctions of $-\Delta$ with corresponding eigenvalues $(\lambda_j)_{j\in\mathbb{N}_0}$, and let a sequence $\big(\xi_j^\bullet\big)_{j\in\mathbb{N}_0}$ of independent, $\mathcal{N}(0,1)$-distributed random variables be given. Then for $s > n/2$ and $m \geq 0$, the series*

$$h^\omega(x) := \sum_{j\in\mathbb{N}} \frac{\varphi_j(x)\, \xi_j^\omega}{(m^2 + \lambda_j/2)^{s/2}}$$

converges and provides a pointwise representation of $h^\bullet \sim \mathring{\mathsf{FGF}}^{\mathsf{M}}_{s,m}$.

REMARK 1.11. (*a*) For Euclidean spaces $\mathsf{M} = \mathbb{R}^n$, the $\mathring{\mathsf{FGF}}^{\mathsf{M}}_{s,m}$ is well studied with particular focus on the massless case $m = 0$. Here some additional effort is required to deal with the kernel of $\big(-\frac{1}{2}\Delta\big)^{s/2}$ which is resolved by factoring out polynomials of degree $\leq s$. The real white noise, the 1d Brownian motion, the Lévy Brownian motion, and the Gaussian Free Field on the Euclidean space are all instances of random fields in the larger family of Fractional Gaussian Fields. The article [37] by Lodhia, Sheffield, Sun, and Watson provides an excellent survey.

Despite the fact that it seems to be regarded as common knowledge (in particular in the physics literature), even in the most prominent case $s = 1$, the Riemannian context is addressed only occasionally, e.g. [10, 22, 28]. In particular, Gelbaum [22] studies the existence on complete Riemannian manifolds of the fractional Brownian motions $\mathsf{FGF}^{\mathsf{M}}_{s,0}$, $s \in (n/2, n/2 + 1)$, and of the massive $\mathsf{FGF}^{\mathsf{M}}_{s,1}$, with the same values of s. Fractional Brownian motions are also constructed on Sierpiński gaskets and related fractals in [6].

(*b*) The particular case of the FGF with $s = 1$ is the *Gaussian Free Field*, discussed and analyzed in detail in the landmark article [50] by Sheffield. The GFF arises as scaling limit of various discrete models of random (hyper-)surfaces over n-dimensional simplicial lattices, e.g. Discrete Gaussian Free Fields (DGFF) or harmonic crystals [50]. The two-dimensional case is particularly relevant, for the GFF is then invariant under conformal transformations of $D \subset \mathbb{R}^2 \cong \mathbb{C}$, and constitutes therefore a useful tool in the study of conformally invariant random objects. For instance, the zero contour lines of the GFF (despite being random distributions, not functions) are well-defined SLE curves [49].

(*c*) Again in the two-dimensional case, the GFF gives rise to an impressive random geometry, the *Liouville Quantum Gravity*. It is a hot topic of current research with plenty of fascinating, deep results — despite the fact that many classical geometric quantities become meaningless, see e.g. [3, 11, 16, 20, 21, 28, 39, 41].

In this paper, our focus will be on the Random Riemannian Geometry in the 'regular' case of Hurst parameter $H := s - n/2 > 0$ in arbitrary dimension. In general, this geometry is not conformally invariant, since neither the Laplace–Beltrami operator nor its powers are conformally covariant. For *compact* manifolds of arbitrary *even* dimension n, we shall address in [14] the conformally invariant case at the critical scale $s = n/2$, a high-dimensional Liouville Quantum Gravity.

1.3. *Higher Order Green Kernel.* The regularity of the Fractional Gaussian Field $h^\bullet$ and the quantitative geometric and functional analytic estimates for the Random Riemannian Geometry $(\mathsf{M}, \mathsf{g}^\bullet)$ will be determined by the Green kernel of order s,

$$G_{s,m}(x,y) := \frac{1}{\Gamma(s)} \int_0^\infty p_t(x,y)\, e^{-m^2 t}\, t^{s-1}\, \mathrm{d}t \tag{1.6}$$

and, in the compact case, by its grounded counterpart

$$\mathring{G}_{s,m}(x,y) := \frac{1}{\Gamma(s)} \int_0^\infty \mathring{p}_t(x,y)\, e^{-m^2 t}\, t^{s-1}\, \mathrm{d}t, \qquad \mathring{p}_t(x,y) := p_t(x,y) - \frac{1}{\mathrm{vol}_{\mathsf{g}}(\mathsf{M})} . \tag{1.7}$$

The latter is also well-behaved in the massless case $m = 0$ whereas the application of the former is restricted to the case of positive mass parameter m. We analyze these Green kernels in detail and derive explicit formulas for model spaces, including Euclidean spaces, tori, hyperbolic spaces, and spheres.

THEOREM 1.12. *For points x, y, let $r := \mathsf{d}(x,y)$. Then,*

(*a*) *For the 1-dimensional torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$,*

$$\mathring{G}^{\mathbb{T}}_{1,0}(r) = \left(r - \frac{1}{2}\right)^2 - \frac{1}{12} , \qquad \mathring{G}^{\mathbb{T}}_{2,0}(r) = -\frac{1}{6}\left(r - \frac{1}{2}\right)^4 + \frac{1}{12}\left(r - \frac{1}{2}\right)^2 - \frac{7}{1440} .$$

(*b*) *For the sphere in 2 and 3 dimensions,*

$$\mathring{G}^{\mathbb{S}^2}_{1,0}(r) = -\tfrac{1}{2\pi}\left(1 + 2\log\sin\tfrac{r}{2}\right) , \qquad \mathring{G}^{\mathbb{S}^2}_{2,0}(r) = \frac{1}{\pi}\int_0^{\sin^2(r/2)} \frac{\log t}{1-t}\, \mathrm{d}t + \frac{1}{\pi} ,$$

$$\mathring{G}^{\mathbb{S}^3}_{1,0}(r) = \tfrac{1}{2\pi^2}\left(-\tfrac{1}{2} + (\pi - r)\cdot \cot r\right) , \qquad \mathring{G}^{\mathbb{S}^3}_{2,0}(r) = \frac{(\pi - r)^2}{4\pi^2} + \frac{1}{8\pi^2} - \frac{1}{12} .$$

(*c*) *For the hyperbolic space in three dimensions and $m > 0$,*

$$G^{\mathbb{H}^3}_{1,m}(r) = \frac{1}{2\pi \sinh r}\, e^{-\sqrt{2m^2+1}\, r} , \qquad G^{\mathbb{H}^3}_{2,m}(r) = \frac{r}{2\pi\sqrt{2m^2+1}\,\sinh r}\, e^{-\sqrt{2m^2+1}\, r} .$$

Of particular interest is the asymptotics of the Green kernel close to the diagonal.

THEOREM 1.13. *Let M be a compact manifold, $m \geq 0$, and $s > n/2$. Then for every $\alpha \in (0,1]$ with $\alpha < s - n/2$ there exists a constant $C = C(\mathsf{M})$ so that*

$$\left|\mathring{G}_{s,m}(x,x) + \mathring{G}_{s,m}(y,y) - 2\,\mathring{G}_{s,m}(x,y)\right|^{1/2} \leq C \cdot \mathsf{d}(x,y)^\alpha .$$

Acknowledgements. The authors would like to thank Matthias Erbar and Ronan Herry for valuable discussions on this project. They are also grateful to Nathanaël Berestycki, and Fabrice Baudoin for respectively pointing out the references [7], and [6, 22], and to Julien Fageot and Thomas Letendre for pointing out a mistake in a previous version of the proof of Proposition 3.10. The authors feel very much indebted to an anonymous reviewer for their careful reading and the many valuable suggestions that have significantly contributed to the improvement of the paper.

2. The Riemannian Manifold. Throughout this paper, (M, g) will be a complete connected n-dimensional smooth Riemannian manifold without boundary, Δ will denote its Laplace–Beltrami operator and $p_t(x,y)$ the associated heat kernel. The latter is symmetric in x, y, and as a function of t, x it solves the heat equation $\frac{1}{2}\Delta u = \frac{\partial}{\partial t} u$. For convenience, we always assume that (M, g) is stochastically complete, i.e.,

$$\int p_t(x,y)\, \mathrm{dvol}_{\mathsf{g}}(y) = 1 , \qquad x \in X,\ t > 0 ,$$

which is a well-known consequence of uniform lower bounds for the Ricci curvature, see e.g. [12, Thm. 5.2.6].

NOTATION 2.1. Throughout the paper, for functions $a, b\colon \mathbb{R} \to (0,\infty)$ and $r_0 \in \mathbb{R}$ apparent from the context we write $a \lesssim b$ if there exist $\varepsilon > 0$ and $c > 0$ so that $a(r) \leq c \cdot b(r)$ for all r so that $|r - r_0| < \varepsilon$, and we set

$$a(r) \asymp b(r) \iff \lim_{r \to r_0} \frac{a(r)}{b(r)} = 1 \qquad \text{and} \qquad a(r) \approx b(r) \iff a \lesssim b \lesssim a .$$

2.1. *Higher Order Green Operators.* For $m > 0$, consider the positive self-adjoint operator

$$A_m \coloneqq m^2 - \tfrac{1}{2}\Delta$$

on $L^2 = L^2(\mathrm{vol}_\mathsf{g})$, and its powers A_m^s defined by means of the Spectral Theorem for all $s \in \mathbb{R}$. On appropriate domains, $A_m^s \circ A_m^r = A_m^{r+s}$ for all $r, s \in \mathbb{R}$. For $s > 0$, the operator A_m^{-s}, called the *Green operator of order s with mass parameter m*, admits the representation

$$A_m^{-s} \coloneqq \frac{1}{\Gamma(s)} \int_0^\infty e^{-m^2 t} t^{s-1} e^{t\Delta/2} \,\mathrm{d}t \qquad \text{on} \quad L^2(\mathrm{vol}_\mathsf{g}) \ . \tag{2.1}$$

LEMMA 2.2. (*i*) *For $s > 0$, the Green operator of order s is an integral operator*

$$\big(A_m^{-s} f\big)(x) = \int G_{s,m}(x,y)\, f(y) \,\mathrm{dvol}_\mathsf{g}(y)$$

with density given by the Green kernel of order s with mass parameter m,

$$G_{s,m}(x,y) \coloneqq \frac{1}{\Gamma(s)} \int_0^\infty e^{-m^2 t}\ t^{s-1}\, p_t(x,y) \,\mathrm{d}t \ , \tag{2.2}$$

where $p_t(x,y)$ is the heat kernel (i.e. the density for the operator $e^{t\Delta/2}$).

(*ii*) *For each $m > 0$, the family $(G_{s,m})_{s>0}$ is a convolution semigroup of kernels, viz. $G_{r+s,m} = G_{r,m} * G_{s,m}$ for $r, s > 0$. In particular, $G_{k,m} = (G_{1,m})^{*k}$ for integer $k \geq 1$.*

(*iii*) *Moreover, $\int G_{s,m}(x, \cdot\,) \,\mathrm{dvol}_\mathsf{g} = m^{-2s}$ for all $x \in \mathsf{M}$, $s > 0$.*

PROOF. (*i*) In light of (2.1), for every $f \in L^2(\mathrm{vol}_\mathsf{g})^+$,

$$(A_m^{-s} f)(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-m^2 t} t^{s-1} \int p_t(x,y)\, f(y) \,\mathrm{dvol}_\mathsf{g}(y) \,\mathrm{d}t \ ,$$

and the conclusion follows by Tonelli's Theorem and the definition (2.2) of $G_{s,m}$. Assertions (*ii*) and (*iii*) are straightforward. ∎

2.2. *The case of manifolds of bounded geometry.* Let $\mathcal{C}_c^\infty$ be the space of all smooth compactly supported functions on M. We recall some definitions of spaces of weakly differentiable functions on M.

2.2.1. *Bessel potential spaces.* Fix $m > 0$, let $p \in [1, \infty)$ and denote by $p' \coloneqq \frac{p}{p-1}$ the Hölder conjugate of $p \in (1, \infty)$. Following [52], we define the *Bessel potential spaces* $L_m^{s,p}$, $s \geq 0$, as the space of all $u \in L^p$ so that $u = A_m^{-s/2} v$ for some $v \in L^p$, endowed with the norm $\|u\|_{L_m^{s,p}} \coloneqq \|v\|_p$. For $s < 0$, we define $L_m^{s,p}$ as the space of all distributions u on M of the form $u = A_m^k v$, where $v \in L_m^{2k+s,p}$ and k is any integer so that $2k + s > 0$, endowed with the norm $\|u\|_{L_m^{s,p}} \coloneqq \|v\|_{L_m^{2k+s,p}}$.

As it turns out, the above definition is well-posed, i.e. independent of k, and we have the following result of R. S. Strichartz'.

LEMMA 2.3 ([52], §4). *The spaces $L_m^{s,p}$, $s \in \mathbb{R}$, are Banach spaces (Hilbert spaces for $p = 2$). The natural inclusion $L_m^{s,p} \subset L_m^{r,p}$, $s > r$, is bounded and dense for every $r, s \in \mathbb{R}$ and $p \in (1, \infty)$. Furthermore, $\mathcal{C}_c^\infty$ is dense in $L_m^{s,p}$ for every $s \in \mathbb{R}$, $m > 0$ and $p \in (1, \infty)$. As a consequence, the L^2-scalar product $\langle \varphi \,|\, \psi \rangle_{L^2}$, $\varphi, \psi \in \mathcal{C}_c^\infty$, extends to a bounded bilinear form between $L_m^{s,p}$ and $L_m^{-s,p'}$, $s > 0$, thus establishing isometric isomorphisms between $L_m^{s,p}$ and $(L_m^{-s,p'})'$, $s \in \mathbb{R}$, $p \in (1, \infty)$. For every $m, s > 0$, the space $L_m^{s,p}$ coincides with the L^p-domain of $(-\Delta)^{s/2}$, and the norm $\|\cdot\|_{L_m^{s,p}}$ is equivalent to the graph-norm $\|\cdot\|_p + \big\|(-\Delta)^{s/2} \cdot\big\|_p$.*

We note that, for $m_1, m_2 > 0$, the spaces $L_{m_1}^{s,p} = L_{m_2}^{s,p}$ coincide setwise, and the corresponding norms are bi-Lipschitz equivalent. For the sake of notational simplicity, we set $H_m^s \coloneqq L_m^{s,2}$ for $s \in \mathbb{R}$, $m > 0$.

2.2.2. *Standard Sobolev spaces.* For a given local chart on M let ∇_{α_i} be the corresponding covariant derivatives. For smooth $f\colon \mathsf{M}\to\mathbb{R}$ and a non-negative integer k, we set $\left|\nabla^0 f\right| \coloneqq |f|$ and let $\left|\nabla^k f\right|$ be defined by

$$\left|\nabla^k f\right|^2 \coloneqq \mathsf{g}^{\alpha_1\beta_1}\cdots\mathsf{g}^{\alpha_k\beta_k}\nabla_{\alpha_1}\cdots\nabla_{\alpha_k} f\cdot\nabla_{\beta_1}\cdots\nabla_{\beta_k} f\ .$$

For $p\in(1,\infty)$, we denote by $E^{k,p}$ the space of all functions $f\in\mathcal{C}^\infty$ so that $\left|\nabla^i f\right|$ is in $L^p=L^p(\mathrm{vol}_{\mathsf{g}})$ for every $0\le i\le k$, and define the Sobolev space $W^{k,p}$ as the completion of $E^{k,p}$ with respect to the norm

$$\|f\|_{W^{k,p}} \coloneqq \sum_{i=0}^k \left\|\left|\nabla^i f\right|\right\|_p\ , \qquad f\in E^{k,p}\ .$$

The space $W^{k,p}_*$ is the closure in $W^{k,p}$ of $\mathcal{C}^\infty_c$.

2.2.3. *Manifolds of bounded geometry.* To simplify the presentation, at some places in the sequel we make the following assumption, corresponding to $\mathscr{H}_\infty$ in [4, Déf. 3].

Assumption 2.4. (M,g) has *bounded geometry,* i.e. the injectivity radius is bounded away from 0, and for every $k\in\mathbb{N}_0$ there exists a constant $C_k=C_{k,\mathsf{g}}$ so that the k^{th}-covariant derivative $\nabla^k R^{\mathsf{g}}$ of the Riemann tensor R^{g} satisfies $\left|\nabla^k R^{\mathsf{g}}\right|_{\mathsf{g}}\le C_k$.

Remark 2.5. It is the main result of [43] that, on an arbitrary smooth differential manifold, the conformal class $[\tilde{\mathsf{g}}]$ of any chosen Riemannian metric $\tilde{\mathsf{g}}$ contains a Riemannian metric g of bouded geometry. Thus, Assumption 2.4 poses no topological restriction on the class of manifolds we consider.

Our main interest lies in compact manifolds and in homogeneous spaces. All these spaces satisfy the above assumption.

By Lemma 2.3 above and e.g. [56, §7.4.5], under Assumption 2.4, we have that $W^{k,p}_*=W^{k,p}$ and $W^{k,p}\cong L^{k,p}_m$ (bi-Lipschitz equivalence) for every integer $k\ge 0$ and $m>0$. Furthermore, $L^{s,p}_m$ for $s\in\mathbb{R}$ may be equivalently defined via localization and pull-back onto $\mathbb{R}^d$, by using geodesic normal coordinates and corresponding fractional Sobolev spaces on $\mathbb{R}^d$, see [56, §§7.2.2, 7.4.5] or [25]. In particular we have the following:

Lemma 2.6. *Under Assumption 2.4, all the standard Sobolev–Morrey and Rellich–Kondrashov embeddings hold for $L^{s,p}_m$.*

Remark 2.7. There exist complete non-compact manifolds with Ricci curvature bounded below for which the whole scale of Sobolev embeddings fails, that is $W^{1,p}\not\hookrightarrow L^q$ for all $1\le q<n$ and $1/p=1/q-1/n$, e.g. [30, Prop. 3.13, p. 30].

We conclude this section with an auxiliary result.

Lemma 2.8. $A_m^{(r-s)/2}\colon H^r_m\longrightarrow H^s_m$ *is an isometry of Hilbert spaces for every $r,s\in\mathbb{R}$ and $m>0$.*

Proof. By duality, it suffices to show the statement for $r,s>0$. In this case, by the definition of H^t_m, $t>0$, and by the semigroup property of $t\mapsto A^t_m$, $t>0$,

$$\left\|A^{(r-s)/2}\varphi\right\|_{H^s_m}=\left\|A^{s/2}_m A^{(r-s)/2}\varphi\right\|_{L^2}=\left\|A^{r/2}_m\varphi\right\|_{L^2}=\|\varphi\|_{H^r_m}\ , \qquad \varphi\in\mathscr{D}\ .$$

The extension to H^s_m follows by the density of $\mathscr{D}$ in H^s_m, Lemma 2.3. ■

2.2.4. *Test functions.* Denote by $\mathscr{D} := \mathcal{C}_c^\infty$ the space of smooth compactly supported functions on M endowed with its canonical LF topology. It is noted in the comments preceding [27, Ch. II, Thm. 10, p. 55] that $\mathscr{D}$ is a nuclear space. We denote by $\mathscr{D}'$ the topological dual of $\mathscr{D}$, and by $\langle \cdot \mid \cdot \rangle = {}_{\mathscr{D}'}\langle \cdot \mid \cdot \rangle_{\mathscr{D}}$ the canonical duality pairing, extending the $L^2(\mathrm{vol}_\mathsf{g})$-scalar product. The weak topology $\sigma(\mathscr{D}', \mathscr{D})$ is the coarsest topology for which all functionals of the form $\langle \cdot \mid \varphi \rangle$, with $\varphi \in \mathscr{D}$, are continuous. We write $\mathscr{D}'_\sigma$ for the space $\mathscr{D}'$ endowed with the weak topology. Recall that a set $B \subset \mathscr{D}$ is bounded if for every neighborhood $U \subset \mathscr{D}$ of the origin in $\mathscr{D}$ there exists $\lambda \geq 0$ such that $B \subset \lambda U$. The strong topology $\beta(\mathscr{D}', \mathscr{D})$ on $\mathscr{D}'$ is the topology of uniform convergence on bounded sets in $\mathscr{D}$, e.g. [55, II.19, Example IV, p. 198]. We write $\mathscr{D}'_\beta$ for the space $\mathscr{D}'$ endowed with the strong topology.

Lemma 2.9. *The space $\mathscr{D}$ embeds continuously into H^s_m for every $s \in \mathbb{R}$ and every $m > 0$.*

Proof. A proof is standard in the case when $s > 0$ is a positive integer. The conclusion for general s follows since the identical inclusion $H^s_m \hookrightarrow H^k_m$ is continuous for every integer $k \leq s$ by the very definition of Bessel potential space. ∎

2.2.5. *Heat-kernel estimates.* We collect here some estimates for the heat kernel on (M, g), which we shall make use of throughout the rest of the work. We also provide estimates on its first and second derivatives, which we need for the Green kernel asymptotics in Section 6. These estimates are sharp.

Lemma 2.10. *Let (M, g) be a Riemannian manifold of bounded geometry. Then:*

(*i*) *there exists a constant $C > 0$, so that for all $x, y \in \mathsf{M}$ and every $t > 0$*

$$p_t(x,y) \leq C(t^{-n/2} \vee 1)\, e^{-\frac{\mathsf{d}^2(x,y)}{Ct}} \; ; \tag{2.3}$$

(*ii*) *there exists a constant $C > 0$, so that for all $x, y \in \mathsf{M}$ and every $t > 0$*

$$|\nabla p_t(x,y)| \leq C\big(t^{-n/2-1/2} \vee 1\big)\, e^{-\frac{\mathsf{d}^2(x,y)}{Ct}} \; ; \tag{2.4}$$

(*iii*) *there exists a constant $C > 0$, so that for all $x, y \in \mathsf{M}$ and every $t > 0$*

$$|\Delta\, p_t(x,y)| \leq C\big(t^{-n/2-1} \vee 1\big)\, e^{-\frac{\mathsf{d}^2(x,y)}{Ct}} \; ; \tag{2.5}$$

(*iv*) *there exists a constant $C > 0$, so that for all $x, y \in \mathsf{M}$ and every $t > 0$*

$$|\nabla_1 \nabla_2\, p_t(x,y)| \leq C\big(t^{-n/2-1} \vee 1\big)\, e^{-\frac{\mathsf{d}^2(x,y)}{Ct}} \; . \tag{2.6}$$

Proof. Throughout the proof $C > 0$ is a constant only depending on (M, g), possibly changing from line to line. (*i*) In light of the bounded geometry assumption we have the Gaussian heat kernel estimate

$$p_t(x,y) \leq \frac{C}{t\,\mathrm{vol}_\mathsf{g}\big(B_{\sqrt{t}\wedge 1}(x)\big)} \left(1 + \frac{\mathsf{d}^2(x,y)}{t}\right)^{\nu_0/2} e^{-\frac{\mathsf{d}^2(x,y)}{4t}}$$

for some $0 < r_0 < \mathrm{inj}(\mathsf{M})$ and $\nu_0 > 0$ [47, Thm. 4.2]. The claim follows since $\mathrm{vol}_\mathsf{g} B_r(x) \geq C r^n$ for all $r < \mathrm{inj}(\mathsf{M})$ by virtue of [9, Prop. 14].

(*ii*) Let $Q = B_{\sqrt{t}}(x) \times [t/2, t]$. Let $u(z,\tau) = p_\tau(z,y)$ on Q. Then by [51, Thm. 1.1] we have

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{\sqrt{t}} + \sqrt{K}\right)\left(1 + \log \frac{\sup_Q u}{u}\right), \tag{2.7}$$

where $-K$, $K \geq 0$, is a lower bound of the Ricci curvature. By [47, Theorem 4.2] we have

$$u(z,\tau) \leq \frac{C}{\mathrm{vol}_\mathsf{g}\big(B_{\sqrt{\tau}\wedge r_0}(z)\big)} e^{-\mathsf{d}^2(z,y)/C\tau} \leq \frac{C}{\mathrm{vol}_\mathsf{g}\big(B_{\sqrt{t}\wedge r_0}(x)\big)},$$

where we used the volume-doubling property and consequently by (2.7) and (2.3)

$$\begin{aligned}|\nabla u(x,t)| &\le C\left(\frac{1}{\sqrt{t}}+\sqrt{K}\right)\left(1+\log\frac{\mathrm{vol}_{\mathsf g}\big(B_{\sqrt{t}\wedge r_0}(x)\big)^{-1}}{u(x,t)}\right)u(x,t)\\ &\le C\left(\frac{1}{\sqrt{t}}+\sqrt{K}\right)\left(1+\log\frac{\mathrm{vol}_{\mathsf g}\big(B_{\sqrt{t}\wedge r_0}(x)\big)^{-1}}{u(x,t)}\right)(t^{-n/2}\vee 1)\,e^{-\frac{\mathsf d^2(x,y)}{Ct}}\,.\end{aligned}$$

In order to estimate $u(x,t)$ from below we use Corollary 1.2 in [36] and obtain for $Kt\le 1$

$$|\nabla u(x,t)|\le \frac{C}{\sqrt{t}}(t^{-n/2}\vee 1)\,e^{-\frac{\mathsf d^2(x,y)}{Ct}}\,e^{Ct}\le C\big(t^{-n/2-1/2}\vee 1\big)\,e^{-\frac{\mathsf d^2(x,y)}{Ct}}\ .$$

For $Kt>1$ we use Corollary 1.7 in [36]

$$|\nabla u(x,t)|\le C\,e^{-\frac{\mathsf d^2(x,y)}{Ct}}\ \le\ C\big(t^{-n/2-1/2}\vee 1\big)\,e^{-\frac{\mathsf d^2(x,y)}{Ct}}\ ,$$

which finishes the proof.

(iii) In light of the bounded geometry assumption, [47, Thm. 4.2] yields

$$|\partial_t p_t(x,y)|\le \frac{C}{t\,\mathrm{vol}_{\mathsf g}\big(B_{\sqrt{t}\wedge r_0}(x)\big)}\left(1+\frac{\mathsf d^2(x,y)}{t}\right)^{\nu_0/2+1}e^{-\frac{\mathsf d^2(x,y)}{4t}}$$

for ν_0 and r_0 as in (i). We estimate the volume of the ball from below as in (i) by applying [9, Prop. 14]. Noting that $\partial_t p_t(x,y)=\Delta p_t(x,y)$ the result follows.

(iv) It follows from [34, Thm. 2.1] that there exists a constant $C>3$ depending on $(\mathsf M,\mathsf g)$ so that, for all $x,y\in\mathsf M$,

$$|\nabla_1\nabla_2\,p_t(x,y)|\le C\,\partial_t p_t(x,y)+C\big(t^{-1}\vee 1\big)p_t(x,y)\ ,\qquad t>0\ .$$

Since $p_t(\,\cdot\,,y)$ is a solution to the heat equation, and using (2.5), we have for all $t\in(0,2]$ and every $x,y\in\mathsf M$,

$$\begin{aligned}|\nabla_1\nabla_2\,p_t(x,y)| &\le C\big(\Delta p_t(\,\cdot\,,y)\big)(x)+C\big(t^{-1}\vee 1\big)p_t(x,y)\\ &\le C\big(t^{-n/2-1}\vee 1\big)e^{-\frac{\mathsf d^2(x,y)}{Ct}}+C\big(t^{-1}\vee 1\big)p_t(x,y)\end{aligned}$$

for some constant $C>0$ only depending on $(\mathsf M,\mathsf g)$ and possibly changing from line to line. Combining this with the heat kernel estimate (2.3) yields the claim for $t\le 2$. For $t\ge 2$ the claim follows from the bound for $t\le 1$ combined with the following inequalities for $t\ge 1$:

$$\begin{aligned}\big|\nabla_x\nabla_y p_{t+1}(x,y)\big| &= \left|\iint \nabla_x p_{1/2}(x,u)\ p_t(u,v)\ \nabla_y p_{1/2}(v,y)\,\mathrm{dvol}_{\mathsf g}(u)\,\mathrm{dvol}_{\mathsf g}(v)\right|\\ &\le \sup_{u\in\mathsf M}|\nabla_x p_{1/2}(x,u)|\cdot\sup_{v\in\mathsf M}|\nabla_y p_{1/2}(y,v)|\cdot\iint p_t(u,v)\,\mathrm{dvol}_{\mathsf g}(u)\,\mathrm{dvol}_{\mathsf g}(v)\le C\ ,\end{aligned}\tag{2.8}$$

which concludes the proof. ∎

2.3. *The case of closed manifolds.* Let us now specialize our constructions to the case when $\mathsf M$ is additionally *closed*, i.e. compact and without boundary.

If $\mathsf M$ is closed, the operator $(m^2-\frac12\Delta)^{-1}$ is compact on $L^2(\mathrm{vol}_{\mathsf g})$, and thus has discrete spectrum. We denote by $(\varphi_j)_{j\in\mathbb N_0}$ the complete L^2-orthonormal system consisting of eigenfunctions of $-\Delta$, each with corresponding eigenvalue λ_j, so that $(\Delta+\lambda_j)\varphi_j=0$ for every j. Since $\mathsf M$ is connected, we have $0=\lambda_0<\lambda_1$ and $\varphi_0\equiv\mathrm{vol}_{\mathsf g}(\mathsf M)^{-1/2}$. Weyl's asymptotic law implies that for some $c>0$,

$$\lambda_j\ge c\,j^{2/n},\qquad j\in\mathbb N\ .\tag{2.9}$$

2.3.1. *Grounding.* If M is closed, we further define the *grounded Green operator of order s with mass parameter m* as the (bounded) self-adjoint operator $\mathring{A}_m^{-s} f := A_m^{-s}(\mathring{f})$ on $L^2(\mathsf{M})$ with

$$\mathring{f} := f - \frac{1}{\mathrm{vol}_{\mathsf{g}}(\mathsf{M})} \int f \,\mathrm{dvol}_{\mathsf{g}} \ .$$

We start by refining the heat-kernel estimates in Lemma 2.10 to the closed case.

LEMMA 2.11 (Heat kernel estimates: compact case). *Let* (M, g) *be a closed Riemannian manifold. Then,*

(*i*) *there exists a constant* $C > 0$*, so that for all* $x, y \in \mathsf{M}$ *and every* $t > 0$

$$p_t(x,y) \le C(t^{-n/2} \vee 1)\, e^{-\frac{\mathrm{d}^2(x,y)}{Ct}} \ , \tag{2.10}$$

$$|\mathring{p}_t(x,y)| \le C(t^{-n/2} \vee 1)\, e^{-\lambda_1 t/2} \ ; \tag{2.11}$$

(*ii*) *for every* $\ell \in \mathbb{N}_0$ *there exists a constant* $C = C(\ell) > 0$*, so that for all* $x, y \in \mathsf{M}$ *and every* $t > 0$

$$\left|\nabla^\ell p_t(x,y)\right| \le C\big(t^{-n/2-\ell/2} \vee 1\big)\, e^{-\frac{\mathrm{d}^2(x,y)}{Ct}}\, e^{-\lambda_1 t/2} \ ; \tag{2.12}$$

(*iii*) *there exists a constant* $C > 0$*, so that for all* $x, y \in \mathsf{M}$ *and every* $t > 0$

$$|\nabla_1 \nabla_2\, p_t(x,y)| \le C\big(t^{-n/2-1} \vee 1\big)\, e^{-\frac{\mathrm{d}^2(x,y)}{Ct}}\, e^{-\lambda_1 t/2} \ . \tag{2.13}$$

PROOF. (*i*) The estimate (2.10) was already shown in Lemma 2.10. We provide here an alternative proof which we subsequently adapt to the case of $\mathring{p}_t$. For $t \ge 1$, the estimate (2.10) immediately follows from the fact that by compactness of M the heat kernel is uniformly bounded on $[1,\infty) \times \mathsf{M} \times \mathsf{M}$. For $t \le 1$ it follows from the celebrated estimate of Li and Yau [35, Cor. 3.1], combined with the fact that $\mathrm{vol}_{\mathsf{g}}(B_{\sqrt{t}}(x)) \ge \frac{1}{C} t^{n/2}$ for each $x \in \mathsf{M}$, which in turn follows from Bishop–Gromov volume comparison and compactness of M, see, e.g., [45, Lem. 9.1.36, p. 269].

Since $-C \le \mathring{p}_t(x,y) \le p_t(x,y)$, the estimate (2.11) for $t \le 2$ follows immediately from the previous estimate. In order to prove (2.11) for $t \ge 2$, note that, for $t \ge 1$,

$$\begin{aligned} |\mathring{p}_{t+1}(x,y)| &= \left| \iint \mathring{p}_{1/2}(x,u)\, \mathring{p}_t(u,v)\, \mathring{p}_{1/2}(v,y) \,\mathrm{dvol}_{\mathsf{g}}(u) \,\mathrm{dvol}_{\mathsf{g}}(v) \right| \\ &\le \sup_{u \in \mathsf{M}} |\mathring{p}_{1/2}(x,u)| \cdot \sup_{v \in \mathsf{M}} |\mathring{p}_{1/2}(y,v)| \cdot \iint \left|\mathring{p}_t(u,v)\right| \mathrm{dvol}_{\mathsf{g}}(u) \,\mathrm{dvol}_{\mathsf{g}}(v) \\ &\le C \iint \left|\mathring{p}_t(u,v)\right| \mathrm{dvol}_{\mathsf{g}}(u) \,\mathrm{dvol}_{\mathsf{g}}(v) \end{aligned}$$

uniformly in $x, y \in \mathsf{M}$. Moreover, note that by the standard spectral calculus for Δ and ultracontractivity of the heat semigroup, see e.g. [12, Thm. 2.1.4], we may express the grounded heat kernel on M as the uniform limit of the series

$$\mathring{p}_t(x,y) = \sum_{j \in \mathbb{N}} e^{-t\lambda_j/2} \varphi_j(x)\, \varphi_j(y) \ , \qquad x, y \in \mathsf{M} \ ,$$

and with this we obtain

$$\begin{aligned} \iint \left|\mathring{p}_t(x,y)\right| \mathrm{dvol}_{\mathsf{g}}(x) \,\mathrm{dvol}_{\mathsf{g}}(y) &= \iint \left| \sum_{j=1}^{\infty} e^{-\lambda_j t/2} \varphi_j(x)\, \varphi_j(y) \right| \mathrm{dvol}_{\mathsf{g}}(x) \,\mathrm{dvol}_{\mathsf{g}}(y) \\ &\le C \sum_{j=1}^{\infty} e^{-\lambda_j t/2} \le C\, e^{-\lambda_1 t/2} \sum_{j=1}^{\infty} e^{(\lambda_1 - \lambda_j)/2} \end{aligned} \tag{2.14}$$

$$= C\, e^{-\lambda_1 t/2}\, e^{\lambda_1/2} \int \mathring{p}_1(x,x)\, \mathrm{dvol}_{\mathsf{g}}(x)$$
$$= C'\, e^{-\lambda_1 t/2} \ .$$

This proves the claim.

(*ii*) It is shown in [53, Eqn. (1.1)] that for every $x, y \in \mathsf{M}$

$$\left|\big(\nabla^\ell \log p_t(\,\cdot\,,y)\big)(x)\right| \leq C_\ell \left(\frac{1}{t} + \frac{\mathsf{d}^2(x,y)}{t^2}\right)^{\ell/2} \ , \qquad t \in (0,2] \ ,$$

for some constant C_ℓ, henceforth possibly changing from line to line. As a consequence,

$$\left|\big(\nabla^\ell p_t(\,\cdot\,,y)\big)(x)\right| \leq C_\ell \left(\frac{1}{t} + \frac{\mathsf{d}^2(x,y)}{t^2}\right)^{\ell/2} p_t(x,y) \ , \qquad t \in (0,2] \ . \tag{2.15}$$

In combination with the heat kernel estimate (2.10) from above this yields the claim for $t \leq 2$. As in part (*i*), the claim for $t \geq 2$ follows from the bound for $t \leq 1$ together with the fact that, for $t \geq 1$,

$$\begin{aligned}
\left|\nabla^\ell_x p_{t+1}(x,y)\right| &= \left|\nabla^\ell_x \mathring{p}_{t+1}(x,y)\right| \\
&= \left| \iint \nabla^\ell_x \mathring{p}_{1/2}(x,u)\ \mathring{p}_t(u,v)\ \mathring{p}_{1/2}(v,y)\, \mathrm{dvol}_{\mathsf{g}}(u)\, \mathrm{dvol}_{\mathsf{g}}(v)\right| \\
&\leq \sup_{u\in\mathsf{M}} |\nabla^\ell_x \mathring{p}_{1/2}(x,u)| \cdot \sup_{v\in\mathsf{M}} |\mathring{p}_{1/2}(y,v)| \cdot \iint \left|\mathring{p}_t(u,v)\right| \mathrm{dvol}_{\mathsf{g}}(u)\, \mathrm{dvol}_{\mathsf{g}}(v) \\
&\leq C\, e^{-\lambda_1 t/2}
\end{aligned}$$

according to the previous estimates (2.15), (2.10), and (2.14).

(*iii*) Let us first note that [34, Thm. 2.1] holds with identical proof also in the case of closed M. Similarly to the proof of Lemma 2.10, it follows from [34, Thm. 2.1] that there exists a constant $C > 3$ depending on (M, g) so that, for all $x, y \in \mathsf{M}$,

$$|\nabla_1 \nabla_2\, p_t(x,y)| \leq C\, \partial_t p_t(x,y) + C\big(t^{-1} \vee 1\big) p_t(x,y) \ , \qquad t > 0 \ .$$

Since $p_t(\,\cdot\,,y)$ is a solution to the heat equation, and using (2.15), we have for all $t \in (0,2]$ and every $x, y \in \mathsf{M}$,

$$\begin{aligned}
|\nabla_1 \nabla_2\, p_t(x,y)| &\leq C\big(\Delta p_t(\,\cdot\,,y)\big)(x) + C\big(t^{-1} \vee 1\big) p_t(x,y) \\
&\leq C\left|\big(\nabla^2 p_t(\,\cdot\,,y)\big)(x)\right| + C\big(t^{-1} \vee 1\big) p_t(x,y) \\
&\leq C\big(t^{-1} \vee 1\big) \left(\frac{\mathsf{d}^2(x,y)}{t} + 1\right) p_t(x,y) + C\big(t^{-1} \vee 1\big) p_t(x,y) \\
&\leq C\big(t^{-1} \vee 1\big) \left(\frac{\mathsf{d}^2(x,y)}{t} + 1\right) p_t(x,y) \ ,
\end{aligned} \tag{2.16}$$

for some constant $C > 0$ depending on (M, g) and possibly changing from line to line. Combining this with the heat kernel estimate (2.3) yields the claim for $t \leq 2$. Again, for $t \geq 2$ the claim follows from the bound for $t \leq 1$ combined with the following inequalities for $t \geq 1$:

$$\begin{aligned}
\left|\nabla_x \nabla_y p_{t+1}(x,y)\right| &= \left| \iint \nabla_x \mathring{p}_{1/2}(x,u)\ \mathring{p}_t(u,v)\ \nabla_y \mathring{p}_{1/2}(v,y)\, \mathrm{dvol}_{\mathsf{g}}(u)\, \mathrm{dvol}_{\mathsf{g}}(v)\right| \\
&\leq \sup_{u\in\mathsf{M}} |\nabla_x \mathring{p}_{1/2}(x,u)| \cdot \sup_{v\in\mathsf{M}} |\nabla_y \mathring{p}_{1/2}(y,v)| \cdot \iint \left|\mathring{p}_t(u,v)\right| \mathrm{dvol}_{\mathsf{g}}(u)\, \mathrm{dvol}_{\mathsf{g}}(v) \\
&\leq C\, e^{-\lambda_1 t/2} \ .
\end{aligned}$$

∎

LEMMA 2.12. *If* M *is closed and* $s > 0$*, then* $\mathring{A}_m^{-s}$ *is an integral operator with density given by the* grounded Green kernel of order s with mass parameter $m \geq 0$*, defined in terms of the* grounded heat kernel*,*

$$\mathring{G}_{s,m}(x,y) \coloneqq \frac{1}{\Gamma(s)} \int_0^\infty e^{-m^2 t}\, t^{s-1}\, \mathring{p}_t(x,y)\,\mathrm{d}t\ , \qquad \mathring{p}_t(x,y) \coloneqq p_t(x,y) - \frac{1}{\mathrm{vol}_{\mathsf{g}}(\mathsf{M})}\ . \tag{2.17}$$

For each $m \geq 0$*, the family* $(\mathring{G}_{s,m})_{s>0}$ *is a convolution semigroup of kernels, and* $\int \mathring{G}_{s,m}(x,\,\cdot\,)\,\mathrm{dvol}_{\mathsf{g}} = 0$ *for all* $x \in \mathsf{M}$*,* $s > 0$*.*

Of particular interest will be $\mathring{G}_{s,0}$, the *massless grounded Green kernel of order s*.

PROOF OF LEMMA 2.12. Let us first observe that $\mathring{G}_{s,m}(x,y)$ as defined above is finite for all $x \neq y$ by virtue of (2.11). We claim that the integral

$$(\mathring{\mathsf{G}}_{s,m} f)(x) \coloneqq \int \mathring{G}_{s,m}(x,y)\, f(y)\,\mathrm{dvol}_{\mathsf{g}}(y) \tag{2.18}$$

is absolutely convergent for every $f \in L^2$ and a.e. x. Indeed, it defines an L^2-function according to

$$\begin{aligned}
&\int \left| \frac{1}{\Gamma(s)} \int \int_0^1 e^{-m^2 t} t^{s-1} \mathring{p}_t(x,y)\,\mathrm{d}t\, f(y) \mathrm{dvol}_{\mathsf{g}}(y) \right|^2 \mathrm{dvol}_{\mathsf{g}}(x) \\
&\quad \leq \int \left(\frac{1}{\Gamma(s)} \int \int_0^1 t^{s-1} \left[p_t(x,y) + \frac{1}{\mathrm{vol}_{\mathsf{g}}(\mathsf{M})} \right] \mathrm{d}t\, |f|(y) \mathrm{dvol}_{\mathsf{g}}(y) \right)^2 \mathrm{dvol}_{\mathsf{g}}(x) \\
&\quad \leq 2e\, \|\mathsf{G}_{s,1} f\|_{L^2}^2 + 2\, \frac{1}{(s\,\Gamma(s))^2 \cdot \mathrm{vol}_{\mathsf{g}}(\mathsf{M})}\, \|f\|_{L^2}^2 \\
&\quad \leq C'\, \|f\|_{L^2}^2 < \infty
\end{aligned}$$

and since, due to (2.12),

$$\begin{aligned}
&\int \left| \frac{1}{\Gamma(s)} \int \int_1^\infty e^{-m^2 t} t^{s-1} \mathring{p}_t(x,y)\,\mathrm{d}t\, f(y) \mathrm{dvol}_{\mathsf{g}}(y) \right|^2 \mathrm{dvol}_{\mathsf{g}}(x) \\
&\quad \leq C \int \left[\frac{1}{\Gamma(s)} \int \int_1^\infty e^{-m^2 t} t^{s-1} e^{-\lambda_1 t/2}\,\mathrm{d}t\, |f|(y) \mathrm{dvol}_{\mathsf{g}}(y) \right]^2 \mathrm{dvol}_{\mathsf{g}}(x) \\
&\quad \leq C'\, \|f\|_{L^2}^2\ .
\end{aligned}$$

Thus,

$$\mathring{\mathsf{G}}_{s,m} \colon L^2 \longrightarrow L^2 \quad \text{is a bounded operator} \tag{2.19}$$

and moreover, (due to the absolute convergence of the integrals) by Fubini's Theorem,

$$\mathring{\mathsf{G}}_{s,m} f(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-m^2 t} t^{s-1} \int \mathring{p}_t(x,y)\, f(y)\,\mathrm{dvol}_{\mathsf{g}}(y)\,\mathrm{d}t = (\mathring{A}_m^{-s} f)(x)\ . \qquad \blacksquare$$

REMARK 2.13. (*a*) For $m > 0$

$$\mathring{G}_{s,m}(x,y) = G_{s,m}(x,y) - \frac{1}{m^{2s}\,\mathrm{vol}_{\mathsf{g}}(\mathsf{M})}\ .$$

(*b*) For each $s > 0$, $m \geq 0$ and $x \in \mathsf{M}$, the distribution $\mathring{G}_{s,m}(x,\,\cdot\,)\mathrm{vol}_{\mathsf{g}}$ is the unique distributional solution to

$$\left(m^2 - \tfrac{1}{2}\Delta\right)^s u = \delta_x - \frac{1}{\mathrm{vol}_{\mathsf{g}}(\mathsf{M})}\,\mathrm{vol}_{\mathsf{g}} \qquad \text{and} \qquad \langle u \,|\, \mathbb{1} \rangle = 0\ . \tag{2.20}$$

Proof. As (a) is straightforward, we only prove (b). It is also standard that $\mathring{G}_{s,m}(x,\,\cdot\,)\mathrm{vol}_{\mathsf{g}}$ is *a* distributional solution to (2.20), thus it suffices to show that the associated homogeneous equations $A^s_m u = 0$ and $\langle u \,|\, \mathbb{1}\rangle = 0$ admit a unique solution for every $s \in \mathbb{R}$.

To this end, denote by $\mathring{\mathscr{D}} := \{\mathring{\varphi} : \varphi \in \mathscr{D}\}$ the space of grounded test functions. Equivalently, we show that $A^s_m \colon \mathring{\mathscr{D}} \to \mathring{\mathscr{D}}$ is a bijection for every $s \in \mathbb{R}$. The fact that $A^k_m(\mathscr{D}) \subset \mathscr{D}$ for integer k holds by the standard Schauder estimates for elliptic operators (for closed manifolds see e.g. [38, Thm. III.5.2 (iii), p. 193]). This is readily extended to $s \in \mathbb{R}$ noting that the integral operator $\mathsf{G}_{s,m}$ with kernel $G_{s,m}(x,\,\cdot\,)$ is a smoothing operator for $s > n/2$.

For $m > 0$, the injectivity on $\mathring{\mathscr{D}}$ (in fact on $L^2(\mathrm{vol}_{\mathsf{g}})$) holds by Lemma 2.8, and the surjectivity by Lemma 2.2. For $m = 0$, the injectivity holds since $\ker A^k_0 = \ker(-\Delta_{\mathsf{g}})^k$ only consists of the constant functions for every non-negative integer k, and the surjectivity holds by Lemma 2.12. We omit the details. ∎

2.3.2. *Eigenfunction expansion.* We conclude the analysis of the closed case by discussing the expansion of the Green kernels $G_{s,m}$ and $\mathring{G}_{s,m}$ in terms of eigenfunctions of the Laplace–Beltrami operator.

Lemma 2.14. *Assume that* M *is closed. Then for all* $m > 0$ *and* $s > n/2$,

$$G_{s,m}(x,y) = \sum_{j\in\mathbb{N}_0} \frac{\varphi_j(x)\,\varphi_j(y)}{(m^2+\lambda_j/2)^s}\ , \qquad x,y \in \mathsf{M}\ , \tag{2.21}$$

where the series is absolutely convergent for every $x, y \in \mathsf{M}$.

Furthermore, for all $m \geq 0$ *and* $s > n/2$,

$$\mathring{G}_{s,m}(x,y) = \sum_{j\in\mathbb{N}} \frac{\varphi_j(x)\,\varphi_j(y)}{(m^2+\lambda_j/2)^s}\ , \qquad x,y \in \mathsf{M}\ . \tag{2.22}$$

(Note that the summation now starts at $j = 1$.*) In particular,*

$$\mathring{G}_{s,0}(x,y) = 2^s \sum_{j\in\mathbb{N}} \frac{\varphi_j(x)\,\varphi_j(y)}{\lambda_j^s}\ , \qquad x,y \in \mathsf{M}\ . \tag{2.23}$$

Proof. By the spectral calculus (e.g. [12, Thm. 2.1.4]), we may express the heat kernel on M as the uniform limit of the series

$$p_t(x,y) = \sum_{j\in\mathbb{N}_0} e^{-t\lambda_j/2}\varphi_j(x)\,\varphi_j(y)\ , \qquad x,y \in \mathsf{M}\ . \tag{2.24}$$

By virtue of (2.2), (2.10), and $s > \frac{n}{2}$ we have that $G_{s,m}(x,x) < \infty$. By Dominated Convergence the representation (2.21) follows for $x = y$. For $x, y \in \mathsf{M}$ we have that the series $\sum_{j\in\mathbb{N}_0} \frac{\varphi_j(x)\,\varphi_j(y)}{(m^2+\lambda_j/2)^s}$ is absolutely convergent due to Cauchy–Schwarz. Hence (2.21) follows again by Dominated Convergence. With the same arguments but using (2.17) and (2.11) instead of (2.2) and (2.10), we can show (2.22). ∎

Remark 2.15. The grounded Green kernel $\mathring{G}_{s,0}(x,y)$ coincides, up to the multiplicative factor 2^s, with the celebrated Minakshisundaram–Pleijel ζ-function $\zeta^{\Delta}_{x,y}(s)$ of the Laplace–Beltrami operator on M, introduced in [42]. The massive grounded Green kernel $\mathring{G}_{s,m}(x,y)$ is therefore the Hurwitz regularization of ζ^{Δ} with parameter m^2.

2.3.3. *Sobolev spaces on compact manifolds.* Again assume that M is closed, and let $(\varphi_j)_{j\in\mathbb{N}_0}$ and $(\lambda_j)_{j\in\mathbb{N}_0}$ be as above. Then for each $m > 0$ and $s \in \mathbb{R}$,

$$H^s_m = \left\{ f \in \mathscr{D}' :\ f = \sum_{j\in\mathbb{N}_0} \alpha_j\varphi_j,\quad \sum_{j=0}^{\infty} \alpha_j^2 \big(m^2+\lambda_j/2\big)^s < \infty \right\}$$

with $\|f\|^2_{H^s_m} = \sum_{j=0}^\infty \alpha_j^2 \left(m^2+\lambda_j/2\right)^s$ and $\langle f \,|\, \psi\rangle = \sum_{j=0}^\infty \alpha_j \,\langle \varphi_j \,|\, \psi\rangle$ for $\psi \in \mathscr{D}$. Note that for all $\psi \in \mathscr{D}$ and $k \in \mathbb{N}$ we have $\sum_{j=0}^\infty \left|\lambda_j^k \,\langle \varphi_j \,|\, \psi\rangle\right|^2 < \infty$.

DEFINITION 2.16. If M is closed we define the *grounded Sobolev spaces* for $m \geq 0$ and $s \in \mathbb{R}$ by

$$\mathring{H}^s_m = \left\{ f \in \mathscr{D}' : \ f = \sum_{j\in\mathbb{N}} \alpha_j \varphi_j, \quad \sum_{j=1}^\infty \alpha_j^2 \left(m^2+\lambda_j/2\right)^s < \infty \right\} ,$$

regarded as a subspace of H^s_m.

LEMMA 2.17. *Assume that* M *is closed.*

(*i*) *For all* $m \geq 0$ *and* $r, s \in \mathbb{R}$,

$$\mathring{A}_m^{-(r-s)/2} = A_m^{-(r-s)/2} \colon \mathring{H}^s_m \longrightarrow \mathring{H}^r_m$$

is an isometry of Hilbert spaces.

(*ii*) *For all* $m > 0$ *and* $s \in \mathbb{R}$,

$$\mathring{H}^s_m = \left\{ f \in H^s_m : \ \langle f \,|\, \mathbb{1}\rangle = 0 \right\} .$$

(*iii*) *For all* $m > 0$ *and* $s \in \mathbb{R}$, *the spaces* $\mathring{H}^s_m$ *and* $\mathring{H}^s_0$ *coincide setwise, and the corresponding norms are bi-Lipschitz equivalent.*

PROOF. (*i*) follows from Lemma 2.8. (*ii*) follows by spectral calculus. (*iii*) For $s \geq 0$,

$$\sum_{j=1}^\infty \alpha_j^2 \left(\lambda_j/2\right)^s \leq \sum_{j=1}^\infty \alpha_j^2 \left(m^2+\lambda_j/2\right)^s \leq \left(\frac{m^2+\lambda_1/2}{\lambda_1/2}\right)^s \cdot \sum_{j=1}^\infty \alpha_j^2 \left(\lambda_j/2\right)^s ,$$

thus

$$\|f\|_{\mathring{H}^s_0} \leq \|f\|_{\mathring{H}^s_m} \leq \left(1+2m^2/\lambda_1\right)^{s/2} \cdot \|f\|_{\mathring{H}^s_0} .$$

Similarly for $s < 0$,

$$\|f\|_{\mathring{H}^s_0} \geq \|f\|_{\mathring{H}^s_m} \geq \left(1+2m^2/\lambda_1\right)^{s/2} \cdot \|f\|_{\mathring{H}^s_0} . \qquad \blacksquare$$

2.4. *The noise distance.* Given any positive numbers s, m, a pseudo-distance $\rho_{s,m}$ on M, called *noise distance* (for reasons which become clear in Corollary 3.12), is defined by

$$\rho_{s,m}(x,y) \coloneqq \left(\frac{1}{\Gamma(s)} \int_0^\infty \int_{\mathsf{M}} e^{-m^2 t}\ t^{s-1} \Big[p_{t/2}(x,z) - p_{t/2}(y,z) \Big]^2 \mathrm{dvol}_{\mathsf{g}}(z)\, \mathrm{d}t \right)^{1/2} . \tag{2.25}$$

Indeed, symmetry and triangle inequality are immediate consequences of the fact that this is the L^2-distance between $p_{\cdot/2}(x,\cdot)$ and $p_{\cdot/2}(y,\,\cdot\,)$ w.r.t. a (possibly infinite) measure on $\mathbb{R}_+ \times \mathsf{M}$. In the case of closed M, the analogous definition for $\mathring{p}_{\cdot/2}(\,\cdot\,,\,\cdot\,)$ results in $\mathring{\rho}_{s,m} = \rho_{s,m}$.

REMARK 2.18. Note that by the symmetry and the Chapman–Kolmogorov property of the heat kernel,

$$\int_{\mathsf{M}} \Big[p_{t/2}(x,z) - p_{t/2}(y,z) \Big]^2 \mathrm{dvol}_{\mathsf{g}}(z) = p_t(x,x) + p_t(y,y) - 2p_t(x,y) .$$

Hence, for all $s, m \in (0,\infty)$ and all $x, y \in \mathsf{M}$ with $G_{s,m}(x,y) < \infty$,

$$\rho_{s,m}(x,y) = \ \Big[G_{s,m}(x,x) + G_{s,m}(y,y) - 2\, G_{s,m}(x,y) \Big]^{1/2} .$$

3. The Fractional Gaussian Field. Let us now define Fractional Gaussian Fields.

THEOREM 3.1. *For $m > 0$ and $s \in \mathbb{R}$, there exists a unique Radon Gaussian measure $\mu_{m,s}$ on $\mathscr{D}'_\sigma$ with characteristic functional*

$$\mathscr{D} \ni \varphi \longmapsto \int_{\mathscr{D}'} e^{\mathrm{i}\,\langle\,\cdot\,|\,\varphi\rangle}\,\mathrm{d}\mu_{m,s}\ , \qquad \varphi \in \mathscr{D}\ ,$$

equal to

$$\chi_{m,s} \colon \varphi \longmapsto \exp\left[-\tfrac{1}{2}\,\|\varphi\|^2_{H^{-s}_m}\right]\ , \qquad \varphi \in \mathscr{D}\ . \tag{3.1}$$

PROOF. Note that $\chi_{m,s}(0) = 1$ and that $\chi_{m,s}$ is positive definite, e.g., [37, Prop. 2.4]. Furthermore, $\chi_{m,s}$ is additionally continuous on $\mathscr{D}$, since $\mathscr{D}$ embeds continuously into H^{-s}_m for every $s \in \mathbb{R}$ and $m > 0$ by Lemma 2.9. Note that $\beta(\mathscr{D}', \mathscr{D})$ is finer than $\sigma(\mathscr{D}', \mathscr{D})$, hence every Radon *probability* measure on $\mathscr{D}'_\beta$ restricts to a Radon probability measure on $\mathscr{D}'_\sigma$. Since $\mathscr{D}$ is nuclear, by Bochner–Minlos Theorem in the form [57, §VI.4.3, Thm. 4.3, p. 410], there exists a Radon probability measure $\mu_{m,s}$ on $\mathscr{D}'_\beta$, and the conclusion follows by restricting this measure to a (non-relabeled) Radon measure on $\mathscr{D}'_\sigma$. ■

Everywhere in the following, $(\Omega, \mathscr{F}, \mathbf{P})$ denotes a probability space supporting countably many i.i.d. Gaussian random variables.

DEFINITION 3.2. Let $m > 0$ and $s \in \mathbb{R}$. An *m-massive Fractional Gaussian Field on* M *with regularity* s, in short: $\mathsf{FGF}^{\mathsf{M}}_{s,m}$, is any $\mathscr{D}'$-valued random field $h^\bullet$ on Ω distributed according to $\mu_{m,s}$.

We omit the superscript M from the notation whenever apparent from context, and write $h^\bullet \sim \mathsf{FGF}_{s,m}$ to denote an m-massive Fractional Gaussian Field with regularity s. Here and henceforth, for random variables $X^\bullet : \omega \mapsto X^\omega$ on Ω the superscript $^\bullet$ will indicate the ω-dependence.

The case $h^\bullet \sim \mathsf{FGF}_{s,m}$ with $s = 0$ is singled out in the scale of all FGF's on M as the only one independent of m. It corresponds to the Gaussian *White Noise* on M induced by the nuclear rigging $\mathscr{D} \subset L^2(\mathrm{vol}_\mathsf{g}) \subset \mathscr{D}'$, where we note that $L^2(\mathrm{vol}_\mathsf{g}) = H^0_m$ for all $m > 0$.

REMARK 3.3. The White Noise $W^\bullet$ on M is the $\mathscr{D}'$-valued, centered Gaussian random field uniquely characterized by *either* one of the following properties, see e.g. the monograph [32]:

$$\begin{aligned} \mathbf{E}\left[e^{\mathrm{i}\langle\varphi\,|\,W^\bullet\rangle}\right] &= e^{-\frac{1}{2}\|\varphi\|^2_{L^2}}\ , && \varphi \in \mathscr{D}\ ;\\ \mathbf{E}\Big[\,\langle\varphi\,|\,W^\bullet\rangle^2\,\Big] &= \|\varphi\|^2_{L^2(\mathrm{vol}_\mathsf{g})}\ , && \varphi \in \mathscr{D}\ ;\\ \mathbf{E}\Big[\,\langle\varphi\,|\,W^\bullet\rangle\cdot\langle\psi\,|\,W^\bullet\rangle\,\Big] &= \int \varphi\,\psi\,\mathrm{d}\mathrm{vol}_\mathsf{g}\ , && \varphi,\psi \in \mathscr{D}\ . \end{aligned}$$

3.1. *Some characterizations.* Let us now characterize the Fractional Gaussian Field $h^\bullet \sim \mathsf{FGF}_{s,m}$ in terms of the associated Gaussian Hilbert space. We recall that a *Gaussian Hilbert space* on $(\Omega, \mathscr{F}, \mathbf{P})$ is a closed linear subspace of $L^2(\Omega)$ consisting of centered Gaussian random variables, cf. e.g. [37, Dfn. 2.5]. We say that a Gaussian Hilbert space $\{X_v : v \in V\}$ is *linearly indexed* by V if V is a linear space and $v \mapsto X_v$ is a linear map.

PROPOSITION 3.4. *Given $h^\bullet \sim \mathsf{FGF}_{s,m}$ on $(\Omega, \mathscr{F}, \mathbf{P})$, the collection*

$$\mathcal{H}_{s,m} \coloneqq \left\{\langle h^\bullet\,|\,f\rangle : f \in H^{-s}_m\right\} \tag{3.2}$$

(with $\langle h^\bullet \,|\, f\rangle$ suitably defined in the proof) is a Gaussian Hilbert space with covariance structure

$$\langle h^\bullet \,|\, f\rangle \sim \mathcal{N}\big(0, \|f\|^2_{H^{-s}_m}\big) \ , \qquad f \in H^{-s}_m \ . \tag{3.3}$$

Vice versa, every Gaussian Hilbert space

$$\tilde{\mathcal{H}}_{s,m} \coloneqq \big\{X^\bullet_f \colon \Omega \longrightarrow \mathbb{R} : f \in H^{-s}_m\big\} \tag{3.4}$$

on $(\Omega, \mathscr{F}, \mathbf{P})$ linearly indexed by H^{-s}_m and satisfying

$$X^\bullet_f \sim \mathcal{N}\big(0, \|f\|^2_{H^{-s}_m}\big) \ , \qquad f \in H^{-s}_m \ , \tag{3.5}$$

is isomorphic to $\mathcal{H}_{s,m}$ as a Hilbert space via the map $X^\bullet_f \mapsto \langle h^\bullet \,|\, f\rangle$.

The space $\mathcal{H}_{s,m}$ is called the Gaussian Hilbert space of $h^\bullet \sim \mathsf{FGF}_{s,m}$.

Corollary 3.5. *Let $h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$ and $\tilde{\mathcal{H}}_{s,m}$ be any Gaussian Hilbert space linearly indexed by H^{-s}_m defined as in* (3.4) *and satisfying* (3.5). *Further suppose that there exists a $\mathscr{D}'$-valued Gaussian field $X^\bullet$ on $(\Omega, \mathscr{F}, \mathbf{P})$ so that $\langle X^\bullet \,|\, \varphi\rangle = X^\bullet_\varphi$ for every $\varphi \in \mathscr{D}$. Then $X^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$.*

Remark 3.6 (Constructions with Schwartz functions). Suppose $\mathsf{M} = \mathbb{R}^n$ is a standard Euclidean space, and denote by $\mathscr{S}$ the space of Schwartz functions on M endowed with its canonical Fréchet topology, and by $\mathscr{S}'_\sigma$ the space of tempered distributions on M endowed with the weak topology $\sigma(\mathscr{S}', \mathscr{S})$. Recall that $\mathscr{S}$ is a nuclear space, and embeds densely and continuously into H^s_m for every $s \in \mathbb{R}$ and $m > 0$. By the very same proof of Theorem 3.1, there exists a centered Gaussian field $X^\bullet$ on $\Omega = \mathscr{S}'_\sigma$ with characteristic functional satisfying (3.1) for every $\varphi \in \mathscr{S}$. By comparison with the massless case, see e.g. the survey [37], the field $X^\bullet$ too would deserve the name of *massive Fractional Gaussian Field* on $\mathsf{M} = \mathbb{R}^n$. In fact, we have $X^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$ in our sense.

Proof. Since the identical embedding $\mathscr{D} \hookrightarrow \mathscr{S}$ is continuous, the space $\mathscr{S}'_\sigma$ of tempered distributions on M embeds identically and continuously (in particular, measurably) into $\mathscr{D}'_\sigma$. Thus, $X^\bullet$ is in particular $\mathscr{D}'$-valued, and it may be regarded as defined on $\Omega = \mathscr{D}'_\sigma$. The conclusion follows in light of Corollary 3.5. ∎

Proof of Proposition 3.4. For every $\varphi \in \mathscr{D}$, the map $t \mapsto \chi_{m,s}(t\varphi)$ as in (3.1) is analytic in t around $t = 0$. Differentiating it twice at $t = 0$ shows that the assignment $\mathscr{D} \ni \varphi \mapsto \langle h^\bullet \,|\, \varphi\rangle$ defines an isometry of $\big(\mathscr{D}, \|\cdot\|_{H^{-s}_m}\big)$ into $L^2(\Omega)$. By density of $\mathscr{D}$ in H^{-s}_m, the latter extends to a linear isometry $H^{-s}_m \to L^2(\Omega)$. Thus, by construction, $\mathcal{H}_{s,m}$ forms a closed linear subspace of $L^2(\Omega)$. By the definition of $\chi_{m,s}$, the random variable $\langle h^\bullet \,|\, \varphi\rangle$ has centered Gaussian distribution with variance $\|\varphi\|^2_{H^{-s}_m}$ for every $\varphi \in \mathscr{D}$. By the H^{-s}_m-continuity in φ of the corresponding characteristic function, the latter distributional characterization extends to H^{-s}_m which yields (3.3).

Vice versa, let $\tilde{\mathcal{H}}_{s,m}$ be as in (3.4) and (3.5). Since the indexing assignment $\iota \colon f \mapsto X^\bullet_f$ is linear, (3.5) shows that it is injective, and therefore an isomorphism of linear spaces. Analogously, $f \mapsto \langle h^\bullet \,|\, f\rangle$ is an isomorphism of linear spaces by (3.3). Thus, the map $X^\bullet_f \mapsto \langle h^\bullet \,|\, f\rangle$ too is an isomorphism of linear spaces, being the composition of $\iota^{-1} \colon \tilde{\mathcal{H}}_{s,m} \to H^{-s}_m$ and $\langle h^\bullet \,|\, \cdot\,\rangle : H^{-s}_m \to \mathcal{H}_{s,m}$. Combining (3.3) and (3.5) shows that $X^\bullet_f \mapsto \langle h^\bullet \,|\, f\rangle$ is additionally an $L^2(\Omega)$-isometry, which concludes the proof. ∎

In particular, we have the following:

Corollary 3.7. *For $s > 0$, $h^\bullet \sim \mathsf{FGF}_{s,m}$ is uniquely characterized as the centered Gaussian process with covariance*

$$\mathrm{Cov}\big[\,\langle h^\bullet \,|\, \varphi\rangle\,, \langle h^\bullet \,|\, \psi\rangle\,\big] = \iint G_{s,m}(x,y)\, \varphi(x)\, \psi(y)\, \mathrm{dvol}_{\mathsf{g}}^{\otimes 2}(x,y) \ , \qquad \varphi, \psi \in \mathscr{D} \subset H^{-s}_m \ . \tag{3.6}$$

PROPOSITION 3.8. *Let $s \in \mathbb{R}$, $m > 0$, and $h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$. Then, the following assertions hold:*

(i) $A^k_m h^\bullet$ *is a well-defined $\mathscr{D}'$-valued random field on $(\Omega, \mathscr{F}, \mathbf{P})$ satisfying $A^k_m h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s-2k,m}$ for every $k \in \mathbb{Z}$;*

(ii) if M is closed, then $A_m^{-(r-s)/2} h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{r,m}$ for every $r \in \mathbb{R}$.

PROOF. (i) Fix $k \in \mathbb{N}$. Since $A_m \colon \mathscr{D} \to \mathscr{D}$, the operator $A^k_m \colon \mathscr{D}' \to \mathscr{D}'$ is well-defined on $\mathscr{D}'$ by transposition. Thus, $A^k_m h^\bullet$ is $\mathbf{P}$-a.s. a well-defined element of $\mathscr{D}'$. By definition of $A^k_m \colon \mathscr{D}' \to \mathscr{D}'$, we have

$$\left\langle A^k_m h^\bullet \,\middle|\, \varphi \right\rangle = \left\langle h^\bullet \,\middle|\, A^k_m \varphi \right\rangle \ , \qquad \varphi \in \mathscr{D} \ . \tag{3.7}$$

By Lemma 2.8, we have $A^k_m f \in H^{-s}_m$ for every $f \in H_m^{-(s-2k)}$. Thus, similarly to the proof of the forward implication in Proposition 3.4, the equality in (3.7) extends from $\mathscr{D}$ to $H_m^{-(s-2k)}$, and

$$\tilde{\mathcal{H}}_{s-2k,m} \coloneqq \left\{ \left\langle A^k_m h^\bullet \,\middle|\, f \right\rangle : f \in H_m^{-(s-2k)} \right\}$$

is a Gaussian Hilbert space on $(\Omega, \mathscr{F}, \mathbf{P})$ linearly indexed by $H_m^{-(s-2k)}$. Furthermore, we conclude again from (3.7) and Lemma 2.8 that

$$\left\langle A^k_m h^\bullet \,\middle|\, \varphi \right\rangle = \left\langle h^\bullet \,\middle|\, A^k_m \varphi \right\rangle \sim \mathcal{N}\left(0, \left\| A^k_m \varphi \right\|^2_{H^{-s}_m}\right) = \mathcal{N}\left(0, \left\| \varphi \right\|^2_{H_m^{-(s-2k)}}\right) \ , \qquad \varphi \in \mathscr{D} \ .$$

Again as in Proposition 3.4, the above equality extends from $\mathscr{D}$ to $H_m^{-(s-2k)}$, and we conclude that $\tilde{\mathcal{H}}_{s-2k,m}$ has covariance structure

$$\left\langle A^k_m h^\bullet \,\middle|\, f \right\rangle \sim \mathcal{N}\left(0, \left\| f \right\|^2_{H_m^{-(s-2k)}}\right) \ , \qquad f \in H_m^{-(s-2k)} \ .$$

By the converse implication in Proposition 3.4, $\tilde{\mathcal{H}}_{s-2k,m}$ is isomorphic as a Hilbert space to the Gaussian Hilbert space $\mathcal{H}_{s-2k,m}$ of an $\mathsf{FGF}^{\mathsf{M}}_{s-2k,m}$. Thus, $A^k_m h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s-2k,m}$ by Corollary 3.5.

(ii) Since M is closed, $A^r_m \colon \mathscr{D} \to \mathscr{D}$ for every $r \in \mathbb{R}$, thus $A^r_m \colon \mathscr{D}' \to \mathscr{D}'$ is well-defined by transposition. The rest of the proof follows exactly as in (i) replacing k by $(s-r)/2$. ■

COROLLARY 3.9. *The following assertions hold:*

(i) all the Fractional Gaussian Fields $h^\bullet_s \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$ for $s \in \mathbb{R}$ and $m > 0$ may be obtained from $h^\bullet_{s-2k} \sim \mathsf{FGF}^{\mathsf{M}}_{s-2k,m}$ as

$$h^\bullet_s \coloneqq A_m^{-2k} h^\bullet_{s-2k} \ ,$$

where k is the only integer so that $s - 2k \in [0, 2)$.

(ii) if M is closed, then all the Fractional Gaussian Fields $h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$ for $s \in \mathbb{R}$ and $m > 0$ may be obtained from the White Noise $W^\bullet$ on M as

$$h^\bullet \coloneqq \left(m^2 - \tfrac{1}{2}\Delta\right)^{-s/2} W^\bullet \ .$$

3.2. *Continuity of the FGF.* The basic property concerning differentiability and Hölder continuity of FGF's is as follows.

PROPOSITION 3.10. *Let $h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$. Then, the following assertions hold:*

(i) Assume that (M, g) has bounded geometry. If $s > n/2 + \alpha$ with $\alpha \in [0, 1)$, then $h^\bullet \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathsf{M})$ a.s.;

(ii) Assume that (M, g) is closed. If $s > n/2 + k + \alpha$ with $k \in \mathbb{N}_0$ and $\alpha \in [0, 1)$, then $h^\bullet \in \mathcal{C}^{k,\alpha}(\mathsf{M})$ a.s.;

(iii) If $s > n/2 + 1$, then $h^\bullet \in W^{1,2}_{\mathrm{loc}}(\mathsf{M})$ a.s.

In particular, the continuity of $h^\bullet$ in the case $s > n/2$ will allow us to rewrite (3.6) in a more comprehensive and suggestive form.

Corollary 3.11. *For each $s > n/2$ the centered Gaussian process $h^\bullet \sim \mathsf{FGF}_{s,m}$ is uniquely characterized by*

$$\mathbf{E}\big[h^\bullet(x)\, h^\bullet(y)\big] = G_{s,m}(x,y)\ , \qquad x,y \in \mathsf{M}\ . \tag{3.8}$$

Corollary 3.12. *For each $s > n/2$, the pseudo-distance $\rho_{s,m}$ is indeed a distance. It is given in terms of the process $h^\bullet \sim \mathsf{FGF}_{s,m}$ by*

$$\rho_{s,m}(x,y) = \mathbf{E}\Big[\big|h^\bullet(x) - h^\bullet(y)\big|^2\Big]^{1/2}\ , \qquad x,y \in \mathsf{M}\ . \tag{3.9}$$

Proof Proposition 3.10. (*i*) Let $h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$ with $s > n/2$. Lemma 2.6 implies that H^s_m embeds continuously into a space of continuous functions on M by Morrey's inequality. As a consequence, $\delta_x \in H^{-s}_m$. Thus, Proposition 3.4 implies that $h^\omega(x) \coloneqq \langle h^\omega \mid \delta_x \rangle$ is $\mathbf{P}$-a.s. well-defined for every fixed $x \in \mathsf{M}$. Together with Corollary 3.7, this proves the representation (3.9) in Corollary 3.12.

Combining (3.9) and Theorem 6.1 we have therefore that

$$\mathbf{E}\Big[\big|h^\bullet(x) - h^\bullet(y)\big|^2\Big]^{1/2} \leq C_\alpha \cdot \mathsf{d}(x,y)^\alpha\ , \qquad x,y \in \mathsf{M}\ ,$$

for some constant $C_\alpha > 0$. In particular, $\omega \mapsto \big(h^\omega(x) - h^\omega(y)\big)$ is a centered Gaussian random variable with covariance dominated by $C_\alpha \cdot \mathsf{d}(x,y)^\alpha$. Therefore, it has finite moments of all orders $p > 1$, and, for every such p, there exists a constant $C_{\alpha,p} > 0$ so that

$$\mathbf{E}\Big[\big|h^\bullet(x) - h^\bullet(y)\big|^p\Big] \leq C_{\alpha,p} \cdot \mathsf{d}(x,y)^{\alpha p}\ , \qquad x,y \in \mathsf{M}\ . \tag{3.10}$$

Since M is smooth, there exists an atlas of charts (U,Φ), with $\Phi \colon U \to \Phi(U) \subset \mathbb{R}^n$ so that

$$C_U^{-1}\, |\Phi(x) - \Phi(y)| \leq \mathsf{d}(x,y) \leq C_U\, |\Phi(x) - \Phi(y)|\ , \qquad x,y \in U\ , \tag{3.11}$$

for some constant $C_U > 0$ possibly depending on U. Define a random field on $\Phi(U)$ by setting $h^\bullet_\Phi \coloneqq h^\bullet \circ \Phi^{-1}$. Combining (3.11) with (3.10),

$$\mathbf{E}\Big[\big|h^\bullet_\Phi(a) - h^\bullet_\Phi(b)\big|^p\Big] \leq C_U \cdot C_{\alpha,p} \cdot |a-b|^{\alpha p}\ , \qquad a,b \in \Phi(U) \subset \mathbb{R}^n\ .$$

By the standard Kolmogorov–Chentsov Theorem, e.g. [46, Thm. I.2.1], we conclude that, for every $\varepsilon > 0$ and every $p > 1$, the function $h^\bullet_\Phi$ satisfies $h^\bullet_\Phi \in \mathcal{C}^{0,\alpha-\varepsilon-n/p}(\Phi(U))$ almost surely for all $\alpha \in (0, s-n/2)$. By arbitrariness of ε and p, and since α ranges in an open interval, we may conclude that $h^\bullet_\Phi \in \mathcal{C}^{0,\alpha}(\Phi(U))$ almost surely for all $\alpha \in (0, s-n/2)$. Finally, since Φ is smooth, it follows that $h^\bullet \in \mathcal{C}^{0,\alpha}(U)$, and therefore that $h^\bullet \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathsf{M})$ almost surely.

(*ii*) Now assume that $h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$ with $s > n/2+k+\alpha$ with $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Note that $A_m^{k/2} h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s-k,m}$ by Proposition 3.8, and $A_m^{-k/2} \colon \mathcal{C}^{0,\alpha}(\mathsf{M}) \to \mathcal{C}^{k,\alpha}(\mathsf{M})$ for every $k \in \mathbb{N}$. Thus the claim follows by the previous part (*i*).

(*iii*): Let K be a bounded convex subset of M with smooth boundary, and denote p^K_t the heat kernel with Neumann boundary conditions on K. Recall that a function $f \in L^2(\mathsf{M})$ belongs to $W^{1,2}(K)$—the form domain for the Neumann heat semigroup on M—if and only if

$$\lim_{t\to 0} \frac{1}{t} \int_K \int_K |f(x) - f(y)|^2\, p^K_t(x,y)\, \mathrm{dvol}_{\mathsf{g}}(x)\, \mathrm{dvol}_{\mathsf{g}}(y) < \infty \tag{3.12}$$

by the very definition of the Neumann heat semigroup on K. Furthermore, the $\lim_{t\to 0}$ is in fact a monotone limit.

In the case $s > n/2+1$, Theorem 6.1 below (applied with $\alpha = 1$) implies that the continuous random function $h^\bullet \sim \mathsf{FGF}^{\mathsf{M}}_{s,m}$ satisfies

$$\mathbf{E}\left[\lim_{t\to 0} \frac{1}{t} \int_K \int_K |h^\bullet(x) - h^\bullet(y)|^2\, p^K_t(x,y)\, \mathrm{dvol}_{\mathsf{g}}(x)\, \mathrm{dvol}_{\mathsf{g}}(y)\right]$$

$$= \lim_{t\to 0} \frac{1}{t} \int_K \int_K \mathbf{E}\Big[|h^\bullet(x) - h^\bullet(y)|^2\Big]\, p_t^K(x,y)\, \mathrm{dvol}_{\mathsf{g}}(x)\, \mathrm{dvol}_{\mathsf{g}}(y)$$
$$\leq \lim_{t\to 0} \frac{C}{t} \int_K \int_K \mathsf{d}(x,y)^2\, p_t^K(x,y)\, \mathrm{dvol}_{\mathsf{g}}(x)\, \mathrm{dvol}_{\mathsf{g}}(y) \leq C'.$$

where the last inequality follows from the Li–Yau estimate [35, Thm. 3.2] on the Neumann heat kernel. Thus

$$\lim_{t\to 0} \frac{1}{t} \int_K \int_K |h^\omega(x) - h^\omega(y)|^2\, p_t^K(x,y)\, \mathrm{dvol}_{\mathsf{g}}(x)\, \mathrm{dvol}_{\mathsf{g}}(y) < \infty$$

for a.e. ω, which by the preceding comment implies $h^\omega \in W^{1,2}(K)$. By arbitrariness of K, the latter implies $h^\omega \in W^{1,2}_{\mathrm{loc}}(\mathsf{M})$. ■

REMARK 3.13. The regularity of $h^\bullet$ provided by Proposition 3.10 is sharp, in the sense that $h^\bullet$ is *not* an element of $\mathcal{C}^{k,\gamma}$ for any $\gamma \in [s - n/2 - k, 1]$.

3.3. *Series Expansions in the Compact Case.* If M is closed, Fractional Gaussian Fields may be approximated by their expansion in terms of eigenfunctions of the Laplace–Beltrami operator Δ. As before in §2.3.2, we denote by $(\varphi_j)_{j\in\mathbb{N}_0} \subset \mathscr{D}$ the complete L^2-orthonormal system consisting of eigenfunctions of Δ, each with corresponding eigenvalue λ_j, so that $(\Delta+\lambda_j)\varphi_j = 0$ for every j. Recall the representations of heat kernel (2.24), Green kernel (2.21), and grounded Green kernel (2.22) in terms of this eigenbasis.

Let now a sequence $\big(\xi_j^\bullet\big)_{j\in\mathbb{N}_0}$ of i.i.d. random variables on a common probability space $(\Omega, \mathscr{F}, \mathbf{P})$ be given with $\xi_j^\bullet \sim \mathcal{N}(0,1)$. For each $\ell > 0$, define a random variable $h_\ell^\bullet : \Omega \to \mathscr{D}$ by

$$h_\ell^\omega(x) := \sum_{j=0}^{\ell} \frac{\varphi_j(x)\, \xi_j^\omega}{(m^2 + \lambda_j/2)^{s/2}} \ . \tag{3.13}$$

THEOREM 3.14. *(i) For every $s \in \mathbb{R}$ and $f \in H_m^{-s}$, the family $(\langle h_\ell^\bullet \,|\, f\rangle)_{\ell\in\mathbb{N}}$ is a centered, L^2-bounded martingale on $(\Omega, \mathscr{F}, \mathbf{P})$.*

(ii) As $\ell \to \infty$, it converges, both a.e. and in L^2, to the random variable $\langle h \,|\, f\rangle^\bullet \in L^2(\Omega)$ given for a.e. ω by

$$\langle h \,|\, f\rangle^\omega := \sum_{j\in\mathbb{N}_0} \frac{\langle \varphi_j \,|\, f\rangle\, \xi_j^\omega}{(m^2 + \lambda_j/2)^{s/2}} \ .$$

(iii) $\langle h \,|\, f\rangle^\bullet$ is a centered Gaussian random variable with variance $\|f\|^2_{H_m^{-s}}$.

PROOF. Assertion (i) and (ii) follow by standard arguments on centered Gaussian variables, e.g. [8, Thm. 1.1.4]. For (iii), observe that by definition, $\langle h \,|\, f\rangle^\bullet$ is a centered Gaussian random variable with variance

$$\mathbf{E}\left[\big(\langle h \,|\, f\rangle^\bullet\big)^2\right] = \sum_{j\in\mathbb{N}_0} \frac{\langle \varphi_j \,|\, f\rangle^2}{(m^2 + \lambda_j/2)^s} = \big\|A_m^{-s/2} f\big\|_2^2 = \|f\|^2_{H_m^{-s}} \ , \tag{3.14}$$

where the first equality holds by orthogonality of $(\varphi_j)_{j\in\mathbb{N}_0}$ and since $\big(\xi_j^\bullet\big)_{j\in\mathbb{N}_0}$ are i.i.d. $\sim \mathcal{N}(0,1)$, the second equality since $(\varphi_j)_{j\in\mathbb{N}_0}$ is a complete L^2-orthonormal system of eigenfunctions of A_m as well, and the third equality by the definition of the norm of H_m^{-s}. ■

COROLLARY 3.15. *The family of random variables*

$$\tilde{\mathcal{H}}_{s,m} := \big\{\langle h \,|\, f\rangle^\bullet : f \in H_m^{-s}\big\} \ , \qquad s \in \mathbb{R} \ , \ m > 0 \ ,$$

is a Gaussian Hilbert space, isomorphic to $\mathcal{H}_{s,m}$ in (3.2) *via the map $\iota\colon \langle h \,|\, f\rangle^\bullet \mapsto \langle h^\bullet \,|\, f\rangle$.*

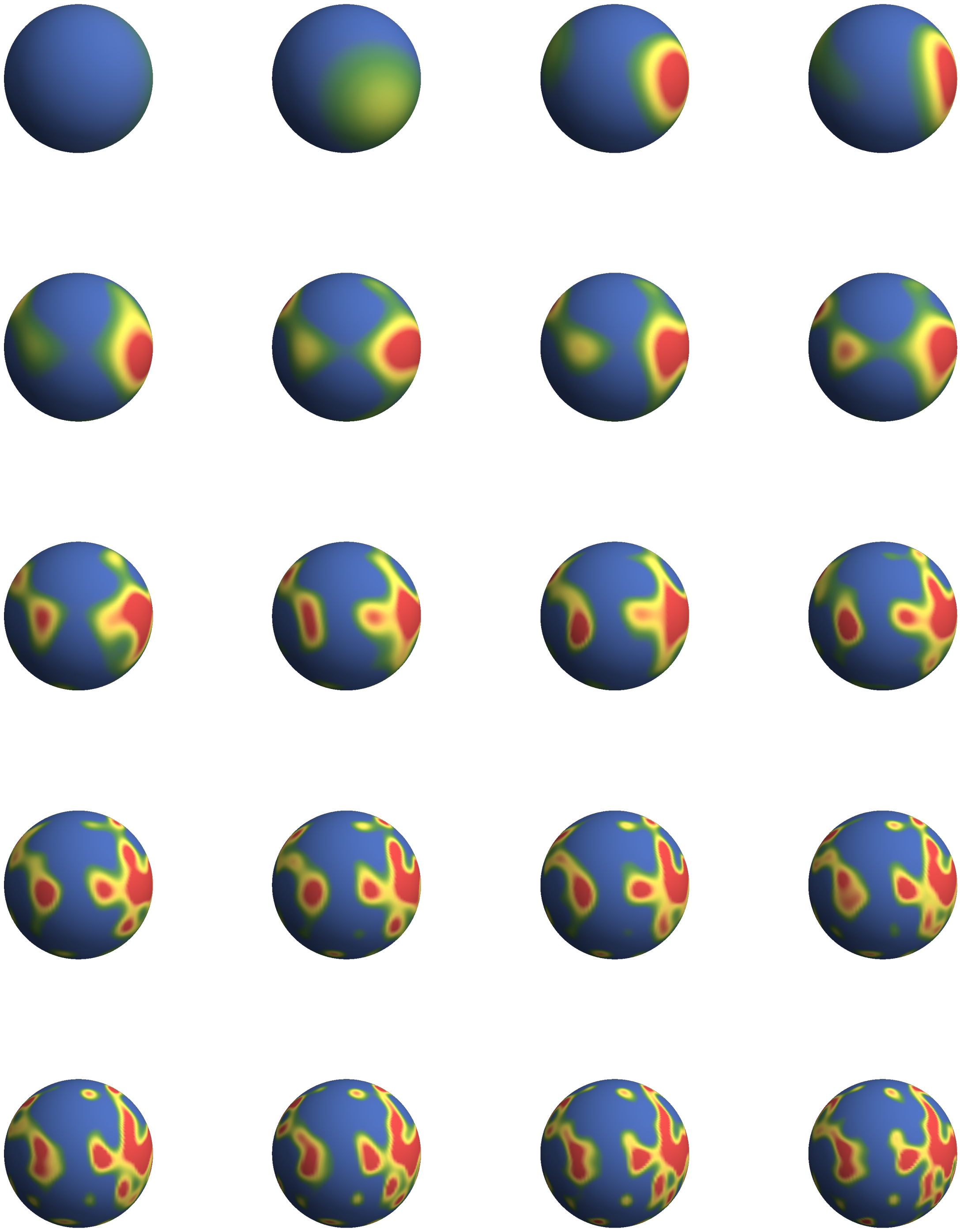


Fig 2: A realization of $h_\ell^\bullet$ in (3.13) on the unit sphere $\mathbb{S}^2$ with, $m = s = 1$ (critical case), and $\ell \in \{1, \dots, 20\}$.

Proof. It is shown in Theorem 3.14(iii) that $\tilde{\mathcal{H}}_{s,m}$ is a Gaussian linear space, closed in $L^2(\Omega)$ by completeness of H_m^{-s} and (3.14), and thus a Gaussian Hilbert space. Since $\mathscr{D}$ embeds continuously into H_m^{-s} for every $s \in \mathbb{R}$, the map $\iota \colon \langle h \,|\, \varphi \rangle^\bullet \mapsto \langle h^\bullet \,|\, \varphi \rangle$ is well-defined for every $\varphi \in \mathscr{D}$. Equation (3.3) together with Theorem 3.14(iii) show that it is as well an isometry, and thus extends to $\tilde{\mathcal{H}}_{s,m}$ by density of $\mathscr{D}$

in H^{-s}_m and (3.14), again for every $s \in \mathbb{R}$. Since $\{\langle h^\bullet \,|\, \varphi\rangle : \varphi \in \mathscr{D}\}$ is dense in $\mathcal{H}_{s,m}$ by construction, as in the proof of Proposition 3.4, the map ι has dense image. Since isometries of Hilbert spaces have closed range, it is as well surjective, and thus an isomorphism of (Gaussian) Hilbert spaces. ■

THEOREM 3.16. *For $s > n/2$, the series*

$$h^\omega(x) := \sum_{j\in\mathbb{N}_0} \frac{\varphi_j(x)\,\xi_j^\omega}{(m^2+\lambda_j/2)^{s/2}}$$

converges in $L^2(\Omega)$ and almost surely on Ω for each $x \in \mathsf{M}$. Moreover it converges on $L^2(\Omega\times\mathsf{M})$ and in $L^2(\mathrm{vol}_\mathsf{g})$ almost surely.

PROOF. The $L^2(\Omega\times\mathsf{M})$ as well as the $L^2(\Omega)$ convergence follow by combining the identities

$$\mathbf{E}\left[\int\left(\sum_{j=\ell+1}^{\ell'} \frac{\varphi_j(x)\,\xi_j^\omega}{(m^2+\lambda_j/2)^{s/2}}\right)^2 \mathrm{dvol}_\mathsf{g}\right] = \sum_{j=\ell+1}^{\ell'} \frac{1}{(m^2+\lambda_j/2)^{s}}\ ,$$
$$\mathbf{E}\left[\left(\sum_{j=\ell+1}^{\ell'} \frac{\varphi_j(x)\,\xi_j^\omega}{(m^2+\lambda_j/2)^{s/2}}\right)^2\right] = \sum_{j=\ell+1}^{\ell'} \frac{\varphi_j(x)^2}{(m^2+\lambda_j/2)^{s}}\ ,$$

and the fact that the terms on the right hand side of both equations converge to 0 as $\ell,\ell'\to\infty$ according to Weyl's asymptotics (2.9) and (2.21) respectively. The almost sure convergence for each x as well as the almost sure convergence for the $L^2(\mathrm{vol}_\mathsf{g})$ sequence follow by Theorem 3.14 and Doob's Martingale Convergence Theorem. ■

3.4. *The Grounded FGF.* Assume now that M is closed. Then, the same arguments used to derive Theorem 3.1 also apply for the grounded norms, and in this case even for $m \geq 0$.

In order to state the next result, let us set $\mathring{\mathscr{D}} := \{\psi \in \mathscr{D} : \langle \mathrm{vol}_\mathsf{g} \,|\, \psi\rangle = 0\}$, and denote by $\mathring{\mathscr{D}}'$ the topological dual of $\mathring{\mathscr{D}}$. We note that $\mathring{\mathscr{D}}$ is a nuclear space when endowed with the subspace topology inherited from $\mathscr{D}$, since every linear subspace of a nuclear space is itself nuclear, e.g. [55, Prop. 50.1, (50.3), p. 514].

THEOREM 3.17. *For $m \geq 0$ and $s \in \mathbb{R}$, there exists a unique Radon Gaussian measure $\mathring{\mu}_{m,s}$ on $\mathring{\mathscr{D}}'$ with characteristic functional given by*

$$\mathring{\chi}_{m,s} \colon \varphi \longmapsto \exp\left[-\tfrac{1}{2}\,\|\varphi\|^2_{\mathring{H}^{-s}_m}\right]\ , \qquad \varphi\in\mathring{\mathscr{D}}\ . \tag{3.15}$$

PROOF. Analogously to Theorem 3.1, it suffices to show that $\mathring{\mathscr{D}}$ embeds continuously into $\mathring{H}^{-s}_m$. In turn, this follows from the continuity of the embedding of $\mathscr{D}$ into H^s_m and Lemma 2.17(*ii*). ■

DEFINITION 3.18. Let $m \geq 0$ and $s \in \mathbb{R}$. A *grounded m-massive Fractional Gaussian Field on M with regularity s*, in short: $\mathring{\mathsf{FGF}}^{\mathsf{M}}_{s,m}$, is any $\mathscr{D}'$-valued random field $h^\bullet$ on Ω distributed according to $\mathring{\mu}_{m,s}$. In the case $m=0$, the field is called a *grounded massless Fractional Gaussian Field on M with regularity s.*

All results for the random fields $\mathsf{FGF}_{s,m}$ have their natural counterparts for $\mathring{\mathsf{FGF}}_{s,m}$, now even admitting $m=0$. In particular, we have the grounded versions of Corollary 3.7 and Theorem 3.16.

COROLLARY 3.19. *For $s > 0$ and $m \geq 0$, the random field $h^\bullet \sim \mathring{\mathsf{FGF}}_{s,m}$ is uniquely characterized as the centered Gaussian process with covariance*

$$\mathrm{Cov}\big[\,\langle h^\bullet \,|\, \varphi\rangle\,,\langle h^\bullet \,|\, \psi\rangle\,\big] = \iint \mathring{G}_{s,m}(x,y)\,\varphi(x)\,\psi(y)\,\mathrm{dvol}_\mathsf{g}^{\otimes 2}(x,y)\ , \qquad \varphi,\psi\in\mathring{\mathscr{D}}\subset\mathring{H}^{-s}_m\ .$$

Corollary 3.20. *For $s > n/2$ and $m \geq 0$, the series*

$$h^\omega(x) := \sum_{j\in\mathbb{N}} \frac{\varphi_j(x)\,\xi_j^\omega}{(m^2+\lambda_j/2)^{s/2}}$$

converges in $L^2(\Omega)$ and almost surely on Ω for each $x \in \mathsf{M}$. Moreover it converges on $L^2(\Omega\times\mathsf{M})$ and in $L^2(\mathrm{vol}_\mathsf{g})$ almost surely.

In particular, $h^\bullet \sim \mathring{\mathsf{FGF}}_{s,0}$ is given by $h^\omega(x) = 2^{s/2}\sum_{j\in\mathbb{N}} \lambda_j^{-s/2}\,\varphi_j(x)\,\xi_j^\omega$ if $s > n/2$.

If $m > 0$, the grounding map $f \mapsto \mathring{f} := f - \frac{1}{\mathrm{vol}_\mathsf{g}(\mathsf{M})}\langle f \,|\, \mathbb{1}\rangle$ allows us to easily switch between the random fields $\mathsf{FGF}^\mathsf{M}_{s,m}$ and $\mathring{\mathsf{FGF}}^\mathsf{M}_{s,m}$, as in the next Lemma.

Lemma 3.21. *For every $s \in \mathbb{R}$ and every $m > 0$,*

(*i*) *given $h^\bullet \sim \mathsf{FGF}_{s,m}$, put $\mathring{h}^\omega := h^\omega - \frac{1}{\mathrm{vol}_\mathsf{g}(\mathsf{M})}\langle h^\omega \,|\, \mathbb{1}\rangle$. Then $\mathring{h}^\bullet \sim \mathring{\mathsf{FGF}}_{s,m}$;*

(*ii*) *given $h^\bullet \sim \mathring{\mathsf{FGF}}_{s,m}$ and independent $\xi \sim \mathcal{N}(0,1)$, put $\hat{h}^\omega := h^\omega + \frac{1}{\sqrt{m^{2s}\,\mathrm{vol}_\mathsf{g}(\mathsf{M})}}\xi^\omega\,\mathbb{1}$. Then $\hat{h}^\bullet \sim \mathsf{FGF}_{s,m}$.*

Proposition 3.22. *Let $\mathring{h}^\bullet \sim \mathring{\mathsf{FGF}}_{s,m}$ on M. If $s > n/2 + k + \alpha$ with $k \in \mathbb{N}_0$ and $\alpha \in [0,1)$, then $\mathring{h}^\bullet \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathsf{M})$ almost surely.*

Proof. Let $\xi \sim \mathcal{N}(0,1)$ be independent of $\mathring{h}^\bullet$. By Lemma 3.21(*ii*), $\mathring{h}^\bullet + \frac{1}{\sqrt{m^{2s}\mathrm{vol}_\mathsf{g}(\mathsf{M})}}\xi^\bullet\,\mathbb{1}$ is distributed as an $\mathsf{FGF}^\mathsf{M}_{s,m}$, and thus it satisfies Proposition 3.10. Since $\frac{1}{\sqrt{m^{2s}\mathrm{vol}_\mathsf{g}(\mathsf{M})}}\xi^\omega\,\mathbb{1} \in \mathscr{D}$ for every ω, the conclusion follows. ■

Remark 3.23. It is worth comparing the grounding of operators and fields presented above with the pinning for fractional Brownian motions in [22], where a *Riesz field* R^s is defined as the centered Gaussian field with covariance

$$\mathbf{E}\left[R^s(x)\,R^s(y)\right] = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\big(p_t(x,y) - p_t(x,o) - p_t(y,o) + p_t(o,o)\big)\,\mathrm{d}t\ , \qquad s \in (n/2, n/2+1)\ ,$$

for some fixed 'origin' $o \in \mathsf{M}$. In particular, while grounding on a compact manifold (M,g) is canonical, the pinning of a Riesz field at $o \in \mathsf{M}$, and hence the properties of the corresponding random Riemannian manifold (see §4 below), would depend on o.

3.5. *Dudley's Estimate.* A crucial role in our geometric estimates and functional inequalities for the Random Riemannian Geometry is played by estimates for the expected maximum of the random field. The fundamental estimate of Dudley provides an estimate in terms of the covering number w.r.t. the pseudo-distance $\rho_{s,m}$, introduced in (2.25).

Notation 3.24. For any pseudo-distance ρ on M, we denote by $N_\rho(\varepsilon)$ the least number of ρ-balls of radius ε which are needed to cover M. When $\rho = \rho_{s,m}$ we write $N_{s,m}(\varepsilon)$ in place of $N_{\rho_{s,m}}(\varepsilon)$.

Theorem 3.25 ([33, Thm. 11.17]). *Fix $s > n/2$ and $m \geq 0$ Then, for $h \sim \mathsf{FGF}^\mathsf{M}_{s,m}$ (and in the compact case also for $h \sim \mathring{\mathsf{FGF}}^\mathsf{M}_{s,m}$),*

$$\mathbf{E}\left[\sup_{x\in\mathsf{M}} h^\bullet(x)\right] \leq 24 \cdot \int_0^\infty \Big(\log N_{s,m}(\varepsilon)\Big)^{1/2}\,\mathrm{d}\varepsilon\ .$$

In Section 6 we will study in detail the asymptotics of the Green kernel close to the diagonal and in particular derive sharp estimates for the noise distance ρ in terms of the Riemannian distance d. This will lead to sharp estimates for the covering numbers $N_{s,m}(\varepsilon)$ and thus in turn to sharp estimates for the expected maximum of the random field.

4. Random Riemannian Geometry. Let a Riemannian manifold (M,g) be given together with a Fractional Gaussian Field $h^\bullet\sim\mathsf{FGF}^{\mathsf{M}}_{s,m}$ with $s>n/2$ and $m>0$. If M is compact, we alternatively can choose $h^\bullet\sim\mathring{\mathsf{FGF}}^{\mathsf{M}}_{s,m}$ with $s>n/2$ and $m\geq 0$. In the sequel, we assume that either M is closed or $m>0$ and (M,g) has bounded geometry.

For almost every $\omega\in\Omega$, by Propositions 3.10 and 3.22, h^ω is a continuous function on M. For each such ω, we consider the Riemannian manifold

$$(\mathsf{M},\mathsf{g}^\omega)\quad\text{with}\quad \mathsf{g}^\omega:=e^{2h^\omega}\,\mathsf{g}\ , \tag{4.1}$$

the new metric being the conformal change of the metric g by the conformal factor h^ω. In other words, we consider the *random Riemannian manifold*

$$\mathsf{M}^\bullet:=(\mathsf{M},\mathsf{g}^\bullet)\quad\text{with}\quad \mathsf{g}^\bullet:=e^{2h^\bullet}\,\mathsf{g} \tag{4.2}$$

with the *random Riemannian metric* $\mathsf{g}^\bullet\colon\omega\mapsto\mathsf{g}^\omega$.

Assuming that M is closed, for a.e. ω, the Riemannian metric g^ω is of class $\mathcal{C}^k$ on M for $k:=\lceil s-n/2\rceil-1\geq 0$, where we set $\lceil a\rceil:=\min(\mathbb{Z}\cap[a,\infty))$. In particular, for $s>n/2+2$, it is almost surely of class $\mathcal{C}^2$, and the Riemannian manifolds M^ω may be studied by smooth techniques. Our main interest in the sequel will be in the case $s\in(n/2,n/2+2]$ where no such techniques are directly applicable and where we have no classical curvature concepts at our disposal.

4.1. *Random Dirichlet Forms and Random Brownian Motions.* Our approach to geometry, spectral analysis, and stochastic calculus on the randomly perturbed Riemannian manifolds $(\mathsf{M},\mathsf{g}^\bullet)$ will be based on Dirichlet-form techniques. Before going into details, let us recall some standard results on the canonical Dirichlet form on the 'un-perturbed' Riemannian manifold.

Remark 4.1. The *canonical Dirichlet form* on the Riemannian manifold (M,g), e.g. [12, §5.1, p. 148], is the closed bilinear form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathrm{vol}_\mathsf{g})$ given by $\mathcal{F}:=W^{1,2}_*$ and

$$\mathcal{E}(\varphi,\psi):=\frac{1}{2}\int\langle\mathrm{d}\varphi\,|\,\mathrm{d}\psi\rangle_{\mathsf{g}_*}\,\mathrm{dvol}_\mathsf{g}=\frac{1}{2}\int\langle\nabla\varphi\,|\,\nabla\psi\rangle_\mathsf{g}\,\mathrm{dvol}_\mathsf{g}\ . \tag{4.3}$$

Here g_* denotes the inverse metric tensor obtained from g by musical isomorphism, d the differential on M, and ∇ the gradient; for functions in $W^{1,2}_*$, differentials and gradients have to be understood in the weak sense. In fact, however, $\mathcal{C}^\infty_c$ is dense in the form domain $\mathcal{F}$ and thus in (4.3) we can restrict ourselves to $\varphi,\psi\in\mathcal{C}^\infty_c$.

The form $(\mathcal{E},\mathcal{F})$ is a regular, strongly local, conservative Dirichlet form properly associated with the standard Brownian motion $\mathbf{B}$ on (M,g), the Markov diffusion process with transition kernel p_t introduced in §2.

The canonical Dirichlet form and the Laplace–Beltrami operator on (M,g) uniquely determine each other by

$$\mathcal{E}(\varphi,\psi)=-\frac{1}{2}\int\Delta\varphi\,\psi\,\mathrm{dvol}_\mathsf{g}\ ,\qquad \varphi,\psi\in\mathcal{C}^\infty_c\ .$$

Under conformal transformations with non-differentiable weights, however, the latter no longer admits a closed expression whereas the former still is easily representable.

Remark 4.2. If $\mathsf{g}'=e^{2f}\mathsf{g}$ is a conformal change of the metric g by means of a smooth weight f, then $\mathsf{g}'_*=e^{-2f}\mathsf{g}_*$, $\mathrm{vol}'_\mathsf{g}=e^{nf}\mathrm{vol}_\mathsf{g}$, and $\nabla'\varphi=e^{-2f}\nabla\varphi$. Thus in particular,

$$\mathcal{E}'(\varphi,\psi):=\frac{1}{2}\int\langle\mathrm{d}\varphi\,|\,\mathrm{d}\psi\rangle_{\mathsf{g}_*}\,e^{(n-2)f}\,\mathrm{dvol}_\mathsf{g}=\frac{1}{2}\int\langle\nabla\varphi\,|\,\nabla\psi\rangle_\mathsf{g}\,e^{(n-2)f}\,\mathrm{dvol}_\mathsf{g}\ ,$$

and $\Delta'\varphi=e^{-2f}\big(\Delta\varphi+(n-2)\,\langle\nabla f\,|\,\nabla\varphi\rangle_\mathsf{g}\big)$.

Now let us turn to the randomly perturbed Riemannian manifolds $(\mathsf{M}, \mathsf{g}^\bullet)$.

THEOREM 4.3. *Let $h^\bullet \sim \mathsf{FGF}_{s,m}$ with $m > 0$ and $s > n/2$. Then,*

(*a*) *for* $\mathbf{P}$*-a.e.* $\omega \in \Omega$, *the quadratic form* $(\mathcal{E}^\omega, \mathcal{C}_c^\infty)$

$$\mathcal{E}^\omega(\varphi, \psi) = \frac{1}{2} \int \langle \nabla\varphi \,|\, \nabla\psi \rangle_{\mathsf{g}}\, e^{(n-2)h^\omega}\, \mathrm{dvol}_{\mathsf{g}} \ , \qquad \varphi, \psi \in \mathcal{C}_c^\infty \subset L^2(e^{nh^\omega}\mathrm{vol}_{\mathsf{g}}) \ , \tag{4.4}$$

is closable on $L^2(e^{nh^\omega}\mathrm{vol}_{\mathsf{g}})$*;*

(*b*) *its closure* $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ *is a regular, irreducible, strongly local Dirichlet form, properly associated with an* $e^{nh^\omega}\mathrm{vol}_{\mathsf{g}}$*-symmetric Markov diffusion process* $\mathbf{B}^\omega$ *on* M*;*

(*c*) *the generator of the closed bilinear form* $(\mathcal{E}^\omega, \mathcal{F}^\omega)$*, denoted by* Δ^ω*, is the unique self-adjoint operator on* $L^2(e^{nh^\omega}\mathrm{vol}_{\mathsf{g}})$ *with* $\mathcal{D}(\Delta^\omega) \subset \mathcal{F}^\omega$ *and*

$$\mathcal{E}^\omega(\varphi, \psi) = -\frac{1}{2} \int (\Delta^\omega \varphi)\, \psi\, e^{nh^\omega} \mathrm{dvol}_{\mathsf{g}} \ , \qquad \varphi \in \mathcal{D}(\Delta^\omega) \ , \ \psi \in \mathcal{F}^\omega \ ; \tag{4.5}$$

(*d*) *the associated intrinsic distance*

$$\mathsf{d}_{\mathcal{E}^\omega}(x,y) := \sup \left\{ |f(x) - f(y)| : f \in \mathcal{F}^\omega \cap \mathcal{C}^0(\mathsf{M}) \ , \ |\nabla f|^2 \le e^{-nh^\omega} \quad \mathrm{vol}_{\mathsf{g}}\text{-}a.e. \right\}$$

coincides with the Riemannian distance d^ω *on* M *given by*

$$\mathsf{d}^\omega(x,y) := \inf \left\{ \int_0^1 e^{h^\omega(\gamma_r)} \sqrt{\mathsf{g}(\dot\gamma_r, \dot\gamma_r)}\, \mathrm{d}r : \ \gamma \in \mathcal{AC}\big([0,1]; \mathsf{M}\big) \ , \ \gamma_0 = x \ , \ \gamma_1 = y \right\} \ . \tag{4.6}$$

PROOF. (*a*) Let ω be given such that h^ω is continuous. Then both $\sigma := e^{nh^\omega}$ and $\rho := e^{(n-2)h^\omega}$ are positive and in L^1_{loc} and so is $1/\rho$. In particular, the weights thus satisfy the so-called Hamza condition. A proof of closability under this condition, in the case $\mathsf{M} = \mathbb{R}^n$, is given in [40, §II.2(a)], and, for general manifolds in the case $U = \mathsf{M}$ and $\sigma \equiv 1$, in [2, Thm. 4.2]. The general case readily follows.

(*b*)+(*c*) For the Markov property, see e.g. [19, Example 1.2.1 and Thm. 3.1.1], for the strong locality and the regularity see e.g. [19, Exercise 3.1.1]. Since the local domain $\mathcal{F}^\omega_{\mathrm{loc}}$ coincides with the local domain $\mathcal{F}$, the irreducibility follows from [5, Thm. 4.5]. The assertions on the associated Markov process and on the generator easily follow.

(*d*) Choosing ω such that h^ω is continuous, the claim follows from [29, Lem. 3.5]. ∎

DEFINITION 4.4. (*a*) The operator Δ^ω is called the *Laplace–Beltrami* or *Laplace operator* on M^ω.

(*b*) The family of operators $\big(e^{t\Delta^\omega/2}\big)_{t>0}$ on $L^2(e^{nh^\omega}\mathrm{vol}_{\mathsf{g}})$ is called the *heat semigroup* on M^ω.

(*c*) The process $\mathbf{B}^\omega$ is called *Brownian motion* on M^ω.

(*d*) A function φ on an open subset $U \subset \mathsf{M}^\omega$ is called *weakly harmonic* if $\varphi \in W^{1,2}_{\mathrm{loc}}(U)$ and $\mathcal{E}^\omega(\varphi, \psi) = 0$ for all $\psi \in \mathcal{C}_c^\infty$ with $\mathrm{supp}(\psi) \subset U$.

THEOREM 4.5. *Let $s > n/2$, $m > 0$, and $h^\bullet \sim \mathsf{FGF}_{s,m}$. Then, for* $\mathbf{P}$*-a.e.* $\omega \in \Omega$*, the following assertions hold:*

(*i*) *every weakly harmonic function on* $U \subset \mathsf{M}^\omega$ *admits a version which is locally Hölder continuous (w.r.t.* d *and, equivalently, w.r.t.* d^ω*);*

(*ii*) *the heat semigroup* $\big(e^{t\Delta^\omega/2}\big)_{t>0}$ *on* M^ω *has an integral kernel* $p^\omega_t(x,y)$ *which is jointly locally Hölder continuous in* t, x, y*;*

(*iii*) *for every starting point, the distribution of the Brownian motion on* M^ω *is uniquely defined.*

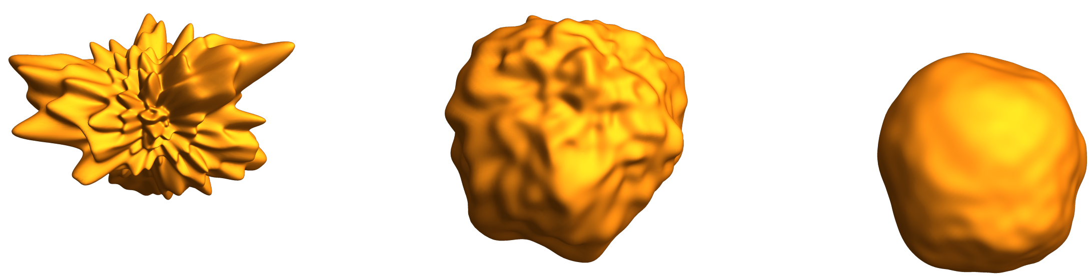
(*iv*) *For all* $x, y \in \mathsf{M}$,

$$\lim_{t\to 0} 2t \log p^\omega_t(x,y) = -\mathsf{d}^\omega(x,y)^2 \ .$$

Proof. Let ω be given such that h^ω is continuous. Then, locally on M, the Dirichlet forms $\mathcal{E}^\omega$ and $\mathcal{E}$ as well as the measures $\mathrm{vol}^\omega_{\mathsf{g}} := e^{nh^\omega}\mathrm{vol}_{\mathsf{g}}$ and $\mathrm{vol}_{\mathsf{g}}$ are comparable. In other words, the 'Riemannian structure' for g^ω is locally uniformly elliptic w.r.t. the structure for g in the sense of [48]. Thus, assertion (i), resp. (ii), follows from either [48, Cor. 5.5] or [54, Cor. 3.3, resp. Prop. 3.1 and Thm. 3.5].

If M is compact, assertion (iii) is a consequence of (ii). For general M, we will choose an exhaustion of M by relatively compact, open sets $B_n \nearrow \mathsf{M}$ which are regular for $\mathcal{E}^\omega$. For instance, according to Wiener's criterion, we can choose the open balls $B_n := B_n(o)$, $n \in \mathbb{N}$, around any fixed point $o \in \mathsf{M}$. Let $\mathcal{E}^{n,\omega}$ denote the Dirichlet form obtained from $\mathcal{E}^\omega$ by imposing Dirichlet boundary conditions on $\mathsf{M}\setminus B_n$, and let $G^{n,\omega}_{1,m}(x,y)$ denote the associated resolvent kernel. Then for any fixed $x \in B_n$ the latter kernel is continuous in $y \in B_n$ (as a consequence of (ii)) and it vanishes as y approaches ∂B_n (due to the regularity of ∂B_n). Thus $\left(G^{n,\omega}_{1,m}\right)_{m>0}$ extends to a Feller resolvent on the compact space $\overline{B_n}$. The associated Feller process $\mathbf{B}^{n,\omega}$ is pointwise well-defined. It will be called Random Brownian Motion with absorption on $\mathsf{M}\setminus B_n$. For any given $k,\ell \in \mathbb{N}$ with $k,\ell \geq n$, the processes $\mathbf{B}^{k,\omega}$ and $\mathbf{B}^{\ell,\omega}$ can be modelled on the same probability space and such that their trajectories coincide until the first hitting time of $\mathsf{M}\setminus B_n$. With a diagonal argument we then construct the process $\mathbf{B}^\omega$ as follows: if it starts in $B_n \setminus B_{n-1,\omega}$, it follows the trajectories of the process $\mathbf{B}^{n+1,\omega}$ until it hits ∂B_n. Then it follows the trajectories of $\mathbf{B}^{n+2,\omega}$ etc. This yields a pointwise well-defined process. By monotonicity of resolvent kernels and Dirichlet forms, it is associated with the monotone increasing limit of Dirichlet forms $\mathcal{E}^\omega = \lim_{n\nearrow\infty}\mathcal{E}^{n,\omega}$.

Assertion (iv) follows from the main result in [44]. ■



(a) $s = 3/2$ (b) $s = 2$ (c) $s = 5/2$

Fig 3: A realization of the random metric $\mathsf{g}^\bullet_\ell = e^{2h^\bullet_\ell}\mathsf{g}$ on $\mathbb{S}^2$, $\ell = 30$.

4.2. *Random Brownian Motions in the $\mathcal{C}^1$-Case.* More precise insights into the analytic and probabilistic structures on the random Riemannian manifold $(\mathsf{M},\mathsf{g}^\bullet)$ can be gained if the regularity parameter s is larger than $n/2+1$. In this case, the conformal weight $h^\bullet$ is a.s. a $\mathcal{C}^1$-function.

To provide an explicit representation for the perturbed Brownian motion, we need some notations and concepts from the abstract theory of Dirichlet forms.

Martingale additive functionals. Denote the Brownian motion on the ('unperturbed') Riemannian manifold (M,g) by

$$\mathbf{B} := \left(\Xi, (\mathscr{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in\mathsf{M}}\right) .$$

Lemma 4.6 (‘Fukushima decomposition’, see [19, §6.3]). *(a) For each continuous $\psi \in W^{1,2}_*$, there exist a unique martingale additive functional $M^{[\psi]}$ and a unique continuous additive functional $N^{[\psi]}$ which is of zero energy such that*

$$\psi(X_t) = \psi(X_0) + M^{[\psi]}_t + N^{[\psi]}_t \qquad t \in [0,\zeta) \qquad P_x\text{-a.s. for q.e. } x \in \mathsf{M} \ . \tag{4.7}$$

The quadratic variation of $M^{[\psi]}$ is given by

$$\langle M^{[\psi]} \rangle_t = \int_0^t \left|\nabla\psi(X_s)\right|^2_{\mathsf{g}} \mathrm{d}s \qquad t \in [0,\zeta) \qquad P_x\text{-a.s. for q.e. } x \in \mathsf{M} \tag{4.8}$$

for any choice of a Borel version of the function $|\nabla\psi|_{\mathsf{g}} \in L^2(\mathsf{M})$.

(b) For each continuous $\psi \in W^{1,2}_{\mathrm{loc}}$, there exists a unique local martingale additive functional $M^{[\psi]} = \big(M^{[\psi]}_t\big)_{t\in[0,\zeta)}$ such that

$$M^{[\psi]}_t = M^{[\psi_n]}_t \qquad t \in [0,\tau_n) \qquad P_x\text{-a.s. for q.e. } x \in \mathsf{M}$$

where, for every $n \in \mathbb{N}$, we let $M^{[\psi_n]}$ be the martingale additive functional associated with a function $\psi_n \in W^{1,2}_*$ such that $\psi = \psi_n$ a.e. on M_n, for some exhausting sequence of relatively compact open sets $\mathsf{M}_n \nearrow \mathsf{M}$, and where $\tau_n \coloneqq \inf\{t \geq 0 : X_t \notin \mathsf{M}_n\}$. As before, the energy $\langle M^{[\psi]}\rangle_t$ for $t \in [0,\zeta)$ is given by (4.8), now with $|\nabla\psi|_{\mathsf{g}} \in L^2_{\mathrm{loc}}(\mathsf{M})$.

(c) For each continuous $\psi \in W^{1,2}_{\mathrm{loc}}$, a super-martingale, multiplicative functional is defined by

$$L^{[\psi]}_t \coloneqq \exp\left(M^{[\psi]}_t - \tfrac{1}{2}\big\langle M^{[\psi]}\big\rangle_t\right) \mathbb{1}_{\{t<\zeta\}} \ . \tag{4.9}$$

For the defining properties of ‘martingale additive functionals’ and of ‘continuous additive functionals of zero energy’ (as well as for the relevant equivalence relations that underlie the uniqueness statements) we refer to the monograph [19].

Example 4.7. If $\mathsf{M} = \mathbb{R}^n$ and $\psi \in \mathcal{C}^2$ then $\big(M^{[\psi]}_t\big)_t$ is the martingale part in the *Itô decomposition*

$$\psi(X_t) = \psi(X_0) + \int_0^t \nabla\psi(X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t \Delta\psi(X_s)\,\mathrm{d}s \qquad P_x\text{-a.s. for all } x \in \mathsf{M} \ .$$

We are now able to provide an explicit construction of the Brownian motion

$$\mathbf{B}^\omega \coloneqq \Big(\Xi, (\mathscr{F}^\omega_t)_{t\geq 0}\,, (X^\omega_t)_{t\geq 0}\,, (P^\omega_x)_{x\in\mathsf{M}_\partial}\,, \zeta^\omega\Big) \tag{4.10}$$

on the randomly perturbed manifold $(\mathsf{M}, \mathsf{g}^\bullet)$ which previously was introduced by abstract Dirichlet form techniques.

Theorem 4.8. *Let $h^\bullet \sim \mathsf{FGF}_{s,m}$ with $m > 0$ and $s > n/2+1$. Then for $\mathbf{P}$-a.e. $\omega \in \Omega$, the process $\mathbf{B}^\omega$ is a time-changed Girsanov transform of the standard Brownian motion $\mathbf{B}$ on (M, g). More precisely:*

(a) For q.e. $x \in \mathsf{M}$, the law P^ω_x is locally absolutely continuous up to life-time ζ^ω w.r.t. the law P_x of $\mathbf{B}$ on the natural filtration $(\mathscr{F}_t)_{t\geq 0}$ of $\mathbf{B}$, viz.

$$\left.\frac{\mathrm{d}P^\omega_x}{\mathrm{d}P_x}\right|_{\mathscr{F}_t\cap\{t<\zeta^\omega\}} = \exp\left(\frac{n-2}{2} M^{\left[h^\omega\right]}_t - \frac{(n-2)^2}{8}\left\langle M^{\left[h^\omega\right]}\right\rangle_t\right) \ , \qquad t \in [0,\zeta^\omega) \ . \tag{4.11}$$

(b) For q.e. $x \in \mathsf{M}$, a trajectory $(X^\omega_t)_{t\in[0,\zeta^\omega)}$ started at x satisfies

$$X^\omega_t = X_{\lambda^\omega_t} \ , \qquad \lambda^\omega_t \coloneqq \inf\{s > 0 : C^\omega_s > t\} \ , \qquad C^\omega_t \coloneqq \int_0^t e^{2h^\omega(X_s)}\,\mathrm{d}s \ . \tag{4.12}$$

(c) *The process* $\mathbf{B}^\omega$ *has life-time* $\zeta^\omega = C^\omega_\infty$.

REMARK 4.9 (On conservativeness). It is not clear to the authors whether the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ is $\mathbf{P}$-a.s. conservative. In particular, the random Brownian motion (4.10) may in principle have *finite* life-time ζ^ω.

PROOF OF THEOREM 4.8. By Proposition 3.10, the random field $h^\bullet$ lies a.s. in $W^{1,2}_{\mathrm{loc}} \cap \mathcal{C}(\mathsf{M})$. Thus, also $e^{(n-2)h^\omega/2} \in W^{1,2}_{\mathrm{loc}} \cap \mathcal{C}(\mathsf{M})$, and we may consider the Girsanov transform $(\mathcal{E}^\phi, \mathcal{F}^\phi)$, e.g. [19, §6.3], of the canonical form $(\mathcal{E}, \mathcal{F})$ by the function $\phi = \phi^\omega := e^{(n-2)h^\omega/2}$, satisfying

$$\mathcal{E}^\phi(\varphi, \psi) = \frac{1}{2}\int \mathsf{g}_*(\mathrm{d}\varphi, \mathrm{d}\psi)\, \phi^2 \mathrm{dvol}_{\mathsf{g}} \ , \qquad \varphi, \psi \in \mathcal{C}^\infty_c \subset L^2(\phi^2 \, \mathrm{vol}_{\mathsf{g}}) \ . \tag{4.13}$$

By standard results in the theory of Dirichlet forms, $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ is a regular Dirichlet form on $L^2(\phi^2 \, \mathrm{vol}_{\mathsf{g}})$, properly associated with the Girsanov transform $\mathbf{B}^\phi$ of the standard Brownian motion $\mathbf{B}$. Indeed, choosing $G_n := B_n(o)$, $n \in \mathbb{N}$, for some fixed $o \in \mathsf{M}$ yields a nondecreasing sequence of (quasi-)open sets with $\bigcup_n G_n = \mathsf{M}$ such that $\phi, 1/\phi$ and $\phi|\nabla\phi| \in L^2(G_n, \mathrm{vol}_g)$. Then, according to [18, Thm. 4.9], the Girsanov-transformed process is properly associated with the quasi-regular Dirichlet form obtained as the closure of $\mathcal{E}^\phi$ with pre-domain

$$\bigcup_{n\in\mathbb{N}} \mathcal{F}_{G_n}$$

where as usual $\mathcal{F}_{G_n} := \{\psi \in \mathcal{F} : \tilde{\psi} = 0 \text{ q.e. on } \mathsf{M} \setminus G_n\}$. Since obviously $\mathcal{C}^\infty_c \subset \bigcup_{n\in\mathbb{N}} \mathcal{F}_{G_n} \subset \mathcal{F}$, this Dirichlet form is even regular.

Now, let us denote by $\left(\mathcal{E}^{\phi,\mu}, \mathcal{F}^{\phi,\mu}\right)$ the time-changed form, e.g. [19, §6.2], of $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ with respect to the measure $\mu = \mu^\omega := e^{2h^\omega} \mathrm{vol}_{\mathsf{g}}$. It is again standard that $\left(\mathcal{E}^{\phi,\mu}, \mathcal{F}^{\phi,\mu}\right)$ is a regular Dirichlet form on $L^2(\phi^2\mu)$, properly associated with the time change $\mathbf{B}^{\phi,\mu}$ of $\mathbf{B}^\phi$ induced by μ. Since $\phi^2\mu = e^{nh^\omega} \mathrm{vol}_{\mathsf{g}}$, the form $\mathcal{E}^{\phi,\mu}$ coincides on $\mathcal{C}^\infty_c$ with the form $\mathcal{E}^\omega$ defined in (4.4). By regularity of both forms we conclude that $\left(\mathcal{E}^{\phi,\mu}, \mathcal{F}^{\phi,\mu}\right) = (\mathcal{E}^\omega, \mathcal{F}^\omega)$ is the canonical form on the Riemannian manifold $\mathsf{M}^\omega = (\mathsf{M}, \mathsf{g}^\omega)$, properly associated with the corresponding Brownian motion $\mathbf{B}^\omega = \mathbf{B}^{\phi,\mu}$.

In order to characterize the law of $\mathbf{B}^\omega$ as in assertion (a), (b), it suffices to note the following. Since $\mathbf{B}$ is conservative, it is noted in e.g. [17, §5 a)] that the process

$$\mathbf{B}^\phi := \left(\Xi^\phi, \left(\mathscr{F}^\phi_t\right)_{t\geq 0}, \left(X^\phi_t\right)_{t\geq 0}, \left(P^\phi_x\right)_{x\in\mathsf{M}_\partial}, \zeta^\phi\right)$$

satisfies $X^\phi_t = X_t$ for $t > 0$ and

$$\left.\frac{\mathrm{d}P^\phi_x}{\mathrm{d}P_x}\right|_{\mathscr{F}_t \cap \{t<\tau_{n-1}\}} = \exp\left(M^{[\log\phi_n]}_t - \tfrac{1}{2}\left\langle M^{[\log\phi_n]}\right\rangle_t\right) \ , \qquad n \in \mathbb{N} \ ,$$

where the functions $\log\phi_n$ are given as in Lemma 4.6(b) for $\log\phi$ in place of ψ, and the stopping times τ_n are defined as $\tau_n := \inf\{t > 0 : X_t \notin \mathsf{M}_n\}$ with M_n again as in Lemma 4.6(b). The conclusion follows by letting n to infinity, since $\mathbf{B}^\omega$ is a time change of $\mathbf{B}^\phi$, and therefore: $P^\omega_x = P^\phi_x$ for each $x \in \mathsf{M}$. Again since $\mathbf{B}^\omega$ is a time change of $\mathbf{B}^\phi$, one has that $X^\omega_t = X^\phi_{\lambda^\omega_t} = X_{\lambda^\omega_t}$ with λ^ω_t as in Equation (4.12) for each $t > 0$, cf. [19, Eqn. (6.2.5)]; assertion (c) is [19, Exercise 6.2.1]. ■

5. Geometric and Functional Inequalities for RRG's. Given a Riemannian manifold (M, g) and the intrinsically defined FGF noise $h^\bullet$, we ask ourselves: how do basic geometric and spectral theoretic quantities of (M, g) change if we switch on the noise? For instance, will $\mathbf{E}\,\mathrm{vol}_{\mathsf{g}^\bullet}(\mathsf{M})$ be smaller or larger than $\mathrm{vol}_{\mathsf{g}}(\mathsf{M})$? How about $\lambda^\bullet_0$, the random spectral bound, or $\lambda^\bullet_1$, the random spectral gap? Can we estimate them in terms of the unperturbed spectral quantities? Can we estimate in average the rate of convergence to equilibrium on the random manifold?

In the following, let a Riemannian manifold (M, g) *of bounded geometry* be given and a random field $h^\bullet \sim \mathsf{FGF}_{s,m}$ with $m > 0$ and $s > n/2$. As before, put $\mathsf{g}^\bullet = e^{2h^\bullet}\mathsf{g}$.

5.1. *Volume, Length, and Distance.* We will compare the random volume, random length, and random distance in the random Riemannian manifold $(M, \mathsf{g}^\bullet)$ with analogous deterministic quantities in geometries obtained by suitable averages of the conformal weight. Recall that $\theta(x) := G_{s,m}(x,x) = \mathbf{E}[h^\bullet(x)^2] \geq 0$ and put

$$\overline{\mathsf{g}}^n := e^{n\,\theta}\mathsf{g}, \qquad \overline{\mathsf{g}}^1 := e^{\theta}\mathsf{g} \ .$$

Further, recall that for given ω with continuous h^ω, the volume of a measurable subset $A \subset \mathsf{M}$ w.r.t. the Riemannian tensor g^ω is given by

$$\mathrm{vol}_{\mathsf{g}^\omega}(A) := \int_A e^{nh^\omega} \mathrm{dvol}_{\mathsf{g}} \ .$$

Similarly, the length of an absolutely continuous curve $\gamma : [0,1] \to \mathsf{M}$ w.r.t. the Riemannian tensor g^ω is given by

$$L_{\mathsf{g}^\omega}(\gamma) := \int_0^1 e^{h^\omega(\gamma_r)}\, |\dot\gamma_r|_{\mathsf{g}}\, \mathrm{d}r \ .$$

PROPOSITION 5.1. *For any measurable $A \subset \mathsf{M}$*

$$\mathbf{E}\big[\mathrm{vol}_{\mathsf{g}^\bullet}(A)\big] = \mathrm{vol}_{\overline{\mathsf{g}}^n}(A) \geq \mathrm{vol}_{\mathsf{g}}(A) \ .$$

In particular,

$$e^{n^2\theta^*/2} \cdot \mathrm{vol}_{\mathsf{g}}(A) \geq \mathbf{E}\big[\mathrm{vol}_{\mathsf{g}^\bullet}(A)\big] \geq e^{n^2\theta_*/2} \cdot \mathrm{vol}_{\mathsf{g}}(A)$$

with $\theta_* := \inf_x G_{s,m}(x,x)$, $\theta^* := \sup_x G_{s,m}(x,x)$.

PROOF. It suffices to note that

$$\mathbf{E}\big[\mathrm{vol}_{\mathsf{g}^\bullet}(A)\big] = \int_A \mathbf{E}[e^{nh^\bullet}]\mathrm{dvol}_{\mathsf{g}} = \int_A e^{n^2 G_{s,m}(x,x)/2} \mathrm{dvol}_{\mathsf{g}}(x) = \mathrm{vol}_{\overline{\mathsf{g}}^n}(A) \ . \qquad \blacksquare$$

PROPOSITION 5.2. *For any absolutely continuous curve $\gamma : [0,1] \to \mathsf{M}$*

$$\mathbf{E}\big[L_{\mathsf{g}^\bullet}(\gamma)\big] = L_{\overline{\mathsf{g}}^1}(\gamma) \geq L_{\mathsf{g}}(\gamma) \ .$$

PROOF. It suffices to note that

$$\mathbf{E}L_{\mathsf{g}^\bullet}(\gamma) = \int_0^1 \mathbf{E}\Big[e^{h^\bullet(\gamma_r)}\Big]\, |\dot\gamma_r|_{\mathsf{g}}\, \mathrm{d}r = \int_0^1 e^{\frac{1}{2}\mathbf{E}[h^\bullet(\gamma_r)^2]}\, |\dot\gamma_r|_{\mathsf{g}}\, \mathrm{d}r = L_{\overline{\mathsf{g}}^1}(\gamma) \ . \qquad \blacksquare$$

PROPOSITION 5.3. *For each $x, y \in \mathsf{M}$*

$$\mathsf{d}_{\overline{\mathsf{g}}^1}(x,y) \geq \mathbf{E}\big[\mathsf{d}_{\mathsf{g}^\bullet}(x,y)\big] \geq \mathsf{d}_{\mathsf{g}}(x,y) \cdot e^{-\mathbf{E}\big[\sup_{z\in\mathsf{M}} h^\bullet(z)\big]} \ .$$

PROOF. Given x and y, let $\overline\gamma$ be any absolutely continuous curve connecting them. Then

$$L_{\overline{\mathsf{g}}^1}(\overline\gamma) = \mathbf{E}\big[L_{\mathsf{g}^\bullet}(\overline\gamma)\big] \geq \mathbf{E}\big[\inf_\gamma L_{\mathsf{g}^\bullet}(\gamma)\big] = \mathbf{E}\big[\mathsf{d}_{\mathsf{g}^\bullet}(x,y)\big].$$

This proves the upper bound.

For the lower bound, let us assume that $\inf_{z\in\mathsf{M}} h^\bullet(z)$ is finite for almost every ω. Otherwise, the lower bound is trivially satisfied. Then $(\mathsf{M}, \mathsf{g}^\bullet)$ is complete and locally compact so that there exists a constant speed geodesic $\gamma^\omega : [0,1] \to \mathsf{M}$ connecting x and y. Then

$$\mathsf{d}_{\mathsf{g}^\omega}(x,y) = \int_0^1 e^{h^\omega(\gamma^\omega_s)} \cdot |\dot\gamma^\omega_s|_{\mathsf{g}}\, \mathrm{d}s \geq \mathsf{d}_{\mathsf{g}}(x,y) \cdot \int_0^1 e^{h^\omega(\gamma^\omega_s)}\, \mathrm{d}s \geq \mathsf{d}_{\mathsf{g}}(x,y) \cdot \inf_{z\in\mathsf{M}} e^{h^\omega(x)} \ .$$

Then, by Jensen's inequality and symmetry of the random field,

$$\mathbf{E}\big[\mathsf{d}_{\mathsf{g}^\bullet}(x,y)\big] \geq \mathsf{d}_{\mathsf{g}}(x,y) \cdot \mathbf{E}\Big[\inf_{z\in\mathsf{M}} e^{h^\bullet(z)}\Big] \geq \mathsf{d}_{\mathsf{g}}(x,y) \cdot e^{-\mathbf{E}\big[\sup_{z\in\mathsf{M}} h^\bullet(z)\big]} \ . \qquad \blacksquare$$

5.2. *Spectral Bound.* The L^2-*spectral bound* for $(\mathsf{M}, \mathsf{g}^\omega)$ is defined by

$$\lambda_0^\omega := \inf \operatorname{spec}(-\Delta_{\mathsf{g}^\omega}) \ .$$

By the standard variational characterization of the spectrum via Rayleigh–Riesz quotients we have that

$$\lambda_0^\omega = \inf \left\{ \frac{\displaystyle\int_\mathsf{M} |\nabla u|_\mathsf{g}^2 \, e^{(n-2)h^\omega} \, \mathrm{dvol}_\mathsf{g}}{\displaystyle\int_\mathsf{M} u^2 \, e^{nh^\omega} \mathrm{dvol}_\mathsf{g}} : u \in \mathcal{C}_c^\infty \right\} . \tag{5.1}$$

Note that λ_0 is not necessarily 0, e.g. $\lambda_0 = \frac{(n-1)^2}{4}$ for the hyperbolic space of curvature -1.

LEMMA 5.4 (Measurability of the spectral bound). *The function $\omega \mapsto \lambda_0^\omega$ is measurable.*

PROOF. Let $\mathcal{C}_c^\infty$ be endowed with the $\mathcal{C}^1$-topology τ_1, and note that $(\mathcal{C}_c^\infty, \tau_1)$ is separable. Further note that, $\mathbf{P}$-almost surely, $(\mathcal{C}_c^\infty, \tau_1)$ embeds continuously into $(\mathcal{F}^\omega, (\mathcal{E}^\omega)_1^{1/2})$, and that this embedding has dense image since $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ is a regular Dirichlet form. Therefore, there exists a countable $\mathbb{Q}$-vector space $D \subset \mathcal{C}_c^\infty$ simultaneously $(\mathcal{E}^\omega)_1^{1/2}$-dense in $\mathcal{F}^\omega$ for $\mathbf{P}$-a.e. ω. As a consequence, the variational characterization (5.1) holds as well when replacing $\mathcal{C}_c^\infty$ by D. Since the integrals' quotient in this characterization is measurable as a function of ω, the corresponding infimum over D is as well a measurable function of ω, since D is countable and the infimum of any countable family of measurable functions is again measurable. ■

PROPOSITION 5.5. *For $n \geq 2$*

$$\left(\mathbf{E}\left[\lambda_0^{\bullet\,-n/2}\right]\right)^{-2/n} \leq \lambda_0^n$$

with λ_0^n the spectral bound for the metric $\overline{\mathsf{g}}^n := e^{n\,\theta}\mathsf{g}$. In particular, whenever $\theta^* < \infty$, then

$$\left(\mathbf{E}\left[\lambda_0^{\bullet\,-n/2}\right]\right)^{-2/n} \leq e^{((n-2)\theta^* - n\theta_*)n/2} \cdot \lambda_0 \ ,$$

and, for homogeneous spaces,

$$\left(\mathbf{E}\left[\lambda_0^{\bullet\,-n/2}\right]\right)^{-2/n} \leq e^{-n\theta} \cdot \lambda_0 \ .$$

PROOF. For each u and a.e. ω

$$\int_\mathsf{M} u^2 e^{nh^\omega} \, \mathrm{dvol}_\mathsf{g} \leq \frac{1}{\lambda_0^\omega} \int_\mathsf{M} |\nabla u|^2 e^{(n-2)h^\omega} \, \mathrm{dvol}_\mathsf{g} \ .$$

Integrating w.r.t. $\mathrm{d}\mathbf{P}(\omega)$ and applying Hölder's inequality yield

$$\int_\mathsf{M} u^2 \cdot \mathbf{E}[e^{nh^\bullet}] \, \mathrm{dvol}_\mathsf{g} \leq \int_\mathsf{M} |\nabla u|_\mathsf{g}^2 \cdot \mathbf{E}\left[\left(\tfrac{1}{\lambda_0^\bullet}\right)^{n/2}\right]^{2/n} \cdot \mathbf{E}\left[e^{(n-2)h^\bullet \cdot \frac{n}{n-2}}\right]^{(n-2)/n} \mathrm{dvol}_\mathsf{g}$$

and thus with $\overline{h} := \frac{n}{2}\theta$,

$$\int_\mathsf{M} u^2 \cdot e^{n\overline{h}} \, \mathrm{dvol}_\mathsf{g} \leq \mathbf{E}\left[\left(\lambda_0^\bullet\right)^{-n/2}\right]^{2/n} \cdot \int_\mathsf{M} |\nabla u|_\mathsf{g}^2 \cdot e^{(n-2)\overline{h}} \, \mathrm{dvol}_\mathsf{g} \ .$$

Since this holds for all u we conclude that $\lambda_0^n \geq \left(\mathbf{E}[(\lambda_0^\bullet)^{-n/2}]\right)^{-2/n}$. ■

REMARK 5.6. Following the argumentation from the proof of Theorem 5.10 below, we can also derive a two-sided, pointwise estimate for the spectral bound, valid for almost every ω:

$$e^{-\alpha \sup |h^\omega|} \leq \frac{\lambda_0^\omega}{\lambda_0} \leq e^{\alpha \sup |h^\omega|} \tag{5.2}$$

with $\alpha := 2(n-1)$ if $n \geq 2$ and $\alpha := 2$ if $n = 1$.

5.3. *Spectral Gap.* In the following we assume that M is closed, and we let $\mathrm{vol}_{\mathrm{g}}^{\omega} = \mathrm{vol}_{\mathrm{g}^\omega} := e^{nh^\omega}\mathrm{vol}_{\mathrm{g}}$. Then, the Laplacian $\Delta_{\mathrm{g}^\omega}$ has compact resolvent and, in particular, it has discrete spectrum. The spectral gap is defined by

$$\lambda_1^\omega := \inf\left(\mathrm{spec}(-\Delta_{\mathrm{g}^\omega}) \setminus \{0\}\right) .$$

Denoting by

$$\pi^\omega f := \frac{1}{\mathrm{vol}_{\mathrm{g}}^{\omega}(\mathsf{M})} \int_{\mathsf{M}} f \mathrm{dvol}_{\mathrm{g}}^{\omega} ,$$

the mean value of f w.r.t. the measure $\mathrm{vol}_{\mathrm{g}}^{\omega}$, the spectral gap has the variational representation

$$\lambda_1^\omega = \inf\left\{ \frac{\displaystyle\int_{\mathsf{M}} |\nabla u|_{\mathrm{g}}^2\, e^{(n-2)h^\omega} \mathrm{dvol}_{\mathrm{g}}}{\displaystyle\int_{\mathsf{M}} (u-\pi^\omega u)^2 \mathrm{dvol}_{\mathrm{g}}^{\omega}} : u \in \mathcal{C}_c^\infty \right\}. \tag{5.3}$$

Hence the spectral gap is the smallest non-zero eigenvalue of the Laplacian and the inverse of the smallest constant for which the Poincaré inequality holds. By the very same proof of the measurability of the random spectral bound (Lemma 5.4) we have as well the following:

Lemma 5.7 (Measurability of the spectral gap). *The function $\omega \mapsto \lambda_1^\omega$ is measurable.*

The function $h^\bullet$ is $\mathbf{P}$-a.s. continuous by Proposition 3.10, thus $\mathbf{P}$-a.s. bounded by compactness of M. As a consequence, the $L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})$-norm is bi-Lipschitz equivalent to the $L^2(\mathrm{vol}_{\mathrm{g}})$-norm. Thus, the spaces $L^2(\mathrm{vol}_{\mathrm{g}})$ and $L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})$ coincide as sets. Again by boundedness of h^ω, the form $\mathcal{E}^\omega$ too is bi-Lipschitz equivalent to $\mathcal{E}$ on $\mathcal{C}_c^\infty$. Set $\mathcal{E}_1(u) := \mathcal{E}(u,u) + \|u\|_{L^2(\mathrm{vol}_{\mathrm{g}})}^2$, and analogously for ω. By the equivalence of the L^2-norms and forms established above, the norm $\mathcal{E}_1^{1/2}$ is bi-Lipschitz equivalent to the norm $(\mathcal{E}_1^\omega)^{1/2}$ on $\mathcal{C}_c^\infty$. Since M is compact, both forms are regular, thus $\mathcal{F}^\omega$ too coincides with $\mathcal{F}$ as a set and the bi-Lipschitz equivalence of $\mathcal{E}_1^{1/2}$ and $(\mathcal{E}_1^\omega)^{1/2}$ extends to $\mathcal{F}$.

Given ω with continuous h^ω, let $P_t^\omega := e^{t\Delta^\omega/2}$, $t>0$, denote the heat semigroup on $L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})$. For each $f \in L^2(\mathrm{vol}_{\mathrm{g}})$, the functions $P_t^\omega f$ will converge as $t \to \infty$ to $\pi^\omega f$. The rate of convergence is determined by λ_1^ω, viz.

$$\left\| P_t^\omega f - \pi^\omega f \right\|_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})} \leq e^{-\lambda_1^\omega t} \cdot \|f\|_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})}$$

or, equivalently,

$$\log \left\| P_t^\omega f - \pi^\omega f \right\|_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})} \leq -\lambda_1^\omega t + \log \|f\|_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})} .$$

Lemma 5.8. *The map $\omega \mapsto \|P_t^\omega f - \pi^\omega f\|_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})}$ is measurable for every $f \in L^2(\mathrm{vol}_{\mathrm{g}})$ and $t>0$.*

Proof. Firstly, let us discuss some heuristics. For $\alpha>0$, set $\mathcal{E}_\alpha^\omega(\,\cdot\,) := \mathcal{E}^\omega(\,\cdot\,) + \|\cdot\|_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})}^2$, and denote by $(G_\alpha^\omega)_{\alpha\geq 0}$ the $L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})$-resolvent semigroup of $(\mathcal{E}^\omega, \mathcal{F}^\omega)$, satisfying (e.g. [40, Thm. I.2.8, p. 18])

$$\mathcal{E}_\alpha^\omega(G_\alpha^\omega u, v) = \langle u \,|\, v\rangle_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})} , \qquad u \in L^2(\mathrm{vol}_{\mathrm{g}}^{\omega}) ,\ v \in \mathcal{F}^\omega .$$

We conclude the measurability in ω of the left-hand side from that of the right-hand side which is clear from the identifications of sets $L^2(\mathrm{vol}_{\mathrm{g}}^{\omega}) = L^2(\mathrm{vol}_{\mathrm{g}})$ and $\mathcal{F}^\omega = \mathcal{F}$. For fixed $t, \alpha > 0$, writing the series expansion of $e^{t\alpha(\alpha G_\alpha - 1)}$ we conclude that

$$\langle P_t^\omega u \,|\, v\rangle_{L^2(\mathrm{vol}_{\mathrm{g}}^{\omega})} , \qquad u \in L^2(\mathrm{vol}_{\mathrm{g}}^{\omega}) ,\ v \in \mathcal{F}^\omega ,$$

is measurable as a function of ω, since $P_t^\omega = \lim_{\alpha\to\infty} e^{t\alpha(\alpha G_\alpha - 1)}$. The measurability of $\omega \mapsto \langle \pi^\omega u \,|\, v\rangle$ may be concluded in a similar way, which would then show the assertion.

In order to make this argument rigorous, we resort to theory of direct integrals of quadratic forms in [13]. In light of Corollary 3.15, we may assume with no loss of generality that $(\Omega, \mathscr{F}, \mathbf{P})$ be the completion of a standard Borel space. Let $D \subset \mathcal{C}_c^\infty$ be the countable $\mathbb{Q}$-vector space simultaneously dense in $(\mathcal{F}^\omega, (\mathcal{E}^\omega)_1^{1/2})$ for $\mathbf{P}$-a.e. $\omega \in \Omega$ constructed in the proof of Lemma 5.4.

Now, let $\omega \mapsto \mathcal{F}^\omega$ be the measurable field of Hilbert spaces with underlying linear space $S := \prod_{\omega\in\Omega} \mathcal{F}^\omega = \mathcal{F}^\Omega$ in the sense of [15, §II.1.3, Dfn. 1, p. 164] with D as a fundamental sequence in the sense of [15, §II.1.3, Dfn. 1(iii), p. 164]. Further let $\omega \mapsto L^2(\mathrm{vol}_\mathrm{g}^\omega)$ be the measurable field of Hilbert spaces with underlying space generated by S as above in the sense of [15, §II.1.3, Prop. 4, p. 167]. In particular, for every $f \in L^2(\mathrm{vol}_\mathrm{g})$, the constant field $\omega \mapsto f \in L^2(\mathrm{vol}_\mathrm{g}^\omega)$ is a measurable vector field. Furthermore, since

$$\int_\Omega \|f\|^2_{L^2(\mathrm{vol}_\mathrm{g}^\omega)} \,\mathrm{d}\mathbf{P}(\omega) = \mathbf{E}\left[\int_\mathsf{M} f^2 \mathrm{dvol}_\mathrm{g}^\omega\right] = \|f\|^2_{L^2(\mathrm{vol}_\mathrm{g})} < \infty \ ,$$

all constant fields are elements of the direct integral of Hilbert spaces $\int_\Omega^\oplus L^2(\mathrm{vol}_\mathrm{g}^\omega)\,\mathrm{d}\mathbf{P}(\omega)$.

It is readily verified that $\omega \mapsto (\mathcal{E}^\omega, \mathcal{F}^\omega)$ is, by construction, a direct integral of quadratic forms in the sense of [13, Dfn. 2.11]. As a consequence, $\omega \mapsto P_t^\omega$ is a measurable field of bounded operators in the sense of [15, §II.2.1, Dfn. 1, p. 179] by [13, Prop. 2.13]. Furthermore, since $\omega \mapsto \mathrm{vol}_\mathrm{g}^\omega(\mathsf{M})$ is measurable, $\omega \mapsto \langle \pi^\omega u \,|\, v\rangle_{L^2(\mathrm{vol}_\mathrm{g}^\omega)}$ is measurable for every $u, v \in D$. Thus, $\omega \mapsto \pi^\omega$ is a measurable field of bounded operators by [15, §II.2.1, Prop. 1, p. 179].

It follows that $\omega \mapsto (P_t^\omega - \pi^\omega)$ is a measurable field of bounded operators. Now fix $f \in L^2(\mathrm{vol}_\mathrm{g})$. Since the constant field $\omega \mapsto f$ is measurable as discussed above, $\omega \mapsto (P_t^\omega - \pi^\omega)f$ too is a measurable vector field, by definition of measurable field of bounded operators. Thus, its norm $\omega \mapsto \|(P_t^\omega - \pi^\omega)f\|_{L^2(\mathrm{vol}_\mathrm{g}^\omega)}$ too is measurable, which concludes the assertion. ∎

LEMMA 5.9. *For every compact manifold* (M, g) *(with continuous, not necessarily smooth metric* g*),*

$$\lambda_1(\mathsf{M}) = \inf\left\{ \max\{\lambda_0(\mathsf{M}_1), \lambda_0(\mathsf{M}_2)\} : \ \mathsf{M}_1, \mathsf{M}_2 \ \textit{non-polar, quasi-open, disjoint} \subset \mathsf{M}\right\} \tag{5.4}$$

where

$$\lambda_0(\mathsf{M}_i) := \inf\left\{ \frac{\displaystyle\int |\nabla v|^2_\mathrm{g} \,\mathrm{dvol}_\mathrm{g}}{\displaystyle\int |v|^2 \,\mathrm{dvol}_\mathrm{g}} : v \in W_*^{1,2} \setminus \{0\} \ , \ \tilde v = 0 \ \textit{q.e. on} \ \mathsf{M}\setminus \mathsf{M}_i \right\} \ . \tag{5.5}$$

Here, as usual in Dirichlet form theory, $\tilde v$ denotes a quasi continuous version of v, and q.e. stands for quasi everywhere, see, e.g., [19, §2.1].

The infimum in (5.4) *is attained for* $\mathsf{M}_1 := \{u > 0\}, \mathsf{M}_2 := \{u < 0\}$ *if u is chosen as an eigenfunction for* $\lambda_1(\mathsf{M})$*. In this case, indeed,*

$$\lambda_1(\mathsf{M}) = \lambda_0(\mathsf{M}_1) = \lambda_0(\mathsf{M}_2) \ .$$

PROOF. Let u be an eigenfunction for $\lambda_1(\mathsf{M})$ and put $\mathsf{M}_1 := \{u > 0\}, \mathsf{M}_2 := \{u < 0\}$. Choosing $v = u^+$ or $v = u^-$ in (5.5) one can verify that $\lambda_0(\mathsf{M}_i) = \lambda_1(\mathsf{M})$ for $i = 1, 2$. This proves the $\geq$-assertion in (5.4).

For the converse estimate, let $v_i \neq 0$ for $i = 1, 2$ be minimizers for $\lambda_0(\mathsf{M}_i)$. Put $\lambda := \lambda_0(\mathsf{M}_1) \vee \lambda_0(\mathsf{M}_2)$ and $u := v_1 + t v_2$ with $t \neq 0$ chosen such that $\int u \,\mathrm{dvol}_\mathrm{g} = 0$. Then

$$\int |\nabla u|^2_\mathrm{g} = \int |\nabla v_1|^2_\mathrm{g} + t^2 \int |\nabla v_2|^2_\mathrm{g} \leq \lambda \int |v_1|^2 + t^2 \lambda \int |v_2|^2 = \lambda \int |u|^2$$

and thus $\lambda_1(\mathsf{M}) \leq \lambda$. ∎

THEOREM 5.10. *For* $\mathbf{P}$*-a.e.* ω*,*

$$e^{-\alpha \sup |h^\omega|} \leq \frac{\lambda_1^\omega}{\lambda_1} \leq e^{\alpha \sup |h^\omega|} \tag{5.6}$$

with $\alpha := 2(n-1)$ *if* $n \geq 2$ *and* $\alpha := 2$ *if* $n = 1$. *In particular,*

$$\mathbf{E}\Big[\big|\log\lambda_1^\bullet - \log\lambda_1\big|\Big] \leq \alpha\,\mathbf{E}\Big[\sup|h^\bullet|\Big] .$$

Proof. Choose a minimizer u for $\lambda_1(\mathsf{M})$ and put $\mathsf{M}_1 := \{u > 0\}, \mathsf{M}_2 := \{u < 0\}$. Then for each ω and each $i = 1, 2$,

$$\begin{aligned}
\lambda_0^\omega(\mathsf{M}_i) &= \inf\left\{\frac{\displaystyle\int |\nabla v|_{\mathrm{g}}^2\, e^{(n-2)h^\omega}\mathrm{dvol}_{\mathrm{g}}}{\displaystyle\int |v|^2 e^{nh^\omega}\mathrm{dvol}_{\mathrm{g}}} : \tilde v = 0 \text{ q.e. on } \mathsf{M}\setminus\mathsf{M}_i\right\}\\
&\leq \frac{\sup_x e^{(n-2)h^\omega(x)}}{\inf_y e^{nh^\omega(y)}}\cdot\inf\left\{\frac{\displaystyle\int |\nabla v|_{\mathrm{g}}^2\mathrm{dvol}_{\mathrm{g}}}{\displaystyle\int |v|^2\,\mathrm{dvol}_{\mathrm{g}}} : \tilde v = 0 \text{ q.e. on } \mathsf{M}\setminus\mathsf{M}_i\right\}\\
&\leq e^{\alpha\sup|h^\omega|}\cdot\lambda_0(\mathsf{M}_i)\\
&= e^{\alpha\sup|h^\omega|}\cdot\lambda_1(\mathsf{M})
\end{aligned}$$

with $\alpha := n + |n-2|$. Hence according to the previous Lemma,

$$\lambda_1^\omega(\mathsf{M}) \leq e^{\alpha\sup|h^\omega|}\cdot\lambda_1(\mathsf{M}) .$$

Interchanging the roles of λ_1^ω and λ_1 and replacing h^ω by $-h^\omega$ yield the reverse inequality. ∎

Corollary 5.11. *For all* $f \in L^2(\mathrm{vol}_{\mathrm{g}})$ *and all* $t > 0$,

$$\mathbf{E}\left[\log\left\|P_t^\bullet f - \pi^\bullet f\right\|_{L^2(\mathrm{vol}_{\mathrm{g}}^\bullet)}\right] \leq -\lambda_1 t\cdot e^{-\alpha\,\mathbf{E}\left[\sup|h^\bullet|\right]} + \log\|f\|_{L^2(\mathrm{vol}_{\mathrm{g}})} + \frac{n^2\,\theta^*}{4} \tag{5.7}$$

with $\theta^* := \sup_x \mathbf{E}\big[h^\bullet(x)^2\big]$ *and* $\alpha := n + |n-2|$.

Proof. With Theorem 5.10 we estimate

$$\lambda_1^\omega t \geq \lambda_1 t\, e^{-\alpha\sup|h^\omega|} .$$

By the convexity we may apply Jensen's inequality and get the estimate

$$\mathbf{E}\big[\lambda_1^\bullet t\big] \geq \lambda_1 t\, e^{-\alpha\,\mathbf{E}\left[\sup|h^\bullet|\right]} .$$

Moreover, again by Jensen's inequality

$$\mathbf{E}\Big[\log\|f\|_{L^2(\mathrm{vol}_{\mathrm{g}}^\bullet)}\Big] \leq \frac{1}{2}\log\mathbf{E}\Big[\|f\|_{L^2(\mathrm{vol}^\bullet)}^2\Big] \leq \frac{1}{2}\log\|f\|_{L^2(\mathrm{vol}_{\mathrm{g}})}^2 + \frac{n^2\,\theta^*}{4} ,$$

which yields the claim. ∎

6. Higher-Order Green Kernels — Asymptotics and Examples.

6.1. *Green Kernel Asymptotics.* The next Theorem illustrates the asymptotic behavior of the higher-order Green kernel $G_{s,m}(x,y)$ close to the diagonal in terms of the Riemannian distance $\mathsf{d}(x,y)$. The statement of the Theorem is sharp, as readily deduced by comparison with the analogous statement for Euclidean spaces, see Equation (6.7) below.

THEOREM 6.1. *Let* (M,g) *be a Riemannian manifold with bounded geometry, and* $s > n/2$*. Then, for every* $\alpha \in (0,1]$ *with* $\alpha < s - n/2$ *there exists a constant* $C_{\alpha,m} > 0$ *so that*

$$\rho_{s,m}(x,y) = |G_{s,m}(x,x) + G_{s,m}(y,y) - 2\,G_{s,m}(x,y)|^{1/2} \ \leq\ C_{\alpha,m} \cdot \mathsf{d}(x,y)^\alpha \ ,$$

for all $m > 0$ *and all* $x, y \in \mathsf{M}$*.*

If M *is additionally closed, then additionally*

$$\rho_{s,m}(x,y) = \left|\mathring{G}_{s,m}(x,x) + \mathring{G}_{s,m}(y,y) - 2\,\mathring{G}_{s,m}(x,y)\right|^{1/2} \ \leq\ C_\alpha \cdot \mathsf{d}(x,y)^\alpha \ ,$$

for all $m \geq 0$*. In this case, the constant* C_α *can be chosen such that*

$$C_\alpha^2 = C \left(\frac{\lambda_1}{4}\right)^{n/2+\alpha-s} \ \frac{\Gamma\big(s-n/2-\alpha\big)}{\alpha^* \cdot \Gamma(s)} \tag{6.1}$$

with $\alpha^* := \alpha$ *whenever* $\alpha \in (0,1/2]$ *and* $\alpha^* := \alpha - 1/2$ *whenever* $\alpha \in (1/2,1]$ *and* $C > 0$ *is a constant only depending on* M*.*

PROOF. Note that

$$\mathring{G}_{s,m}(x,x) + \mathring{G}_{s,m}(y,y) - 2\,\mathring{G}_{s,m}(x,y) = G_{s,m}(x,x) + G_{s,m}(y,y) - 2\,G_{s,m}(x,y) \ , \qquad m > 0 \ .$$

Thus it suffices to prove the claim for $\mathring{G}_{s,m}$.

Assume first that M is closed. Throughout the proof, $C > 0$ denotes a finite constant, only depending on M but possibly changing from line to line. For $x, y \in \mathsf{M}$ denote by $([x,y]_r)_{r\in[0,1]}$ any constant speed distance-minimizing geodesic joining x to y.

Assume first that $\sigma := 2\alpha \in (0,1]$. Then,

$$\begin{aligned}
&\sup_{\substack{x,y\in\mathsf{M}\\ x\neq y}} \left[\frac{\Gamma(s)}{\mathsf{d}(x,y)^\sigma} \left|\mathring{G}_{s,m}(x,x) + \mathring{G}_{s,m}(y,y) - 2\,\mathring{G}_{s,m}(x,y)\right|\right] \leq \\
&\leq 2 \sup_{\substack{x,y\in\mathsf{M}\\ x\neq y}} \left[\int_0^\infty \frac{|p_t(x,x) - p_t(x,y)|}{\mathsf{d}(x,y)} \cdot \mathsf{d}(x,y)^{1-\sigma} \cdot e^{-m^2 t}\ t^{s-1}\,\mathrm{d}t\right] \\
&\leq 2 \sup_{\substack{x,y\in\mathsf{M}\\ x\neq y}} \left[\int_0^\infty e^{-m^2 t}\ t^{s-1} \cdot \mathsf{d}(x,y)^{1-\sigma} \int_0^1 |\nabla p_t(x,[x,y]_r)|\,\mathrm{d}r\,\mathrm{d}t\right] .
\end{aligned}$$

By (2.12)

$$\begin{aligned}
&\sup_{\substack{x,y\in\mathsf{M}\\ x\neq y}} \left[\Gamma(s)\,\mathsf{d}(x,y)^{-\sigma} \left|\mathring{G}_{s,m}(x,x) + \mathring{G}_{s,m}(y,y) - 2\,\mathring{G}_{s,m}(x,y)\right|\right] \\
&\qquad \leq C \sup_{\substack{x,y\in\mathsf{M}\\ x\neq y}} \left[\mathsf{d}(x,y)^{1-\sigma} \int_0^\infty e^{-(m^2+\lambda_1/2)t}\ t^{s-1}\ (t^{-n/2-1/2} \vee 1)\ \cdot\right. \\
&\qquad\qquad \left. \cdot \int_0^1 \exp\left(-\frac{r^2\,\mathsf{d}(x,y)^2}{Ct}\right) \mathrm{d}r\,\mathrm{d}t\right] \\
&\qquad \leq C \sup_{\substack{x,y\in\mathsf{M}\\ x\neq y}} \left[\int_0^\infty e^{-\lambda_1 t/2}\ t^{s-1+(1-\sigma)/2}\ (t^{-n/2-1/2} \vee 1)\ \cdot\right. \\
&\qquad\qquad \left. \cdot \int_0^1 \left(\frac{r^2\,\mathsf{d}(x,y)^2}{t}\right)^{(1-\sigma)/2} \exp\left(-\frac{r^2\,\mathsf{d}(x,y)^2}{Ct}\right) r^{\sigma-1}\,\mathrm{d}r\,\mathrm{d}t\right] \\
&\qquad \leq \frac{C}{\sigma} \int_0^\infty e^{-\lambda_1 t/4}\ t^{s-(n+\sigma)/2-1}\,\mathrm{d}t \ = \ \frac{C}{\sigma}\left(\frac{4}{\lambda_1}\right)^{s-(n+\sigma)/2} \Gamma\big(s-(n+\sigma)/2\big) \ .
\end{aligned} \tag{6.2}$$

For the last *in*equality, we used the fact that the function $R \mapsto R^{(1-\sigma)/2}\exp(-R/C)$ is uniformly bounded on $(0,\infty)$, independently of $\sigma \in (0,1]$.

Assume now that $\sigma := 2\alpha \in (1,2]$. Then, similarly to the previous case,

$$\sup_{\substack{x,y\in \mathsf{M}\\ x\neq y}} \left[\frac{\Gamma(s)}{\mathsf{d}(x,y)^\sigma}\left|\mathring{G}_{s,m}(x,x)+\mathring{G}_{s,m}(y,y)-2\,\mathring{G}_{s,m}(x,y)\right|\right]$$

$$\leq \sup_{\substack{x,y\in \mathsf{M}\\ x\neq y}} \int_0^\infty \frac{|p_t(x,x)+p_t(y,y)-2p_t(x,y)|}{\mathsf{d}(x,y)^\sigma}\, e^{-m^2 t} t^{s-1}\,\mathrm{d}t$$

$$\leq \sup_{\substack{x,y\in \mathsf{M}\\ x\neq y}} \int_0^\infty e^{-m^2 t}\, t^{s-1}\, \mathsf{d}(x,y)^{1-\sigma}\int_0^1 |\nabla_2\, p_t(x,[x,y]_\rho)-\nabla_2\, p_t(y,[x,y]_\rho)|\,\mathrm{d}\rho\,\mathrm{d}t$$

$$\leq \sup_{\substack{x,y\in \mathsf{M}\\ x\neq y}} \int_0^\infty e^{-m^2 t}\, t^{s-1}\, \mathsf{d}(x,y)^{2-\sigma}\int_0^1\int_0^1 |\nabla_1\nabla_2\, p_t([x,y]_\varrho,[x,y]_\rho)|\,\mathrm{d}\rho\,\mathrm{d}\varrho\,\mathrm{d}t\ .$$

By (2.13), similarly

$$\sup_{\substack{x,y\in \mathsf{M}\\ x\neq y}} \left[\frac{\Gamma(s)}{\mathsf{d}(x,y)^\sigma}\left|\mathring{G}_{s,m}(x,x)+\mathring{G}_{s,m}(y,y)-2\,\mathring{G}_{s,m}(x,y)\right|\right]$$

$$\leq C\sup_{\substack{x,y\in \mathsf{M}\\ x\neq y}} \int_0^\infty\int_0^1\int_0^1 \exp\left(-\frac{(\rho-\varrho)^2\,\mathsf{d}^2(x,y)}{Ct}\right)\mathrm{d}\rho\,\mathrm{d}\varrho\,\cdot$$

$$\cdot\,\mathsf{d}(x,y)^{2-\sigma}\, e^{-(m^2+\lambda_1/2)t}\, t^{s-1}\,(t^{-n/2-1}\vee 1)\,\mathrm{d}t$$

$$\leq C\sup_{\substack{x,y\in \mathsf{M}\\ x\neq y}} \int_0^\infty\int_0^1\int_0^1 \left(\frac{(\rho-\varrho)^2\,\mathsf{d}^2(x,y)}{t}\right)^{1-\sigma/2}\cdot$$

$$\cdot\exp\left(-\frac{(\rho-\varrho)^2\,\mathsf{d}^2(x,y)}{Ct}\right)|\rho-\varrho|^{\sigma-2}\,\mathrm{d}\rho\,\mathrm{d}\varrho\,\cdot$$

$$\cdot\, t^{1-\sigma/2}\, e^{-\lambda_1 t/2}\, t^{s-1}(t^{-n/2-1}\vee 1)\,\mathrm{d}t$$

$$\leq \frac{C}{\sigma(\sigma-1)}\int_0^\infty e^{-\lambda_1 t/4}\, t^{s-(n+\sigma)/2-1}\,\mathrm{d}t = \frac{C}{\sigma(\sigma-1)}\left(\frac{4}{\lambda_1}\right)^{s-(n+\sigma)/2}\Gamma\big(s-(n+\sigma)/2\big)\ .$$

Assume now that M has bounded geometry. The proof holds in a similar way to the case of closed M, having care to replace the application of (2.12) with (2.4) and (2.13) with (2.6). ∎

COROLLARY 6.2. *Let M be a compact manifold. Then, there exists a constant $C>0$ such that for all $m\geq 0$ and all $x,y\in\mathsf{M}$,*

$$\rho_{s,m}(x,y)\ \leq\ \begin{cases} C\cdot\left(\frac{\lambda_1}{2}\right)^{-s/2}\cdot\mathsf{d}(x,y), & s\geq \frac{n}{2}+2\ ,\\ \frac{C}{\sqrt{s-n/2-1}}\cdot\mathsf{d}(x,y), & s\in(\frac{n}{2}+1,\frac{n}{2}+2]\ ,\\ \frac{C}{s-n/2}\cdot\mathsf{d}^{s/2-n/4}(x,y), & s\in(\frac{n}{2},\frac{n}{2}+1]\ .\end{cases}$$

The estimate in the third case is not sharp. The previous Theorem provides estimates $\rho_{s,m}\leq C_\alpha\,\mathsf{d}^\alpha$ for every $\alpha<s-n/2$. (As $\alpha\to s-n/2$, however, the constant C_α will diverge.)

PROOF. The eigenfunction representation (2.22) of $\mathring{G}_{s,m}$ yields that

$$\rho^2_{s,m}(x,y)=\sum_{j=1}^\infty (m^2+\lambda_j/2)^{-s}\left[\varphi_j^2(x)+\varphi_j^2(y)-2\varphi_j(x)\varphi_j(y)\right]\ .$$

Hence, $\rho^2_{s,m}(x,y) \le \rho^2_{s,0}(x,y)$ for all x,y,s,m under consideration. Moreover, for all $x,y\in \mathsf{M}$ the function

$$s \mapsto (\lambda_1/2)^s \cdot \rho^2_{s,0}(x,y) \text{ is decreasing.} \tag{6.3}$$

Therefore, the first case $s \ge \frac{n}{2}+2$ follows from the choice $s=\frac{n}{2}+2$ which is included in the second case.

In the second case $s\in(\frac{n}{2}+1,\frac{n}{2}+2]$, with the choice $\alpha=1$ the previous Theorem provides the estimate

$$\frac{\rho^2_{s,m}(x,y)}{\mathsf{d}^2(x,y)} \le C_1^2 \le C\,{\lambda_1}^{n/2+1-s}\,\frac{\Gamma\big(s-n/2-1\big)}{\Gamma(s)} \le \frac{C'}{s-n/2-1}\ .$$

In the third case $s\in(\frac{n}{2},\frac{n}{2}+1]$, with the choice $\alpha=\frac{1}{2}(s-\frac{n}{2})\in(0,1/2]$ the previous Theorem provides the estimate

$$\frac{\rho^2_{s,m}(x,y)}{\mathsf{d}^{s-n/2}(x,y)} \le C_\alpha^2 \le C\,{\lambda_1}^{n/4-s/2}\,\frac{\Gamma\big(s/2-n/4\big)}{(s-n/2)\,\Gamma(s)} \le \frac{C'}{(s-n/2)^2}\ . \qquad \blacksquare$$

6.2. *Supremum estimates.* Now let us combine Dudley's estimate, Theorem 3.25, for the supremum of the Gaussian field with our Hölder estimate, Corollary 6.2, for the noise distance.

THEOREM 6.3. *For every compact manifold* M *there exists a constant* $C=C(\mathsf{M})$ *such that for every* $h^\bullet \sim \mathsf{F\mathring{G}F}^{\mathsf{M}}_{s,m}$ *with any* $m\ge 0$,

$$\mathbf{E}\bigg[\sup_{x\in\mathsf{M}} h^\bullet(x)\bigg] \le \begin{cases} C\cdot(\lambda_1/2)^{-s/2}, & s\ge \frac{n}{2}+1\ ,\\ C\cdot(s-n/2)^{-3/2}, & s\in\big(\frac{n}{2},\frac{n}{2}+1\big]\ .\end{cases}$$

PROOF. Recall the Notation 3.24 for the covering number of a pseudo-metric, and let $\rho=\rho_{s,m}$ be as in (2.25). For the Riemannian distance d on the compact manifold M,

$$N_{\mathsf{d}}(\varepsilon) \le \big(C\cdot\varepsilon^{-n}\big)\vee 1$$

for some constant $C>0$.

In the case $s\in(\frac{n}{2},\frac{n}{2}+1]$, Corollary 6.2 yields $\rho\le C_s\,\mathsf{d}^\alpha$ with $\alpha\coloneqq\frac12(s-\frac{n}{2})$ and $C_s\coloneqq C/(s-n/2)$, and thus

$$B^{(\rho)}_\varepsilon(x) \supset B^{(\mathsf{d})}_{(\varepsilon/C_s)^{1/\alpha}}(x)\ , \qquad \varepsilon>0,\quad x\in\mathsf{M}\ .$$

This implies

$$N_{s,m}(\varepsilon) \le N_{\mathsf{d}}\big((\varepsilon/C_s)^{1/\alpha}\big) \le \big(C\cdot(\varepsilon/C_s)^{-n/\alpha}\big)\vee 1\ .$$

Hence,

$$\begin{aligned}\int_0^\infty \big(\log N_{s,m}(\varepsilon)\big)^{1/2}\,\mathrm{d}\varepsilon &\le \int_0^{C^{\alpha/n}\cdot C_s}\left(c-\frac{n}{\alpha}\log\frac{\varepsilon}{C_s}\right)^{1/2}\mathrm{d}\varepsilon = C_s\cdot\int_0^{C^{\alpha/n}}\Big(c-\frac{n}{\alpha}\log\varepsilon\Big)^{1/2}\mathrm{d}\varepsilon\\ &\le \frac{C_s}{\alpha^{1/2}}\cdot\int_0^{C^{1/n}}\big(c'-n\log\varepsilon\big)^{1/2}\,\mathrm{d}\varepsilon = \frac{C_s}{\alpha^{1/2}}\cdot\ C' = \frac{C''}{(s-n/2)^{3/2}}\ .\end{aligned}$$

In the case $s>n/2+1$, the monotonicity property (6.3) and the estimate from Corollary 6.2 (for $s=n/2+1$) imply

$$\rho_{s,m}(x,y) \le (\lambda_1/2)^{(n/2+1-s)/2}\cdot\rho_{n/2+1,0}(x,y) \le C\,(\lambda_1/2)^{(n/2+1-s)/2}\cdot\mathsf{d}^{1/2}(x,y)\ .$$

Hence, following the previous argumentation we obtain

$$\begin{aligned}\int_0^\infty\big(\log N_{s,m}(\varepsilon)\big)^{1/2}\,\mathrm{d}\varepsilon &\le C\,(\lambda_1/2)^{(n/2+1-s)/2}\cdot\int_0^{C^{1/n}}\big(c-2n\log\varepsilon\big)^{1/2}\,\mathrm{d}\varepsilon\\ &\le C'\,(\lambda_1/2)^{(n/2+1-s)/2} = C''\,(\lambda_1/2)^{-s/2}\ .\end{aligned} \qquad \blacksquare$$

6.3. *Examples.*

6.3.1. *Euclidean space.* On the n-dimensional Euclidean space, the Green kernels are given by

$$G^{\mathbb{R}^n}_{s,m}(x,y) \coloneqq G^n_{s,m}(|x-y|)$$

with

$$G^n_{s,m}(r) \coloneqq \frac{1}{(2\pi)^{n/2}\,\Gamma(s)} \int_0^\infty e^{-r^2/2t}\, e^{-m^2 t}\, t^{s-n/2-1}\, \mathrm{d}t \ . \tag{6.4}$$

Note that $G^n_{s,m}(r) \le G^n_{s,m}(0) < \infty$ if $s > n/2$ whereas $G^n_{s,m}(r) \approx \log\frac{1}{r}$ as $r \to 0$ if $s = n/2$ and $G^n_{s,m}(r) \approx \frac{1}{r^{n-2s}}$ if $s < n/2$. Closed expressions for $G^n_{1,m}(r)$ are available for odd n, e.g.

$$G^1_{1,m}(r) = \frac{1}{\sqrt{2}m} e^{-\sqrt{2}m\,r}, \qquad G^3_{1,m}(r) = \frac{1}{2\pi\, r}\, e^{-\sqrt{2}m\,r}, \qquad G^5_{1,m}(r) = \frac{(1+\sqrt{2}mr)}{4\pi^2 r^3}\, e^{-\sqrt{2}m\,r} \ . \tag{6.5}$$

From this, with the relations formulated below, various other explicit expressions can be derived, for instance, $G^3_{2,m}(r) = \frac{1}{2\pi\,\sqrt{2}m}\, e^{-\sqrt{2}m\,r}$ and, more generally,

$$G^n_{\frac{n+1}{2},m}(r) = \frac{1}{(2\pi)^{\frac{n-1}{2}}\,\Gamma(\frac{n+1}{2})\,\sqrt{2}m}\, e^{-\sqrt{2}m\,r} \ .$$

Lemma 6.4. *For $m,s,r>0$ and $n\in\mathbb{N}$, the Green kernels $G^n_{s,m}(r)$ satisfy the relations*

$$G^n_{s,am}(r) = a^{n-2s}\, G^n_{s,m}(ar) \ , \qquad a>0 \ , \tag{6.6a}$$

$$G^n_{s+a,m}(r) = \frac{1}{(2\pi)^a}\,\frac{\Gamma(s)}{\Gamma(s+a)}\, G^{n-2a}_{s,m}(r) \ , \qquad -s<a<n/2 \ , \tag{6.6b}$$

$$sm^2 G^n_{s+1,m}(r) = (s-n/2)\, G^n_{s,m}(r) + \frac{r^2}{2(s-1)}\, G^n_{s-1,m}(r) \ , \qquad s>1 \ . \tag{6.6c}$$

Proof. The first two formulas follow by change of variable in the integral representation (6.4). The third one follows by integration by parts via

$$\int_0^\infty e^{-r^2/2t} e^{-m^2 t} t^{s-n/2}\ \mathrm{d}t = \frac{1}{m^2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}(e^{-r^2/2t} t^{s-n/2}) e^{-m^2 t}\ \mathrm{d}t \ . \qquad \blacksquare$$

Theorem 6.5. *For $m>0$, the asymptotics of the higher order Green kernel as $r\to 0$ is as follows*

$$G^n_{s,m}(0) - G^n_{s,m}(r) \ \asymp\ \begin{cases} -\dfrac{\Gamma(n/2-s)}{2^s\,\pi^{n/2}\,\Gamma(s)}\cdot r^{2s-n} & \text{if } s\in(n/2,n/2+1) \ , \\ \dfrac{1}{2^{n/2}\,\pi^{n/2}\,\Gamma(s)}\cdot r^2\log\dfrac{1}{r} & \text{if } s=n/2+1 \ , \\ \dfrac{\Gamma(s-n/2-1)}{2^{n/2+1}\, m^{2s-n-2}\,\pi^{n/2}\,\Gamma(s)}\cdot r^2 & \text{if } s>n/2+1 \ , \end{cases} \tag{6.7}$$

where $\asymp$ is as in Notation 2.1.

Proof. For convenience, we provide two proofs. The first one is based on direct calculations.

For proving the claim in the case $s>n/2+1$, consider

$$\begin{aligned} \lim_{r\to0} r^{-2}\cdot(2\pi)^{n/2}\Gamma(s)\big[G^n_{s,m}(0)-G^n_{s,m}(r)\big] &= \lim_{r\to0}\int_0^\infty \frac{1-e^{-r^2/2t}}{r^2}\, e^{-m^2 t}\, t^{s-n/2-1}\,\mathrm{d}t \\ &= \frac{1}{2}\cdot\int_0^\infty e^{-m^2 t}\, t^{s-n/2-2}\,\mathrm{d}t = \frac{1}{2}\Gamma\left(s-\frac{n}{2}-1\right) m^{-2s+n+2} \ , \end{aligned}$$

since by assumption $s > 1 + \frac{n}{2}$. In the case $n/2 < s < n/2+1$, consider

$$\begin{aligned}\lim_{r\to 0} r^{-2s+n}(2\pi)^{n/2}\Gamma(s)\cdot\left[G^n_{s,m}(0)-G^n_{s,m}(r)\right] &= \lim_{r\to 0} r^{-2s+n}\cdot\int_0^\infty \left(1-e^{-r^2/2t}\right)e^{-m^2t}\,t^{s-n/2-1}\,\mathrm{d}t\\ &= \lim_{r\to 0}\int_0^\infty \left(1-e^{-1/2t}\right)e^{-(mr)^2t}\,t^{s-n/2-1}\,\mathrm{d}t\\ &= \int_0^\infty \left(1-e^{-1/2t}\right)t^{s-n/2-1}\,\mathrm{d}t\\ &= 2^{n/2-s}\int_0^\infty \left(1-e^{-u}\right)u^{n/2-s-1}\,\mathrm{d}u\\ &= -\frac{2^{n/2-s}}{n/2-s}\int_0^\infty e^{-u}\,u^{n/2-s}\,\mathrm{d}u\\ &= -2^{n/2-s}\Gamma(n/2-s)\ .\end{aligned}$$

(For the third equality above, we used the monotonicity of the integrand in r, and for the fifth, we used integration by parts.) In the case $s = \frac{n}{2}+1$, applying De l'Hôpital twice yields

$$\begin{aligned}&\lim_{r\to 0}\frac{(2\pi)^{n/2}\Gamma(s)}{r^2\log 1/r}\cdot\left[G^n_{s,m}(0)-G^n_{s,m}(r)\right]\\ &=\lim_{r\to 0}\frac{1}{r^2\log 1/r}\cdot\int_0^\infty\left(1-e^{-r^2/2t}\right)e^{-m^2t}\,\mathrm{d}t\\ &=-\lim_{r\to 0}\frac{1}{r(1+2\log r)}\int_0^\infty re^{-r^2/2t}e^{-m^2t}t^{-1}\,\mathrm{d}t\\ &=\lim_{r\to 0}\frac{r}{2}\int_0^\infty re^{-r^2/2t}e^{-m^2t}t^{-2}\,\mathrm{d}t \qquad \left[\frac{r^2}{2t}=u\ ,\quad -\frac{r^2}{2t^2}\,\mathrm{d}t=\mathrm{d}u\right]\\ &=\lim_{r\to 0}\int_0^\infty e^{-\frac{m^2r^2}{2u}}e^{-u}\,\mathrm{d}u = 1\ .\end{aligned}$$

An alternative proof of the claims may be obtained from the representation [58, Eqn. (15), p. 183] of the Green kernel $G^n_{s,m}(r)$ in terms of the modified Bessel functions K_α for $\alpha\in\mathbb{R}$:

$$G^n_{s,m}(r) = \frac{2}{(2\pi)^{n/2}\,\Gamma(s)}\left(\frac{r}{\sqrt{2}m}\right)^{s-n/2}K_{s-n/2}(\sqrt{2}m\,r)\ , \tag{6.8}$$

and the known asymptotics for K_α and its derivatives. ■

Remark 6.6. For all integer values of s and n, explicit expressions for $G^n_{s,m}$ may be obtained from (6.8) in terms of the *reverse Bessel polynomials*, e.g. [26, §II.1, Eqn.s (7)–(9)], in view of the characterization in terms of such polynomials of the Bessel function K_α for semi-integer α, e.g. [26, §III.1].

6.3.2. *Torus.* Let $\mathbb{T}=\mathbb{R}/\mathbb{N}$ be the circle of length 1.

Proposition 6.7. *For all $s,m>0$,*

$$G^{\mathbb{T}}_{s,m}(x,y) = \sum_{j\in\mathbb{Z}} G^{\mathbb{R}}_{s,m}(x,y+j)\ . \tag{6.9}$$

In particular, $G^{\mathbb{T}}_{s,m}(x,y) = G^{\mathbb{T}}_{s,m}\big(\mathsf{d}_{\mathbb{T}}(x,y)\big)$ with $\mathsf{d}_{\mathbb{T}}(x,y)=\min\{|x-y|,1-|x-y|\}$ for $x,y\in[0,1]$ and

$$G^{\mathbb{T}}_{1,m}(r) = \frac{\cosh\left(\sqrt{2}m(r-1/2)\right)}{\sqrt{2}m\cdot\sinh(m/\sqrt{2})}\ . \tag{6.10}$$

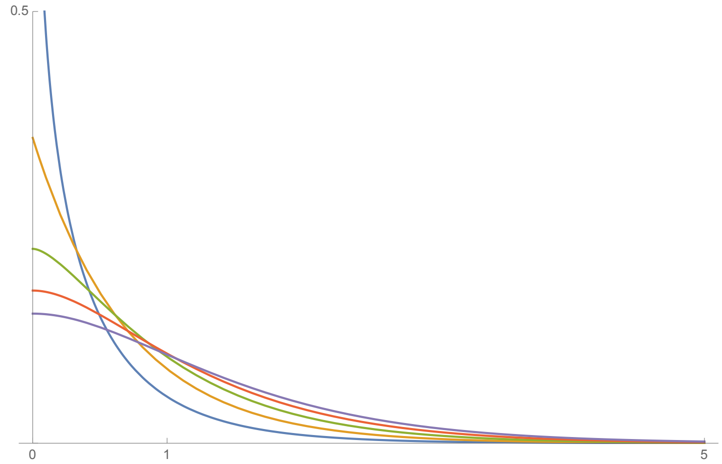


Fig 4: The Green kernels $G^1_{s,1}$ for $2s = 1,\ldots,5$ (in reverse order w.r.t. the value at 0). Note that $\lim_{r\to 0} G^1_{1/2,1}(r) = +\infty$.

PROOF. The first claim is an immediate consequence of the analogous formula for the heat kernel:

$$p_t^{\mathbb{T}}(x,y) = \sum_{j\in\mathbb{Z}} p_t^{\mathbb{R}}(x,y+j)\ .$$

The second claim follows from the first one combined with (6.5) according to

$$\begin{aligned} G^{\mathbb{T}}_{1,m}(r) &= \frac{1}{\sqrt{2}m}\sum_{k\in\mathbb{N}_0} e^{-\sqrt{2}m(r+k)} + \frac{1}{\sqrt{2}m}\sum_{k\in\mathbb{N}_0} e^{-\sqrt{2}m[(1-r)+k]} \\ &= \frac{1}{\sqrt{2}m\left(1-e^{-\sqrt{2}m}\right)}\left(e^{-\sqrt{2}mr}+e^{-\sqrt{2}m(1-r)}\right) = \frac{\cosh\left(\sqrt{2}m(r-1/2)\right)}{\sqrt{2}m\cdot\sinh(m/\sqrt{2})} \end{aligned}$$

for $r\in[0,1/2]$. ∎

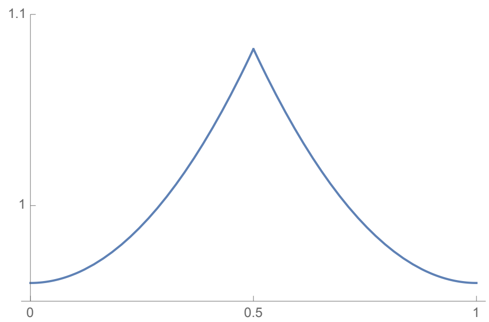


Fig 5: The Green kernel $G^{\mathbb{T}}_{1,1}(\frac{1}{2},y)$ with $y\in[0,1)$.

THEOREM 6.8. *For $m=0$ and integer $s\geq 1$,*

$$\mathring{G}^{\mathbb{T}}_{s,0}(r) = (-1)^{s-1}\frac{2^s}{(2s)!}B_{2s}(r)\ ,\qquad s\in\mathbb{N}\ ,\qquad r\in[0,1/2]\ ,$$

where B_n denotes the n^{th} Bernoulli polynomial.

In particular,

$$\mathring{G}^{\mathbb{T}}_{1,0}(r) = \left(r-\frac{1}{2}\right)^2 - \frac{1}{12}\ , \tag{6.11}$$

$$\mathring{G}^{\mathbb{T}}_{2,0}(x,y) = -\frac{1}{6}\left(r-\frac{1}{2}\right)^4 + \frac{1}{12}\left(r-\frac{1}{2}\right)^2 - \frac{7}{1440}\ , \tag{6.12}$$

$$\mathring{G}^{\mathbb{T}}_{3,0}(x,y) = \frac{1}{90}\left(r-\frac{1}{2}\right)^6 - \frac{1}{72}\left(r-\frac{1}{2}\right)^4 + \frac{7}{1440}\left(r-\frac{1}{2}\right)^2 - \frac{31}{120960}\ . \tag{6.13}$$

Further observe that

$$\lim_{r\to 0}\frac{1}{r}\Big(\mathring{G}^{\mathbb{T}}_{1,0}(0) - \mathring{G}^{\mathbb{T}}_{1,0}(r)\Big) \ = \ \lim_{r\to 0}\frac{1}{r}\Big(G^{\mathbb{T}}_{1,m}(0) - G^{\mathbb{T}}_{1,m}(r)\Big) \ = \ \lim_{r\to 0}\frac{1}{r}\Big(G^{\mathbb{R}}_{1,m}(0) - G^{\mathbb{R}}_{1,m}(r)\Big) \ = \ 1$$

for all $m > 0$, and

$$\lim_{r\to 0}\frac{1}{r^2}\Big(\mathring{G}^{\mathbb{T}}_{2,0}(0) - \mathring{G}^{\mathbb{T}}_{2,0}(r)\Big) \ = \ \frac{1}{6} \qquad \text{whereas} \qquad \lim_{r\to 0}\frac{1}{r^2}\Big(G^{\mathbb{R}}_{2,m}(0) - G^{\mathbb{R}}_{2,m}(r)\Big) \ = \ \frac{1}{2\sqrt{2}\,m}\ .$$

PROOF. For convenience, we provide two proofs. Recall the eigenfunction representation (2.22) for the grounded Green kernel,

$$\mathring{G}_{s,m}(x,y) = \sum_{j\in\mathbb{N}} \frac{\varphi_j(x)\,\varphi_j(y)}{(m^2+\lambda_j/2)^s}\ , \qquad x,y\in \mathsf{M}\ .$$

For the torus, we have $\lambda_{2k-1} = \lambda_{2k} = (2\pi k)^2$ for $k\in\mathbb{N}$ with $\varphi_{2k-1}(x) = \sqrt{2}\sin\big(2k\pi x\big)$, and $\varphi_{2k}(x) = \sqrt{2}\cos\big(2k\pi x\big)$. Choosing $m=0$, $y=0$, and $x=r$ thus yields

$$\mathring{G}^{\mathbb{T}}_{s,0}(r) = \frac{1}{2^{s-1}}\sum_{k\in\mathbb{N}} \frac{1}{(\pi k)^{2s}}\cos\big(2k\pi r\big)\ , \qquad r\in[0,1/2]\ , \tag{6.14}$$

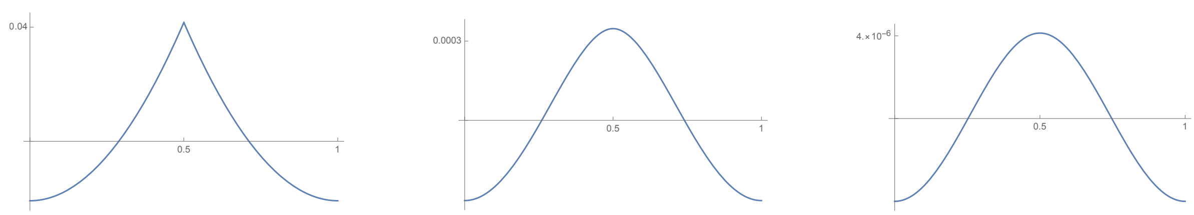
and the conclusion follows by e.g. [23, 1.443.1].

An alternative proof of the claim can be obtained in the following way. For $s=1$, the right-hand side of (6.14) is indeed the Fourier series for the function given in (6.11). The values of $f_s \coloneqq G^{\mathbb{T}}_{s,0}$ for all other $s\in\mathbb{N}$ can then be derived from there and from the facts that

$$f''_{s+1} = -2\,f_s, \qquad f'_s(1/2) = 0, \qquad \int_0^{1/2} f_s(r)\,\mathrm{d}r = 0\ .$$

The first claim follows from (2.20). Moreover, (6.11) can be derived from (6.10) by passing to the limit as $m\to 0$:

$$\mathring{G}^{\mathbb{T}}_{1,0}(x,y) = \lim_{m\to 0}\left[G^{\mathbb{T}}_{1,m}(x,y) - \frac{1}{m^2}\right]\ . \qquad \blacksquare$$



(a) $s=1$ (b) $s=2$ (c) $s=3$

Fig 6: The grounded Green kernel $\mathring{G}^{\mathbb{T}}_{s,0}(\frac{1}{2},y)$ with $y\in[0,1)$ for $s=1,2,3$.

6.3.3. *Hyperbolic Space.* For the hyperbolic space $\mathbb{H}^n$ of curvature -1, a closed expression for the Green kernels is available in dimension 3.

PROPOSITION 6.9. *For all $s, m, r > 0$,*

$$G^{\mathbb{H}^3}_{s,m}(r) = \frac{r}{\sinh r}\,\frac{1}{(2\pi)^{3/2}\,\Gamma(s)}\int_0^\infty e^{-(m^2+1/2)t}\,e^{-r^2/(2t)}\,t^{s-5/2}\,\mathrm{d}t \;=\; \frac{r}{\sinh r}\cdot G^{\mathbb{R}^3}_{s,\sqrt{m^2+1/2}}(r) \tag{6.15}$$

with $G^{\mathbb{R}^3}_{s,m}(r)$ denoting the Green kernel for $\mathbb{R}^3$ as discussed above.

Thus, for instance, $G^{\mathbb{H}^3}_{2,m}(r) = \frac{1}{2\pi\sqrt{2m^2+1}}\,\frac{r}{\sinh r}e^{-\sqrt{2m^2+1}\,r}$.

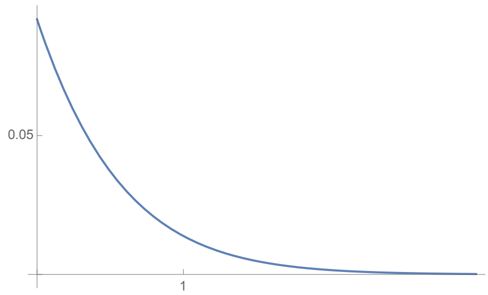


Fig 7: The Green kernel $G^{\mathbb{H}^3}_{2,1}$.

PROOF. The claim is an immediate consequence of the closed expression for the heat kernel on $\mathbb{H}^3$ given e.g. in [12, Eqn. (5.7.3)]. ■

REMARK 6.10. Integro-differential representations for $G^{\mathbb{H}^n}_{s,m}$, $n \geq 4$, may be obtained in light of the analogous representations for the heat kernel $p^{\mathbb{H}^n}_t$ in [24].

COROLLARY 6.11. *The Green kernel $G^{\mathbb{H}^3}_{s,m}$ on $\mathbb{H}^3$ has asymptotic behavior close to the diagonal similar to $G^{\mathbb{R}^3}_{s,m}$. More precisely, if $C(s,m)$ denote the constants in the asymptotic formula* (6.7) *for the Euclidean Green kernel, then*

$$G^{\mathbb{H}^3}_{s,m}(0) - G^{\mathbb{H}^3}_{s,m}(r) \asymp \begin{cases} C\big(s,\sqrt{m^2+1/2}\big)\cdot r^{2s-3} & \textit{if } s \in (3/2, 3/2+1)\ , \\ C\big(s,\sqrt{m^2+1/2}\big)\cdot r^2\log\frac{1}{r} & \textit{if } s = 3/2+1\ , \\ \Big(C\big(s,\sqrt{m^2+1/2}\big)+\frac{1}{6}\Big)\cdot r^2 & \textit{if } s > 3/2+1\ . \end{cases} \tag{6.16}$$

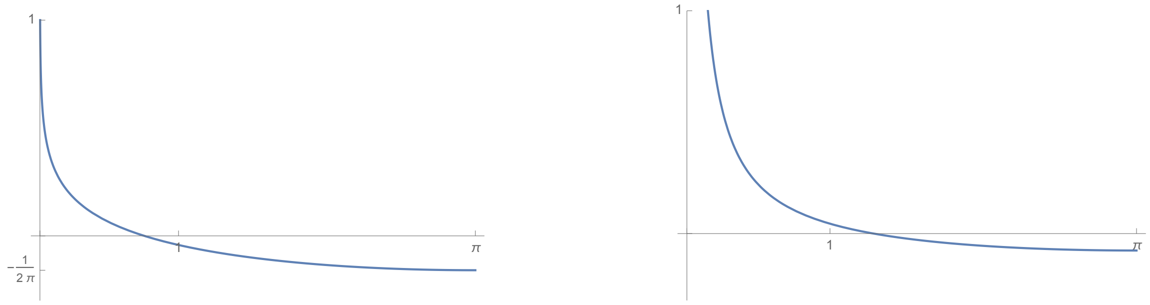
PROOF. It suffices to compute the Taylor expansion of $G^{\mathbb{H}^3}_{s,m}(r)$ in the form (6.15) around $r = 0$. For $s \in (3/2, 3/2+1]$ the Taylor expansion of $r/\sinh(r)$ only provides terms of order $O(r^2)$, which are smaller than the leading order of $G^{\mathbb{R}^3}_{s,\sqrt{m^2+1/2}}(r)$ as $r \to 0$ computed in (6.7). When $s > 3/2+1$, the same Taylor expansion provides a further additive factor $r^2/6$. ■

6.3.4. *Sphere.* For the unit sphere we can derive explicit formulas for the grounded Green kernel of any order $s \in \mathbb{N}$ in any dimension, based on the observation (2.20), the well-known representation of the radial Laplacian on spheres, and symmetry arguments. We present the results in some of the most important cases.

THEOREM 6.12. *For the sphere in 2 and 3 dimensions,*

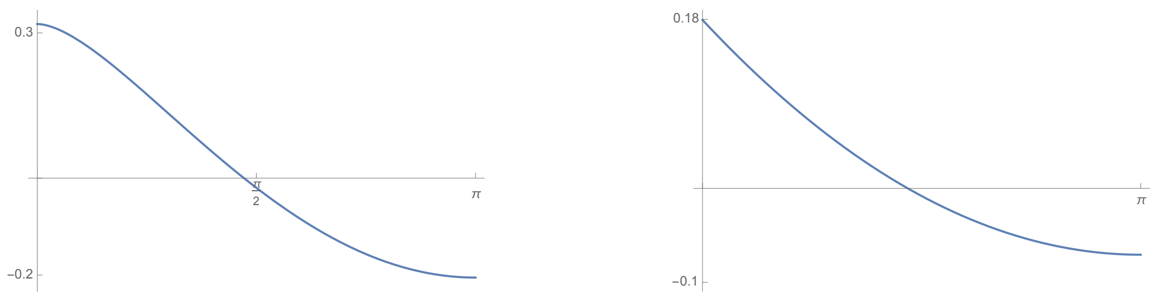
$$\mathring{G}^{\mathbb{S}^2}_{1,0}(r) = -\,\frac{1}{2\pi}\left(1+2\log\sin\frac{r}{2}\right)\ , \qquad \mathring{G}^{\mathbb{S}^3}_{1,0}(r) = \frac{1}{2\pi^2}\left(-\frac{1}{2}+(\pi-r)\cdot\cot r\right)\ , \tag{6.17}$$

$$\mathring{G}^{\mathbb{S}^2}_{2,0}(r) = \frac{1}{\pi}\int_0^{\sin^2(r/2)} \frac{\log t}{1-t}\, \mathrm{d}t + \frac{1}{\pi}\ , \qquad \mathring{G}^{\mathbb{S}^3}_{2,0}(r) = \frac{(\pi-r)^2}{4\pi^2} + \frac{1}{8\pi^2} - \frac{1}{12}\ . \tag{6.18}$$



(a) $\mathring{G}^{\mathbb{S}^2}_{1,0}$ (b) $\mathring{G}^{\mathbb{S}^3}_{1,0}$

Fig 8: The grounded Green kernels on $\mathbb{S}^n$ for $s=1$ and $n=2,3$.



(a) $\mathring{G}^{\mathbb{S}^2}_{2,0}$ (b) $\mathring{G}^{\mathbb{S}^3}_{2,0}$

Fig 9: The grounded Green kernels on $\mathbb{S}^n$ for $s=2$ and $n=2,3$.

Observe that for all $m>0$ as $r\to 0$,

$$\mathring{G}^{\mathbb{S}^2}_{1,0}(r) \asymp G^{\mathbb{R}^2}_{1,m}(r) \asymp -\frac{1}{\pi}\log r\ , \qquad \mathring{G}^{\mathbb{S}^3}_{1,0}(r) \asymp G^{\mathbb{H}^3}_{1,m}(r) \asymp G^{\mathbb{R}^3}_{1,m}(r) \asymp \frac{1}{2\pi\, r}\ ,$$

and

$$\mathring{G}^{\mathbb{S}^3}_{2,0}(r) - \mathring{G}^{\mathbb{S}^3}_{2,0}(0) \ \asymp\ G^{\mathbb{H}^3}_{2,m}(r) - G^{\mathbb{H}^3}_{2,m}(0) \ \asymp\ G^{\mathbb{R}^3}_{2,m}(r) - G^{\mathbb{R}^3}_{2,m}(0) \ \asymp\ -\frac{1}{2\pi}\, r\ .$$

Proof. Recall that for a radially symmetric function $f(\,\cdot\,) = u\big(\mathsf{d}(x,\,\cdot\,)\big)$ on the n-sphere, the Laplacian and the volume integral are given by

$$\Delta f(y) = u''(r) + (n-1)\cot(r)\, u'(r) = \frac{1}{\sin^{n-1}(r)}\Big(\sin^{n-1}(r)\, u'(r)\Big)' \text{ with } r=\mathsf{d}(x,y)$$

and $\int_{\mathbb{S}^n} f\,\mathrm{dvol} = c_n\int_0^\pi u(r)\,\sin^{n-1}(r)\,\mathrm{d}r$. The representations in (6.17) thus follow from the fact that the functions u_2 and u_3 given by the respective right-hand sides of (6.17) are the unique solutions on the interval $(0,\pi)$ to the second-order differential equation

$$u_n''(r) + (n-1)\cot(r)\, u_n'(r) = \frac{2}{\mathrm{vol}(\mathbb{S}^n)}\ , \qquad \lim_{r\to 0} r^{n-1} u_n'(\pi - r) = 0\ , \qquad \int_0^\pi u_n(r)\sin^{n-1}(r)\,\mathrm{d}r = 0\ ,$$

which may be easily verified. Indeed, the function $u=u_2$ given above satisfies $u'(r)=-\frac{1}{2\pi}\cot\frac{r}{2}$ and thus

$$(u'(r)\cdot\sin r)'=-\tfrac{1}{2\pi}\,(1+\cos r)'=\tfrac{1}{2\pi}\sin r\ ,$$

hence $\Delta u=\frac{1}{2\pi}=\frac{2}{\mathrm{vol}(\mathbb{S}^2)}$. Moreover, $\int_0^\pi u(r)\,\sin(r)\,\mathrm{d}r=0$.

Similarly, $u=u_3$ satisfies $u'(r)=-\frac{1}{2\pi^2}\left(\cot r+(\pi-r)\frac{1}{\sin^2 r}\right)$ and thus

$$(u'(r)\cdot\sin^2 r)'=-\tfrac{1}{2\pi^2}\,(\cos r\,\sin r+\pi-r)'=\tfrac{1}{\pi^2}\sin^2 r\ ,$$

hence $\Delta u=\frac{1}{\pi^2}=\frac{2}{\mathrm{vol}(\mathbb{S}^3)}$. Moreover, $\int_0^\pi u(r)\,\sin^2(r)\,\mathrm{d}r=0$.

The representations in (6.18) follow from the fact that the functions v_2 and v_3 given by the respective right-hand sides of (6.18) are the unique solutions to

$$v_n''(r)+(n-1)\cot(r)\,v_n'(r)=-2u_n(r),\qquad \lim_{r\to 0} r^{n-1}v_n'(\pi-r)=0,\qquad \int_0^\pi v_n(r)\sin^{n-1}(r)\,\mathrm{d}r=0$$

with $u_n=\mathring{G}^{\mathbb{S}^n}_{1,0}$ for $n=2,3$ as specified above. To verify this, observe that v_2 satisfies $v_2'(r)\,\sin r=\frac{2}{\pi}\sin^2\frac{r}{2}\,\log\sin^2\frac{r}{2}$ and thus $(v_2'(r)\,\sin r)'\,\frac{1}{\sin r}=-2u_2$. Moreover,

$$\int_0^\pi\left(v_2(r)-\frac{1}{\pi}\right)\sin(r)\,\mathrm{d}r=\frac{2}{\pi}\int_0^1\int_0^t\frac{\log r}{1-r}\,\mathrm{d}r\,\mathrm{d}t=-\frac{2}{\pi}=-\frac{1}{\pi}\int_0^\pi\sin(r)\,\mathrm{d}r\ .$$

Similarly, v_3 as defined above satisfies

$$-\frac{1}{\sin^2 r}\Big(v_3'(r)\,\sin^2 r\Big)'=\frac{1}{2\pi^2\,\sin^2 r}\Big((\pi-r)\,\sin^2 r\Big)'=\frac{1}{2\pi^2}\Big(-1+2(\pi-r)\,\cot r\Big)=2u_3\ .\qquad\blacksquare$$

REMARK 6.13. The expression for $\mathring{G}^{\mathbb{S}^2}_{1,0}$ is in fact well known (see e.g. [31, Eqn. (9)]) and may equivalently be derived by means of complex geometry.

References.

[1] Abramowitz, M. and Stegun, I. A. *Handbook of Mathematical Functions.* with Formulas, Graphs, and Mathematical Tables. Courier Corporation, Apr. 1972.

[2] Albeverio, S., Brasche, J., and Röckner, M. Dirichlet forms and generalized Schrödinger operators. In Holden, H. and Jensen, A., editors, *Schrödinger Operators – Proceedings of the Nordic Summer School in Mathematics – Sandbjerg Slot, Sønderborg, Denmark, August 1-12, 1988*, volume 345 of *Lecture Notes in Physics*, pages 1–42. Springer-Verlag, 1989.

[3] Andres, S. and Kajino, N. Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions. *Probab. Theory Relat. Fields*, 166:713–752, 2016.

[4] Aubin, T. Espaces de Sobolev sur les Variétés Riemanniennes. *Bull. Sc. math.*, 100:149–173, 1976.

[5] Barlow, M. T., Chen, Z.-Q., and Murugan, M. Stability of EHI and regularity of MMD spaces. *arXiv:2008.05152v1*, 2020.

[6] Baudoin, F. and Lacaux, C. Fractional Gaussian fields on the Sierpinski gasket and related fractals. *arXiv:2003.04408*, 2020.

[7] Berestycki, N. Diffusion in planar Liouville quantum gravity. *Ann. I. H. Poincaré B*, 51(3):947–964, 2015.

[8] Bogachev, V. I. *Gaussian Measures*, volume 62 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., 1998.

[9] Croke, C. Some isoperimetric inequalities and eigenvalue estimates *Ann. Sci. Ecole Norm. Sup.*, 13(4):419–435, 1980.

[10] Dang, N. V. Wick squares of the Gaussian Free Field and Riemannian rigidity. *arXiv:1902.07315*, 2019.

[11] David, F., Kupiainen, A., Rhodes, R., and Vargas, V. Liouville quantum gravity on the Riemann sphere. *Commun. Math. Phys.*, 342(3):869–907, 2016.

[12] Davies, E. B. *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[13] Dello Schiavo, L. Ergodic Decomposition of Dirichlet Forms via Direct Integrals and Applications. *Potential Anal.*, 43 pp., 2021.

[14] Dello Schiavo, L., Herry, R., Kopfer, E., and Sturm, K.-T. Conformally Invariant Random Fields, Quantum Liouville Measures, and Random Paneitz Operators on Riemannian Manifolds of Even Dimension. *arXiv:2105.13925.*

[15] Dixmier, J.: Von Neumann Algebras. North-Holland (1981)

[16] Duplantier, B., Miller, J., and Sheffield, S. Liouville quantum gravity as a mating of trees. *Astérisque*, 2021+. In press.

[17] Eberle, A. Girsanov-type transformations of local Dirichlet forms: An analytic approach. *Osaka J. Math.*, 33(2):497–531, 1996.

[18] Fitzsimmons, P. J. Absolute continuity of symmetric diffusions. *Ann. Probab.*, 25(1):230–258, 1997.

[19] Fukushima, M., Oshima, Y., and Takeda, M. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics.* de Gruyter, extended edition, 2011.

[20] Garban, C., Rhodes, R., and Vargas, V. On the heat kernel and the Dirichlet form of Liouville Brownian motion. *Electron. J. Probab.*, 19(96):1–25, 2014.

[21] Garban, C., Rhodes, R., and Vargas, V. Liouville Brownian Motion. *Ann. Probab.*, 44(4):3076–3110, 2016.

[22] Gelbaum, Z. A. Fractional Brownian Fields over Manifolds. *Trans. Amer. Math. Soc.*, 366(9):4781–4814, 2014.

[23] Gradshteyn, I. S. and Ryzhik, I. M. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[24] Grigor'yan, A. and Noguchi, M. The Heat Kernel on Hyperbolic Space. *Bull. Lond. Math. Soc.*, 30:643–650, 1998.

[25] Große, N. and Schneider, C. Sobolev spaces on Riemannian manifolds with bounded geometry: General coordinates and traces. *Math. Nachr.*, 286(16):1586–1613, 2013.

[26] Grosswald, E. *Bessel Polynomials*, volume 698 of *Lecture Notes in Mathematics.* Springer-Verlag, 1978.

[27] Grothendieck, A. Produits Tensoriels Topologiques et Espaces Nucléaires. *Mem. Am. Math. Soc.*, 16, 1955.

[28] Guillarmou, C., Rhodes, R., and Vargas, V. Polyakov's formulation of $2d$ bosonic string theory. *Publ. math. IHES*, 130:111–185, 2019.

[29] Han, B.-X. and Sturm, K.-T. Curvature-dimension conditions under time change. *Ann. Matem. Pura Appl.*, 201(2):801-822, 2021.

[30] Hebey, E. *Sobolev Spaces on Riemannian Manifolds.* Springer-Verlag, 1996.

[31] Kimura, Y. and Okamoto, H. Vortex Motion on a Sphere. *J. Phys. Soc. Japan*, 56(12):4203–4206, 1987.

[32] Kuo, H.-H. *White Noise Distribution Theory.* Probability and Stochastics Series. CRC Press, 1996.

[33] Ledoux, M. and Talagrand, M. *Probability in Banach Spaces: Isoperimetry and Processes*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge – A Series of Modern Surveys in Mathematics.* Springer, 2^{nd} edition, 2006.

[34] Li, J.. Gradient Estimate for the Heat Kernel of a Complete Riemannian Manifold and Its Applications. *J. Funct. Anal.*, 97:293–310, 1991.

[35] Li, P. and Yau, S.-T. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156:153–201, 1986.

[36] Li, L. and Zhang, Z. On Li-Yau Heat Kernel Estimate. *Acta Math. Sinica*, 37(8):1205–1218, 2021.

[37] Lodhia, A., Sheffield, S., Sun, X., and Watson, S.S. Fractional Gaussian fields: A survey. *Probab. Surveys*, 13:1–56, 2016.

[38] Lawson, H. B. Jr., and Michelsohn, M.-L. *Spin Geometry.* Princeton Mathematical Series. Princeton University Press, 1989.

[39] Le Gall, J.-F. Brownian geometry. *Jpn. J. Math.*, 14(2):135–174, 2019.

[40] Ma, Z.-M. and Röckner, M. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Graduate Studies in Mathematics. Springer, 1992.

[41] Miller, J. and Sheffield, S. Liouville quantum gravity and the Brownian map I: the $\mathrm{QLE}(8/3, 0)$ metric. *Invent. Math.*, 219(1):75–152, 2020.

[42] Minakshisundaram, S. and Pleijel, Å. Some Properties of the Eigenfunctions of the Laplace-Operator on Riemannian Manifolds. *Can. J. Math.*, 1(3):242–256, 1949.

[43] Müller, O. and Nardmann, M. Every conformal class contains a metric of bounded geometry. *Math. Ann.*, 363(1-2):143–174, 2015.

[44] Norris, J. R. Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds. *Acta Math.*, 179:79–103, 1997.

[45] Petersen, P. *Riemannian Geometry.* Graduate Texts in Mathematics 171. Springer, 2006.

[46] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion.* Grundlehren der mathematischen Wissenschaften 293. Springer, 1991.

[47] Saloff-Coste, L. A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices*, 2:27–38, 1992.

[48] Saloff-Coste, L. Uniformly Elliptic Operators on Riemannian Manifolds. *J. Differ. Geom.*, 36:417–450, 1992.

[49] Schramm, O. and Sheffield, S. A contour line of the continuum Gaussian free field. *Probab. Theory Relat. Fields*, 157:47–80, 2013.

[50] Sheffield, S. Gaussian free fields for mathematicians. *Probab. Theory Relat. Fields*, 139:521–541, 2007.

[51] Souplet, P. and Zhang, Q. Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds. *Bulletin of the London Mathematical Society*, 38(6):1045–1053, 2006.

[52] Strichartz, R. S. Analysis of the Laplacian on the complete Riemannian manifold. *J. Funct. Anal.*, 52:48–79, 1983.

[53] Stroock, D. W. and Turetsky, J. Upper Bounds on Derivatives of the Logarithm of the Heat Kernel. *Comm. Anal. Geom.*, 6(4):669–685, 1998.

[54] Sturm, K.-T. Analysis on local Dirichlet spaces III. The Parabolic Harnack Inequality. *J. Math. Pures Appl.*, 75:273–297, 1996.

[55] Trèves, F. *Topological Vector Spaces, Distributions and Kernels*, volume 25 of *Pure and Applied Mathematics.* Academic Press, 1967

[56] Triebel, H. *Theory of Function Spaces – Volume II*, volume 84 of *Monographs in Mathematics.* Birkhäuser, 1992.

[57] Vakhania, N. N., Tarieladze, V. I., and Chobanyan, S. A. *Probability Distributions on Banach Spaces*, volume 14 of *Mathematics and its Applications (Soviet Series)*. D. Reidel Publishing Co., Dordrecht, 1987.

[58] Watson, G. A. *Treatise on the Theory of Bessel Functions*. Cambridge University Press, 2nd edition, 1944.

Institute of Science and Technology Austria
Am Campus 1
3400 Klosterneuburg
Austria
E-mail: lorenzo.delloschiavo@ist.ac.at

Institut für Angewandte Mathematik
Rheinische Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany
E-mail: eva.kopfer@iam.uni-bonn.de
E-mail: sturm@uni-bonn.de