Curvature-dimension conditions under time change

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Abstract

We derive precise transformation formulas for synthetic lower Ricci bounds under time change. More precisely, for local Dirichlet forms we study how the curvature-dimension condition in the sense of Bakry-Émery will transform under time change. Similarly, for metric measure spaces we study how the curvature-dimension condition in the sense of Lott-Sturm-Villani will transform under time change.

Keywords: metric measure space, curvature-dimension condition, time change, Bakry-Émery theory, Dirichlet form.

Contents

1 Introduction

A. Bakry and Émery [5] formulated a powerful criterion for obtaining equilibration and regularity results for the Markov semigroups associated with local Dirichlet forms. Let us briefly recall their concept. A Dirichlet form $\mathcal{E}$, densely defined on

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some $L^2(X,\mathfrak{m})$, satisfies the $\mathrm{BE}(k,N)$ condition with some function $k \in L^\infty_{\mathrm{loc}}(X,\mathfrak{m})$ and some number $N \in [1,\infty]$ if

$$\frac{1}{2}\int \Gamma(f)\Delta\varphi\, \mathrm{d}\mathfrak{m} - \int \Gamma(f,\Delta f)\varphi\, \mathrm{d}\mathfrak{m} \geq \int \Big(k\,\Gamma(f) + \frac{1}{N}(\Delta f)^2\Big)\varphi\, \mathrm{d}\mathfrak{m}. \tag{1.1}$$

for all suitable functions f and $\varphi \geq 0$ on X. Here Δ denotes the generator associated with $\mathcal{E}$ and Γ the carré du champ operator. Estimate (1.1) can be regarded as an abstract formulation of Bochner's inequality on Riemannian manifolds. Thus, in this Eulerian approach to curvature-dimension conditions, $k(x)$ will be considered as a synthetic lower bound for the "Ricci curvature at $x \in X$" and N as an upper bound for the "dimension" of X.

From the very beginning of this theory, the transformation formula for the Bakry-Émery condition $\mathrm{BE}(k,N)$ under *drift transformation* played a key role. Most importantly in the case $N=\infty$, this states that the Dirichlet form

$$\mathcal{E}^*(u) := \int \Gamma(u)\,\mathrm{d}\mathfrak{m}^* \text{ on } L^2(X,\mathfrak{m}^*) \text{ with } \mathfrak{m}^* := e^{-V}\,\mathfrak{m}$$

satisfies $\mathrm{BE}(k^*,\infty)$ with $k^* := k + h_V$ where $h_V(x) := \inf_f \frac{1}{\Gamma(f)}\big[\Gamma\big(\Gamma(V,f),f\big) - \frac{1}{2}\Gamma\big(\Gamma(f),V\big)\big]$ denotes the lower bound for the Hessian of V at $x \in X$ for any sufficiently regular function V on X.

The goal of this paper now is to analyze the transformation property of the Bakry-Émery condition under *time change*. That is, we will pass from the original Dirichlet form $\mathcal{E}$ on $L^2(X,\mathfrak{m})$ to a new one defined as

$$\mathcal{E}'(u) := \int \Gamma(u)\,\mathrm{d}\mathfrak{m} \text{ on } L^2(X,\mathfrak{m}') \text{ with } \mathfrak{m}' := e^{2w}\,\mathfrak{m}$$

for some $w \in L^\infty_{\mathrm{loc}}(X,\mathfrak{m})$. Our main result provides a Bakry-Émery condition for this transformed Dirichlet form provided the original Dirichlet form satisfies a Bakry-Émery condition with finite N.

Theorem 1.1. *Assume that $\mathcal{E}$ satisfies the $\mathrm{BE}(k,N)$ condition for some $k \in L^\infty_{\mathrm{loc}}$ and some $N \in [1,\infty)$, and that $w \in \mathrm{D}_{\mathrm{loc}}(\boldsymbol{\Delta}) \cap L^\infty_{\mathrm{loc}}$ with $\boldsymbol{\Delta} w = \boldsymbol{\Delta}_{sing} w + \Delta_{ac} w\,\mathfrak{m}$ and $\boldsymbol{\Delta}_{sing} w \leq 0$. Then for any $N' \in (N,\infty]$ and $k' \in L^\infty_{\mathrm{loc}}$, the time-changed Dirichlet form $\mathcal{E}'$ on $L^2(X,\mathfrak{m}')$ satisfies the $\mathrm{BE}(k',N')$ condition provided*

$$k' \leq e^{-2w}\Big[k - \frac{(N-2)(N'-2)}{N'-N}\Gamma(w) - \Delta_{ac} w\Big]. \tag{1.2}$$

Corollary 1.2. *If in addition k' is bounded from below, say $k' \geq K'$ for some $K' \in \mathbb{R}$, then the time changed Dirichlet form $\mathcal{E}'$ and the associated heat semigroup $(\mathrm{P}'_t)_{t\geq 0}$ satisfy the following gradient estimate*

$$\Gamma'(P'_t f) + \frac{1-e^{-2K't}}{N'K'}(\Delta' P'_t f)^2 \leq e^{-2K't} P'_t\big(\Gamma'(f)\big). \tag{1.3}$$

Remark 1.3. Generator and carré du champ operator of the time-changed Dirichlet form $\mathcal{E}'$ on $L^2(X,\mathfrak{m}')$ are given by

$$\Delta' = e^{-2w}\Delta, \quad \Gamma' = e^{-2w}\Gamma.$$

Moreover, the associated Brownian motion $(\mathbb{P}'_x, \mathrm{B}'_t)$ (c.f. Chapter 6 [10]) is given by $\mathbb{P}'_x = \mathbb{P}_x$ and

$$\mathrm{B}'_t = \mathrm{B}_{\tau_t}, \quad \tau_t = \int_0^t e^{-2w(\mathrm{B}'_s)} ds, \qquad \sigma_t = \int_0^t e^{2w(\mathrm{B}_s)}\, \mathrm{d}s, \quad \mathrm{B}_t = \mathrm{B}'_{\sigma_t}. \tag{1.4}$$

Note that heat semigroup $(P'_t)_{t\geq 0}$ and Brownian motion $(\mathbb{P}'_x, \mathrm{B}'_t)$ are linked to each other by

$$P'_t f(x) = \mathbb{E}'_x[f(\mathrm{B}'_{2t})].$$

B. A different approach, the so-called Lagrangian approach, to synthetic lower Ricci bounds was proposed in the works of Lott, Villani [16] and Sturm [20]. Here the objects under consideration are metric measure spaces. Such a space $(X, \mathrm{d}, \mathfrak{m})$ satisfies the curvature-dimension condition $\mathrm{CD}(K,\infty)$ – meaning that its Ricci curvature is bounded from below by K – if the Boltzmann entropy $\mathrm{Ent}(.,\mathfrak{m})$ is weakly K-convex on the Wasserstein space $\mathcal{P}_2(X)$. More refined curvature-dimension conditions $\mathrm{CD}(K,N)$ and $\mathrm{CD}^*(K,N)$ with finite $N \in [1,\infty)$ were introduced in [21] and [20]. Combined with the requirement of Hilbertian energy functional, this led to the conditions $\mathrm{RCD}(K,N)$ and $\mathrm{RCD}^*(K,N)$ [2], which fortunately turned out to be equivalent to each other [**?**].

Also from the very beginning of this theory, the transformation formula for the curvature-dimension conditions $\mathrm{CD}(K,N)$, $\mathrm{CD}^*(K,N)$ $\mathrm{RCD}(K,N)$ under *drift transformation* played a key role. Most easily formulated in the case $N=\infty$, it states that the condition $\mathrm{CD}(K,\infty)$ for a given metric measure space $(X,\mathrm{d},\mathfrak{m})$ and the L-convexity of V on X imply the condition $\mathrm{CD}(K+L,\infty)$ for the transformed metric measure space $(X, \mathrm{d}, e^{-V}\mathfrak{m})$. The same holds with RCD in the place of CD.

Subject of the investigations in this paper is the *time-changed metric measure space* $(X,\mathrm{d}',\mathfrak{m}')$ where $\mathfrak{m}' = e^{2w}\mathfrak{m}$ for some $w \in L^\infty_{\mathrm{loc}}(X,\mathfrak{m})$ and

$$\mathrm{d}'(x,y) := \sup\left\{\phi(x)-\phi(y) : \phi \in \mathrm{D}_{\mathrm{loc}}(\mathcal{E}) \cap C(X), |\mathrm{D}\phi| \leq e^w \ \mathfrak{m}\text{-a.e. in } X\right\}$$

for $x, y \in X$. Assuming that w is continuous $\mathfrak{m}$-a.e. on X this allows for a dual representation as

$$\mathrm{d}'(x,y) = \inf\left\{\int_0^1 \mathrm{e}^{\bar{w}(\gamma_s)}\,|\dot\gamma_s|\,\mathrm{d}s : \gamma \in \mathrm{AC}([0,1],X), \gamma_0 = x, \gamma_1 = y\right\}$$

where $\bar{w}(x) := \limsup_{y\to x} w(y)$ denotes the upper semicontinuous envelope of w. Our main result provides the transformation formula for the curvature-dimension condition under time change.

Theorem 1.4. *Let* $(X,\mathrm{d},\mathfrak{m})$ *be a* $\mathrm{RCD}(K,N)$ *space and let* $w \in \mathrm{D}_{\mathrm{loc}}(\mathbf{\Delta}) \cap L^\infty_{\mathrm{loc}}(X)$ *be continuous* $\mathfrak{m}$*-a.e. with* $\mathbf{\Delta} w = \mathbf{\Delta}_{sing} w + \Delta_{ac} w\, \mathfrak{m}$ *and* $\mathbf{\Delta}_{sing} w \leq 0$*. Then the time-changed metric measure space* $(X,\mathrm{d}',\mathfrak{m}')$ *satisfies the* $\mathrm{RCD}(K',N')$ *condition for any* $N' \in (N,+\infty]$ *and* $K' \in \mathbb{R}$ *such that*

$$K' \leq e^{-2w}\Big[K - \frac{(N-2)(N'-2)}{N'-N}|\mathrm{D}w|^2 - \Delta_{ac} w\Big].$$

Theorem 1.4 is a more or less immediate consequence of Theorem 1.1 and the fact that the Eulerian and the Lagrangian curvature-dimension conditions, $\mathrm{BE}(K,N)$ and $\mathrm{RCD}(K,N)$, are equivalent to each other as proven in [9].

Remark 1.5. The first derivation of the transformation formula for the (Eulerian) curvature-dimension condition $\mathrm{BE}(K,N)$ under conformal transformation as well as under time change was presented in [23] by the second author in the setting of regular Dirichlet forms admitting a nice core of sufficiently smooth functions ("Γ-calculus in the sense of Bakry-Émery-Ledoux").

Combining the techniques and results in [12] and [17], the first author [13, 14] proved the transformation formula for the Lagrangian curvature-dimension condition $\mathrm{RCD}(K,N)$ under conformal transformation when the reference function w is bounded and smooth enough. Together with the well-known transformation formula for $\mathrm{RCD}(K,N)$ under drift transformations, this result also provides a transformation formula for $\mathrm{RCD}(K,N)$ under time change.

The focus of the current paper is on proving the transformation formula for the (Eulerian or Lagrangian) curvature-dimension condition under time change in a setting of great generality (Dirichlet forms or metric measure spaces) and with minimal regularity and boundedness assumptions on w.

C. One of the important applications of time-change is the "convexification" of non-convex subsets $\Omega \subset X$ of an $\mathrm{RCD}(K,N)$-space $(X,\mathrm{d},\mathfrak{m})$ as introducted by the second author and Lierl [15]. For sublevel sets of regular semi-convex functions V, they proved convexity after suitable conformal transformations while control of the curvature bound under these transformations follows from the work [13] of the first author. Unfortunately, these previous results do not apply to the most natural potential, the signed distance function $V = \mathrm{d}(.,\Omega) - \mathrm{d}(.,X\setminus\Omega)$ due to lack of regularity. The more general results of the current paper, will apply to a suitable truncation of the signed distance function and thus provide the following Convexification Theorem.

Theorem 1.6. *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{RCD}(K,N)$ space and Ω be a bounded ℓ-convex domain in (X,d) with $\mathfrak{m}(\partial\Omega)=0$ and $\mathfrak{m}^+(\partial\Omega)<\infty$. Then for any $N' \in (N,+\infty]$, there exists a Lipschitz function w such that the time-changed metric measure space $(\overline{\Omega},\mathrm{d}^w,\mathfrak{m}^w)$ is a $\mathrm{RCD}(K',N')$ space for some $K' \in \mathbb{R}$.*

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2 Time change and the Bakry-Émery condition

This section is devoted to study synthetic lower Ricci bounds under time change in the setting of Dirichlet forms. More precisely, we will derive the transformation formula for the Bakry-Émery condition under time change.

2.1 Dirichlet forms and the $\mathrm{BE}(K,N)$ condition

In this part, we recall some basic facts about Dirichlet form theory and the Bakry-Émery theory. Firstly we make some basic assumptions on the Dirichlet form, see also [18] for examples satisfying these conditions.

Assumption 2.1. We assume that

a) (X,τ) is a topological space, $(X,\mathcal{B})$ is a measurable space and $\mathfrak{m}$ is a σ-finite Radon measure with full support (i.e. $\operatorname{supp}\mathfrak{m} = X$); $\mathcal{B}$ is the $\mathfrak{m}$-completion of the Borel σ-algebra generated by τ; and $L^p(X,\mathfrak{m})$ will denote the space of L^p-integrable functions on $(X,\mathcal{B},\mathfrak{m})$;

b) $\mathcal{E}(\cdot) : L^2(X,\mathfrak{m}) \mapsto [0,\infty]$ is a strongly local, quasi-regular, symmetric Dirichlet form with domain $\mathbb{V} := \mathrm{D}(\mathcal{E}) = \big\{f \in L^2(X,\mathfrak{m}) : \mathcal{E}(f) < \infty\big\}$; denote by $(P_t)_{t>0}$ the heat semi-group generated by $\mathcal{E}$;

c) there exists an increasing sequence of ("cut-off") functions with compact support $(\chi_\ell)_{\ell\geq 1} \subset \mathbb{V}_\infty$ such that $0 \leq \chi_\ell \leq 1$, $\Gamma(\chi_\ell) \leq C$ for all ℓ and $\chi_\ell \to 1$, $\Gamma(\chi_\ell) \to 0$ as $\ell \to \infty$, cf. [19];

d) $\mathcal{E}$ satisfies the Bakry-Émery condition $\mathrm{BE}(K,\infty)$ for some $K \in \mathbb{R}$.

To formulate the latter, recall that $\mathbb{V}_\infty := \mathrm{D}(\mathcal{E}) \cap L^\infty(X,\mathfrak{m})$ is an algebra with respect to pointwise multiplication. We say that $\mathcal{E}$ admits a carré du champ if there exists a quadratic continuous map $\Gamma : \mathbb{V} \to L^1(X,\mathfrak{m})$ such that

$$\int_X \Gamma(f)\varphi \,\mathrm{d}\mathfrak{m} = \mathcal{E}(f, f\varphi) - \frac{1}{2}\mathcal{E}(f^2,\varphi) \qquad \text{for all } f \in \mathbb{V}, \varphi \in \mathbb{V}_\infty.$$

By polarization, we define $\Gamma(f,g) := \frac{1}{4}\big(\Gamma(f+g) - \Gamma(f-g)\big)$ and obtain $\mathcal{E}(f,g) = \int \Gamma(f,g)\,\mathrm{d}\mathfrak{m}$ for all $f,g \in \mathbb{V}$. It is known that Γ is local in the sense that $\Gamma(f-g) = 0$ $\mathfrak{m}$-a.e. on the set $\{f = g\}$.

The Dirichlet form $\mathcal{E}$ induces a densely defined selfadjoint operator $\Delta : \mathrm{D}(\Delta) \subset \mathbb{V} \mapsto L^2$ satisfying $\mathcal{E}(f,g) = -\int g\Delta f \,\mathrm{d}\mathfrak{m}$ for all $g \in \mathbb{V}$. Put

$$\Gamma_2(f;\varphi) := \frac{1}{2}\int \Gamma(f)\Delta\varphi \,\mathrm{d}\mathfrak{m} - \int \Gamma(f,\Delta f)\varphi \,\mathrm{d}\mathfrak{m}$$

and $\mathrm{D}(\Gamma_2) := \Big\{(f,\varphi) : f,\varphi \in \mathrm{D}(\Delta),\ \Delta f \in \mathbb{V},\ \varphi, \Delta\varphi \in L^\infty\Big\}$.

Definition 2.2 (Bakry-Émery condition)**.** Given a function $k \in L^\infty$ and a number $N \in [1,\infty]$, we say that the Dirichlet form $\mathcal{E}$ satisfies the $\mathrm{BE}(k,N)$ condition if it admits a carré du champ and if

$$\frac{1}{2}\int \Gamma(f)\Delta\varphi \,\mathrm{d}\mathfrak{m} - \int \Gamma(f,\Delta f)\varphi \,\mathrm{d}\mathfrak{m} \geq \int \Big(k\Gamma(f) + \frac{1}{N}(\Delta f)^2\Big)\varphi \,\mathrm{d}\mathfrak{m}. \tag{2.1}$$

for all $(f,\varphi) \in \mathrm{D}(\Gamma_2)$, $\varphi \geq 0$.

Remark 2.3. Since by our standing assumption the Dirichlet form $\mathcal{E}$ satisfies $\mathrm{BE}(K,\infty)$ for some $K \in \mathbb{R}$, the "space of test functions"

$$\mathrm{TestF}(\mathcal{E}) := \big\{ f \in \mathrm{D}(\Delta) : \ \Delta f \in \mathbb{V}^\infty, \, \Gamma(f) \in L^\infty \big\}$$

is dense in $\mathbb{V}$ (c.f. Section 2 [3] and Remark 2.5 therein). Hence, the $\mathrm{BE}(k,N)$ condition will follow if (2.1) holds true for all $f \in \mathrm{TestF}(\mathcal{E})$ and all non-negative $\varphi \in \mathrm{D}(\Delta) \cap L^\infty$ with $\Delta\varphi \in L^\infty$.

Lemma 2.4. *For every $f \in \mathrm{D}(\Delta)$, we have $\Gamma(f)^{1/2} \in \mathbb{V}$ and*

$$\mathcal{E}\Big(\Gamma(f)^{1/2}\Big) \leq \int (\Delta f)^2 \, d\mathfrak{m} - K \cdot \mathcal{E}(f).$$

Proof. By self-improvement, the Bakry-Émery inequality $\mathrm{BE}(K,\infty)$ as introduced above implies the stronger L^1-version

$$\int \Gamma(f)^{1/2} \Delta\varphi \, \mathrm{d}\mathfrak{m} - \int \frac{1}{\Gamma(f)^{1/2}} \Gamma(f, \Delta f) \varphi \, \mathrm{d}\mathfrak{m} \geq K \int \Gamma(f)^{1/2} \varphi \, \mathrm{d}\mathfrak{m}.$$

for all $f, \varphi \in \mathrm{D}(\Delta)$ with $\Delta f \in \mathbb{V}$, see [17]. Choosing $\varphi = P_t(\Gamma(f)^{1/2})$ and then letting $t \to 0$ yields the claim for $f \in \mathrm{D}(\Delta)$ with $\Delta f \in \mathbb{V}$. Since the class of these f's is dense in $\mathrm{D}(\Delta)$, the claim follows. $\square$

Definition 2.5. i) We say that $f \in \mathbb{V}^e$ if there exists a Cauchy sequence $(f_n)_n \subset \mathbb{V}$ w.r.t. the semi-norm $\mathcal{E}(\cdot)$ and such that $f_n \to f$ $\mathfrak{m}$-a.e. Then we define $\mathcal{E}(f) := \lim_{n\to\infty} \mathcal{E}(f_n)$. Similarly, Γ can be extended to $\mathbb{V}^e$.

ii) We say that $f \in \mathbb{V}_{\mathrm{loc}}$ if for any bounded open set U, there is $\bar{f} \in \mathbb{V}$ such that $f = \bar{f}$ on U. Then a function $\Gamma(f) \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$ can be defined unambiguously by $\Gamma(f) := \Gamma(\bar{f})$ on U.

Similarly, we define the spaces $\mathrm{D}_{\mathrm{loc}}(\Delta)$ and $\mathrm{TestF}_{\mathrm{loc}}(\mathcal{E})$.

Definition 2.6 (Local weak Bakry-Émery condition)**.** Given a function $k \in L^\infty_{\mathrm{loc}}$ and a number $N \in [1,\infty]$, we say that the Dirichlet form $\mathcal{E}$ satisfies the $\mathrm{BE}_{\mathrm{loc}}(k,N)$ condition if it admits a carré du champ and if

$$-\frac{1}{2} \int \Gamma\big(\Gamma(f), \varphi\big) \, \mathrm{d}\mathfrak{m} - \int \Gamma(f, \Delta f) \varphi \, \mathrm{d}\mathfrak{m} \geq \int \Big(k\Gamma(f) + \frac{1}{N}(\Delta f)^2 \Big) \varphi \, \mathrm{d}\mathfrak{m}. \tag{2.2}$$

for all $f \in \mathrm{D}_{\mathrm{loc}}(\Delta) \cap L^\infty_{\mathrm{loc}}$ with $\Delta f \in \mathbb{V}_{\mathrm{loc}}$ and all non-negative $\varphi \in \mathbb{V}^\infty$ with compact support and $\Gamma(\varphi) \in L^\infty$.

Note that our standing assumption $\mathrm{BE}(K,\infty)$ implies that $\Gamma(f)^{1/2} \in \mathbb{V}_{\mathrm{loc}}$ for each $f \in \mathrm{D}_{\mathrm{loc}}(\Delta)$. Thus for functions f and φ as above, the term $-\frac{1}{2}\int \Gamma\big(\Gamma(f), \varphi\big) \, \mathrm{d}\mathfrak{m}$ is well-defined.

Lemma 2.7. *$\mathcal{E}$ satisfies $\mathrm{BE}(k,N)$ for $k \in L^\infty$ if and only if it satisfies $\mathrm{BE}_{\mathrm{loc}}(k,N)$.*

Proof. Assume that $\mathrm{BE}(k,N)$ holds true and let f and φ be given as in Definition 2.6. Choose $f' \in \mathrm{D}(\Delta) \cap L^\infty$ with $\Delta f' \in \mathbb{V}$ such that $f = f'$ on a neigborhood of $\{\varphi \neq 0\}$. Choose uniformly bounded, nonnegative $\varphi_n \in \mathrm{D}(\Delta)$ with $\Gamma(\varphi_n), \Delta\varphi_n \in L^\infty$ such that $\varphi_n \to \varphi$ a.e. on X and in $\mathbb{V}$ as $n \to \infty$. (For instance, put $\varphi_n = P_{1/n}\varphi$.) Then (2.1) implies

$$-\frac{1}{2}\int \Gamma\big(\Gamma(f'),\varphi_n\big)\,\mathrm{d}\mathfrak{m} - \int \Gamma(f',\Delta f')\varphi_n\,\mathrm{d}\mathfrak{m} \geq \int \Big(k\Gamma(f') + \frac{1}{N}(\Delta f')^2\Big)\varphi_n\,\mathrm{d}\mathfrak{m} > -\infty$$

for all n. Passing to the limit $n \to \infty$ yields (2.2) with f' in the place of f. Since by assumption $f = f'$ on a neigborhood of $\{\varphi \neq 0\}$, this yields the claim (2.2).

Conversely, assume that $\mathrm{BE}_{\mathrm{loc}}(k,N)$ holds true and let f and φ be given as in Definition 2.2. Put $\varphi_n = P_{1/n}\varphi$ and $\varphi_{\ell,n} = \chi_\ell \cdot P_{1/n}\varphi$ with $(\chi_\ell)_\ell$ being the cut-off functions from assumption 2.1. According to the $\mathrm{BE}_{\mathrm{loc}}(k,N)$ assumption, (2.2) holds with $\varphi_{\ell,n}$ in the place of φ. Passing to the limit $\ell \to \infty$ yields (2.2) with φ_n in the place of φ $(\forall n)$. This, however, is equivalent to (2.1), again with φ_n in the place of φ. Finally passing to the limit $n \to \infty$ yields (2.1) for the given φ. □

Remark 2.8. From the proof of the preceding Lemma, it is obvious that the class of f's to be considered for (2.2) can equivalently be restricted to $f \in \mathrm{D}_{\mathrm{loc}}(\Delta) \cap L^\infty_{\mathrm{loc}}$ with $\Delta f \in \mathbb{V}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$.

2.2 Self-improvement of the Bakry-Émery condition

The formulation of the subsequent results on the self-impovement property will require the theory of differential structures of Dirichlet forms as introduced by Gigli in [12]. In order to shorten the length of the paper, we will skip the introduction of (co)tangent modules, list the results directly and ignore subtle differences.

Proposition 2.9 (Section 2.2, [12])**.** *Given a strongly local, symmetric Dirichlet form $\mathcal{E}$ admitting a carré du champ Γ defined on $\mathbb{V}^e$ as above. Then there exists a L^∞-Hilbert module $L^2(TM)$ satisfying the following properties.*

i) *$L^2(TM)$ is a Hilbert space equipped with the norm $\|\cdot\|$ such that the following correspondence (embedding) holds*

$$\mathbb{V}^e \ni f \mapsto \nabla f \in L^2(TM), \qquad \|\nabla f\|^2 = \int \Gamma(f)\,\mathrm{d}\mathfrak{m}.$$

ii) *$L^2(TM)$ is a module over the commutative ring $L^\infty(X,\mathfrak{m})$.*

iii) *The norm $\|\cdot\|$ is induced by a pointwise inner product $\langle\cdot,\cdot\rangle$ satisfying*

$$\langle \nabla f, \nabla g\rangle = \Gamma(f,g) \qquad \mathfrak{m} - a.e.$$

and

$$\langle h\nabla f, \nabla g\rangle = h\langle \nabla f, \nabla g\rangle \qquad \mathfrak{m} - a.e.$$

for any $f, g \in \mathbb{V}^e$.

iv) *$L^2(TM)$ is generated by $\{\nabla g : g \in \mathbb{V}^e\}$ in the following sense. For any $v \in L^2(TM)$, there exists a sequence $v_n = \sum_{i=1}^{M_n} a_{n,i}\nabla g_{n,i}$ with $a_{n,i} \in L^\infty$ and $g_{n,i} \in \mathbb{V}^e$, such that $\|v - v_n\| \to 0$ as $n \to \infty$.*

By Corollary 3.3.9 [12], for any $f \in \mathrm{D}(\Delta)$ there is a continuous symmetric $L^\infty(M)$-bilinear map $\mathrm{Hess}_f(\cdot,\cdot)$ defined on $[L^2(TM)]^2$, with values in $L^0(X,\mathfrak{m})$. In particular, if $f,g,h \in \mathrm{TestF}$ (c.f. Lemma 3.2 [17], Theorem 3.3.8 [12]), $\mathrm{Hess}_f(\cdot,\cdot)$ is given by the following formula:

$$2\mathrm{Hess}_f(\nabla g,\nabla h) = \Gamma(g,\Gamma(f,h)) + \Gamma(h,\Gamma(f,g)) - \Gamma(f,\Gamma(g,h)). \tag{2.3}$$

Combining Theorem 1.4.11 and Proposition 1.4.10 in [12], we obtain the following structural results. As a consequence, we can compute $\mathrm{Hess}_f(\cdot,\cdot)$ and $\Gamma(\cdot,\cdot)$ using local coordinate.

Proposition 2.10. *Denote by $L^2(TM)$ the tangent module associated with $\mathcal{E}$. Then there exists a unique decomposition (up to $\mathfrak{m}$-null sets) $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of X such that*

- *a) For any $n \in \mathbb{N}$ and any $B \subset E_n$ with positive measure, $L^2(TM)$ has an orthonormal basis $\{e_{i,n}\}_{i=1}^n$ on B,*
- *b) For every subset B of E_∞ with finite positive measure, there exists an orthonormal basis $\{e_{i,B}\}_{i\in\mathbb{N}\cup\{\infty\}} \subset L^2(TM)|_B$ which generates $L^2(TM)|_B$,*

where we say that a countable set $\{v_i\}_i \subset L^2(TM)$ is orthonormal on B if $\langle v_i, v_j\rangle = \delta_{ij}$ $\mathfrak{m}$-a.e. on B. By definition, the local dimension $\dim_{\mathrm{loc}}(x) \in \mathbb{N}\cup\{\infty\}$ is n if $x \in E_n$.

Proposition 2.11. *Let $\mathcal{E}$ be a Dirichlet form satisfying the $\mathrm{BE}(k,N)$ condition for some $k \in L^\infty$ and some number $N \in [1,\infty]$ and let $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ be the decomposition given by Proposition 2.10. Then $\mathfrak{m}(E_n) = 0$ for $n > N$, and for any $(f,\varphi) \in \mathrm{D}(\Gamma_2)$, we have*

$$\Gamma_2(f;\varphi) \;\geq\; \int \Big(k\Gamma(f) + |\mathrm{Hess}_f|_{\mathrm{HS}}^2 + \frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{trHess}_f - \Delta f)^2\Big)\varphi \,\mathrm{d}\mathfrak{m} \tag{2.4}$$

where $\frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{trHess}_f - \Delta f)^2$ is taken 0 on E_N by definition.

The same estimate (2.4) also holds true for all all $f \in \mathrm{D}_{\mathrm{loc}}(\Delta)\cap L^\infty_{\mathrm{loc}}$ with $\Delta f \in \mathbb{V}_{\mathrm{loc}}$ and all nonnegative $\varphi \in \mathbb{V}^\infty$ with compact support and $\Gamma(\varphi) \in L^\infty$ provided $\mathcal{E}$ satisfies the $\mathrm{BE}_{\mathrm{loc}}(k,N)$ condition for some $k \in L^\infty_{\mathrm{loc}}$ and $N \in [1,\infty]$.

Proof. The proof for constant $k = K$ was given in [14], Proposition 3.2 and Theorem 3.3. In fact, the proof there only relies on a so-called self-improvement technique in Bakry-Émery theory, which can also be applied to $\mathrm{BE}(k,N)$ case without difficulty. Also the extension via localization is straightforward. □

In order to proceed, we briefly recall the notion of measure-valued Laplacian $\mathbf{\Delta}$ as introduced in [11, 17]. We say that $f \in \mathrm{D}(\mathbf{\Delta}) \subset \mathbb{V}^e$ if there exists a signed Borel measure $\mu = \mu_+ - \mu_-$ charging no capacity zero sets such that

$$\int \overline{\varphi}\,\mathrm{d}\mu = -\int \Gamma(\varphi.f)\,\mathrm{d}\mathfrak{m}$$

for any $\varphi \in \mathbb{V}$ with quasi-continuous representative $\overline{\varphi} \in L^1(X,|\mu|)$. If μ is unique, we denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll \mathfrak{m}$, we also denote its density by Δf if there is no ambiguity.

Proposition 2.12 (See Lemma 3.2 [17]). *Let $\mathcal{E}$ be a Dirichlet form satisfying the* $\mathrm{BE}_{\mathrm{loc}}(k, N)$ *condition. Then for any $f \in \mathrm{TestF}_{\mathrm{loc}}(\mathcal{E})$, we have $\Gamma(f) \in \mathrm{D}_{\mathrm{loc}}(\boldsymbol{\Delta})$ and*

$$\frac{1}{2}\boldsymbol{\Delta}\Gamma(f) - \Gamma(f, \Delta f)\,\mathfrak{m} \geq \Big(k\Gamma(f) + |\mathrm{Hess}_f|^2_{\mathrm{HS}} + \frac{1}{N - \dim_{\mathrm{loc}}}(\mathrm{tr}\mathrm{Hess}_f - \Delta f)^2\Big)\,\mathfrak{m}.$$

In particular, the singular part of the measure $\boldsymbol{\Delta}\Gamma(f)$ is non-negative.

2.3 $\mathrm{BE}(K, N)$ condition under time change

We define the time-change of the Dirichlet form $\mathcal{E}$ in the following way.

Definition 2.13 (Time change). Given a function $w \in L^\infty_{\mathrm{loc}}(X, \mathfrak{m})$, define the weighted measure $\mathfrak{m}^w := e^{2w}\mathfrak{m}$ and the time-changed Dirichlet form $\mathcal{E}^w$ on $L^2(X, \mathfrak{m}^w)$ by

$$\mathcal{E}^w(f) := \int \Gamma(f)\,\mathrm{d}\mathfrak{m} \qquad \forall f \in \mathbb{V}^w$$

with $\mathrm{D}(\mathcal{E}^w) := \mathbb{V}^w := \mathbb{V}^e \cap L^2(X, \mathfrak{m}^w)$. Note that indeed $\mathcal{E}^w(f)$ does not depend on w and $(\mathbb{V}^w)^e = \mathbb{V}^e$.

We leave it to the reader to verify the following simple but fundamental properties.

Lemma 2.14. *i) $\mathcal{E}^w$ is a strongly local, symmetric Dirichlet form.*

ii) $\mathcal{E}^w$ admits a carré du champ defined on $(\mathbb{V}^w)_{\mathrm{loc}} = \mathbb{V}_{\mathrm{loc}}$ by $\Gamma^w := e^{-2w}\Gamma$.

iii) Furthermore, $\mathrm{D}_{\mathrm{loc}}(\Delta^w) = \mathrm{D}_{\mathrm{loc}}(\Delta)$ and $\Delta^w f = e^{-2w}\Delta f$.

iv) If in addition $w \in \mathbb{V}_{\mathrm{loc}}$ then $\mathrm{TestF}_{\mathrm{loc}}(\mathcal{E}^w) = \mathrm{TestF}_{\mathrm{loc}}(\mathcal{E})$.

Our first main result will provide the basic estimate for the Bakry-Émery condition under time change.

Theorem 2.15. *Let $w \in \mathrm{D}_{\mathrm{loc}}(\boldsymbol{\Delta}) \cap L^\infty_{\mathrm{loc}}$ be given and assume that $\mathcal{E}$ satisfies $\mathrm{BE}_{\mathrm{loc}}(k, N)$ condition for some $N \in [2, \infty)$ and $k \in L^\infty_{\mathrm{loc}}$. Then for any $N' \in (N, \infty]$, any $f \in \mathrm{TestF}_{\mathrm{loc}}(\mathcal{E})$ and any non-negative $\varphi \in \mathbb{V}_\infty$ with compact support, we have*

$$\begin{aligned}-\int \Big[\frac{1}{2}\Gamma^w\big(\Gamma^w(f), \varphi\big) + \Gamma^w\big(f, \Delta^w f\big)\varphi\Big]\,\mathrm{d}\mathfrak{m}^w&\\ \geq \int \overline{\Gamma^w(f)\varphi}\,\mathrm{d}\kappa + \frac{1}{N'}\int (\Delta^w f)^2\varphi\,\mathrm{d}\mathfrak{m}^w& \qquad (2.5)\end{aligned}$$

where

$$\kappa := e^{-2w}\Big(k - \frac{(N-2)(N'-2)}{N' - N}\Gamma(w)\Big)\,\mathfrak{m}^w - \boldsymbol{\Delta}w.$$

Proof. By Lemma 2.14 we know

$$
\begin{aligned}
& -\frac{1}{2}\int \Gamma^w\big(\Gamma^w(f),\varphi\big)\,\mathrm{d}\mathfrak{m}^w - \int \Gamma^w(f,\Delta^w f)\varphi\,\mathrm{d}\mathfrak{m}^w \\
= & -\frac{1}{2}\int \Gamma\big(e^{-2w}\Gamma(f),\varphi\big)\,\mathrm{d}\mathfrak{m} - \int \Gamma(f,e^{-2w}\Delta f)\varphi\,\mathrm{d}\mathfrak{m} \\
= & \Big(-\frac{1}{2}\int \Gamma\big(\Gamma(f),\varphi\big)e^{-2w}\,\mathrm{d}\mathfrak{m} + \int \Gamma(f)\Gamma(w,\varphi)e^{-2w}\,\mathrm{d}\mathfrak{m}\Big) - \Big(\int \Gamma(f,\Delta f)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} \\
& -2\int \Delta f\Gamma(f,w)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m}\Big) \\
= & \Big(-\frac{1}{2}\int \Gamma\big(\Gamma(f),e^{-2w}\varphi\big)\,\mathrm{d}\mathfrak{m} - \int \Gamma\big(\Gamma(f),w\big)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m}\Big) + \int \Gamma(f)\Gamma(w,\varphi)e^{-2w}\,\mathrm{d}\mathfrak{m} \\
& - \int \Gamma(f,\Delta f)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} + 2\int \Delta f\Gamma(f,w)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} \\
= & \underbrace{\Big\{-\frac{1}{2}\int \Gamma\big(\Gamma(f),e^{-2w}\varphi\big)\,\mathrm{d}\mathfrak{m} - \int \Gamma(f,\Delta f)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m}\Big\}}_{\Gamma_2(f;e^{-2w}\varphi)} + \Big(\int \Gamma\big(w,e^{-2w}\Gamma(f)\varphi\big)\,\mathrm{d}\mathfrak{m} \\
& - \int \Gamma\big(w,e^{-2w}\Gamma(f)\big)\varphi\,\mathrm{d}\mathfrak{m}\Big) - \int \Gamma\big(\Gamma(f),w\big)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} + 2\int \Delta f\Gamma(f,w)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} \\
= & \Gamma_2(f;e^{-2w}\varphi) + \int \Gamma\big(w,e^{-2w}\Gamma(f)\varphi\big)\,\mathrm{d}\mathfrak{m} - \Big(\int \Gamma\big(w,\Gamma(f)\big)\varphi e^{-2w}\,\mathrm{d}\mathfrak{m} - 2\int \Gamma(w)\Gamma(f)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m}\Big) \\
& - \int \Gamma\big(\Gamma(f),w\big)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} + 2\int \Delta f\Gamma(f,w)e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} \\
= & (I)+(II)+(III)+(IV) - \Big[\int \frac{(N-2)(N'-2)}{N'-N}\Gamma(w,f)^2e^{-2w}\varphi\,\mathrm{d}\mathfrak{m} + \int \overline{\varphi\Gamma^w(f)}\,\mathrm{d}\boldsymbol{\Delta} w\Big]
\end{aligned}
$$

where

$$
(I) \;=\; \Gamma_2(f;e^{-2w}\varphi) - \int \Big(|\mathrm{Hess}_f|^2_{\mathrm{HS}} + \frac{1}{N-\dim_{\mathrm{loc}}}(\Delta f - \mathrm{tr}\mathrm{Hess}_f)^2\Big)\varphi e^{-2w}\,\mathrm{d}\mathfrak{m},
$$

$$
\begin{aligned}
(II) \;=\; & \int \Big(|\mathrm{Hess}_f|^2_{\mathrm{HS}} + 2\Gamma(f)\Gamma(w) + (\dim_{\mathrm{loc}}-2)\Gamma(f,w)^2 - 2\Gamma\big(w,\Gamma(f)\big) \\
& +2\Gamma(f,w)\mathrm{tr}\mathrm{Hess}_f\Big)\varphi e^{-2w}\,\mathrm{d}\mathfrak{m},
\end{aligned}
$$

$$
(III) = \frac{1}{N'-\dim_{\mathrm{loc}}}\int \Big(\Delta f - \mathrm{tr}\mathrm{Hess}_f + (2-\dim_{\mathrm{loc}})\Gamma(f,w)\Big)^2\varphi e^{-2w}\,\mathrm{d}\mathfrak{m},
$$

and

$$
\begin{aligned}
(IV) \;=\; & \int \Big[\Big(\frac{1}{N-\dim_{\mathrm{loc}}} - \frac{1}{N'-\dim_{\mathrm{loc}}}\Big)(\Delta f - \mathrm{tr}\mathrm{Hess}_f)^2 \\
& +2\Big(1-\frac{2-\dim_{\mathrm{loc}}}{N'-\dim_{\mathrm{loc}}}\Big)(\Delta f - \mathrm{tr}\mathrm{Hess}_f)\Gamma(f,w) \\
& +\Big(\frac{(N-2)(N'-2)}{N'-N} - (\dim_{\mathrm{loc}}-2) - \frac{(\dim_{\mathrm{loc}}-2)^2}{N'-\dim_{\mathrm{loc}}}\Big)\Gamma(f,w)^2\Big]\varphi e^{-2w}\,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

By Proposition 2.12 we obtain

$$(I) \geq \int k\Gamma(f)\overline{e^{-2w}\varphi}\,\mathrm{d}\mathfrak{m}.$$

By Lemma 2.16 below we get

$$(II) + (III) \geq \frac{1}{N'} \int \left(\Delta f\right)^2 \varphi e^{-2w}\,\mathrm{d}\mathfrak{m}.$$

As for the last term (IV), it can be checked that the function in the bracket is positive definite, so $(IV) \geq 0$.

Combining the computations above, we complete the proof. ☐

Lemma 2.16. *For any $f \in \mathrm{TestF}_{\mathrm{loc}}(\mathcal{E})$, we have*

$$A_1 + \frac{1}{N' - \dim_{\mathrm{loc}}} A_2^2 \geq \frac{1}{N'} \left(\Delta f\right)^2 \quad \mathfrak{m} - a.e.$$

where

$$\begin{aligned} A_1 := |\mathrm{Hess}_f|_{\mathrm{HS}}^2 \quad &+ \quad 2\Gamma(f)\Gamma(w) + (\dim_{\mathrm{loc}} - 2)\Gamma(f,w)^2 \\ &- \quad 2\Gamma(w, \Gamma(f)) + 2\Gamma(f,w)\mathrm{trHess}_f \end{aligned}$$

and

$$A_2 := \Delta f - \mathrm{trHess}_f - (\dim_{\mathrm{loc}} - 2)\Gamma(f,w).$$

Proof. By Proposition 2.11, there exits an orthonormal basis $\{e_i\}_i \subset L^2(TM)$. Then we denote $\mathrm{Hess}_f(e_i, e_j)$ by $(\mathrm{Hess}_f)_{ij}$ and denote $\langle \nabla g, e_i \rangle$ by g_i for any $g \in \mathbb{V}^e$. We define a matrix $H := (H_{ij})$ by $H_{ij} = (\mathrm{Hess}_f)_{ij} - w_i f_j - w_j f_i + \Gamma(f,w)\delta_{ij}$. Then we have

$$\begin{aligned} \sum_{i,j} H_{ij}^2 \quad &= \quad \sum_{i,j} \left((\mathrm{Hess}_f)_{ij} - w_i f_j - w_j f_i + \Gamma(f,w)\delta_{ij}\right)^2 \\ &= \quad |\mathrm{Hess}_f|_{\mathrm{HS}}^2 + 2\Gamma(f)\Gamma(w) + (\dim_{\mathrm{loc}} - 2)\Gamma(f,w)^2 \\ &\qquad - 2\Gamma(w, \Gamma(f)) + 2\Gamma(f,w)\mathrm{trHess}_f \quad = \quad A_1 \end{aligned}$$

where $\mathrm{trHess}_f = \sum_i (\mathrm{Hess}_f)_{ii}$. It can also be seen that

$$\mathrm{tr}H = \sum_i H_{ii} = \mathrm{trHess}_f + (\dim_{\mathrm{loc}} - 2)\Gamma(f,w) = \Delta f - A_2.$$

Finally, we obtain

$$\begin{aligned} A_1 + \frac{1}{N' - \dim_{\mathrm{loc}}} A_2^2 \quad &= \quad \|H\|_{\mathrm{HS}}^2 + \frac{1}{N' - \dim_{\mathrm{loc}}} \left(\mathrm{tr}H - \Delta f\right)^2 \\ &\geq \quad \frac{1}{\dim_{\mathrm{loc}}} \left(\mathrm{tr}H\right)^2 + \frac{1}{N' - \dim_{\mathrm{loc}}} \left(\mathrm{tr}H - \Delta f\right)^2 \geq \frac{1}{N'} \left(\Delta f\right)^2 \end{aligned}$$

which is the thesis. ☐

Theorem 2.17 (BE(k, N) condition under time change)**.** *Let $\mathcal{E}$ be a Dirichlet form satisfying the* $\mathrm{BE}_{\mathrm{loc}}(k, N)$ *condition for some $N \in [2, \infty)$ and $k \in L^\infty_{\mathrm{loc}}(X, \mathfrak{m})$. Assume that $w \in \mathrm{D}_{\mathrm{loc}}(\mathbf{\Delta}) \cap L^\infty_{\mathrm{loc}}$ with $\mathbf{\Delta} w = \mathbf{\Delta}_{sing} w + \Delta_{ac} w\, \mathfrak{m}$ and $\mathbf{\Delta}_{sing} w \le 0$. Moreover, assume that for some $N' \in (N, \infty]$ and $K' \in \mathbb{R}$*

$$K' \le e^{-2w} \Big[k - \frac{(N-2)(N'-2)}{N'-N} \Gamma(w) - \Delta_{ac} w \Big] \tag{2.6}$$

$\mathfrak{m}$-a.e. on X. Then the time-changed Dirichlet form $\mathcal{E}^w$ on $L^2(X, \mathfrak{m}^w)$ satisfies the $\mathrm{BE}(K', N')$ *condition.*

In particular, we have the following gradient estimate

$$\Gamma^w(P^w_t f) + \frac{1 - e^{-2K't}}{N'K'} (\Delta^w P^w_t f)^2 \le e^{-2K't} P^w_t \big(\Gamma^w(f)\big) \qquad \mathfrak{m}^w\text{-a.e.}$$

for all $f \in \mathrm{D}(\mathcal{E}^w)$.

Proof. Given the estimate (2.5) from the previous Theorem, we iteratively will extend the class of functions for which it holds true.

i) Our first claim is that (2.5) holds for all $f \in \mathrm{D}_{\mathrm{loc}}(\Delta) \cap L^\infty_{\mathrm{loc}}$ with $\Delta f \in \mathbb{V}_{\mathrm{loc}}$ and all compactly supported, nonnegative $\varphi \in \mathbb{V}^\infty$ with $\Gamma(\varphi) \in L^\infty$. Indeed, given such f and φ, choose $f' \in \mathrm{D}(\Delta) \cap L^\infty$ with $\Delta f \in \mathbb{V}$ such that $f = f'$ on a neighborhood of $\{\varphi \neq 0\}$. Choose $f_n \in \mathrm{TestF}(\mathcal{E})$ with $f_n \to f'$ in $\mathrm{D}(\Delta)$ and $\Delta f_n \to \Delta f'$ in $\mathbb{V}$. (For instance, put $f_n = P_{1/n} f'$.) Applying (2.5) with f_n in the place of f and passing to the limit $n \to \infty$ yields the claim. Indeed,

$$\Gamma^w(\Gamma^w(f_n), \varphi) = e^{-4w} \Big[\Gamma(\Gamma(f_n), \varphi) - 2\Gamma(f_n) \cdot \Gamma(w, \varphi) \Big]$$

which according to Lemma 2.4 for $n \to \infty$ converges to

$$e^{-4w} \Big[\Gamma(\Gamma(f), \varphi) - 2\Gamma(f) \cdot \Gamma(w, \varphi) \Big] = \Gamma^w(\Gamma^w(f), \varphi)$$

since $f_n \to f$ in $\mathrm{D}(\Delta)$.

ii) Our next claim is that (2.5) holds for all $f \in \mathrm{D}_{\mathrm{loc}}(\Delta^w) \cap L^\infty_{\mathrm{loc}}(\mathfrak{m}^w)$ with $\Delta^w f \in (\mathbb{V}^w)^\infty_{\mathrm{loc}}$ and all compactly supported, nonnegative $\varphi \in (\mathbb{V}^w)^\infty$ with $\Gamma^w(\varphi) \in L^\infty(\mathfrak{m}^w)$. Indeed, the conditions on f and on φ will not depend on w as long as $w \in \mathbb{V}^\infty_{\mathrm{loc}}$ which is the case by assumption. This is obvious in the case of the conditions on φ. For the conditions on f, note that $\Delta^w = e^{-2w} \Delta$ and $\Gamma^w(\Delta^w) \le 2e^{-4w} \big[\Gamma(\Delta f) + 4(\Delta f)^2\, \Gamma(w)\big]$.

iii) Taking into account the assumptions on $\mathbf{\Delta} w$ and on K', according to Lemma 2.7 together with Remark 2.8 the assertion of the second claim already proves $\mathrm{BE}(K', N')$.

iv) The gradient estimate is a standard consequence of $\mathrm{BE}(K', N')$, see [9]. ☐

Remark 2.18. Let $e^{2w} = \rho$ and $N' = \infty$ in Theorem 2.17. Then condition (2.6) becomes

$$K' \le K \rho^{-1} + \frac{1}{2} \Delta \rho^{-1} - N \Gamma(\rho^{-\frac{1}{2}}) = \rho^{-\frac{N}{2}} \Big(K - \frac{1}{N-2} \Delta \Big) \rho^{\frac{N}{2} - 1}.$$

Furthermore, when $N = 2$, the condition is $K' \rho \le K - \frac{1}{2} \Delta \ln \rho$.

3 Time change and the Lott-Sturm-Villani condition

In this section, we will study synthetic lower Ricci bounds under time change in the setting of metric measure spaces. More precisely, we will derive the transformation formula for the curvature-dimension condition of Lott-Sturm-Villani under time change.

3.1 Metric measure spaces and time change

Assumption 3.1. In this section we will assume that the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ fulfils the following conditions:

i) (X, d) is a complete and separable geodesic space;

ii) $\mathfrak{m}$ is a d-Borel measure and $\operatorname{supp}\mathfrak{m} = X$;

iii) $(X, \mathrm{d}, \mathfrak{m})$ satisfies the Riemannian curvature-dimension condition $\mathrm{RCD}(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty]$.

Given such a metric measure space $(X, \mathrm{d}, \mathfrak{m})$, the energy is defined on $L^2(X, \mathfrak{m})$ by

$$\begin{aligned}\mathcal{E}(f) \;&:=\; \inf\Big\{ \liminf_{n\to\infty} \int_X \operatorname{lip}(f_n)^2 \mathrm{d}\mathfrak{m} : f_n \in \operatorname{Lip}_b(X),\ f_n \to f \text{ in } L^2(X, \mathfrak{m}) \Big\} \\ &=\; \int_X |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}\end{aligned}$$

where $\operatorname{lip}(f)(x) := \limsup_{y\to x} |f(x) - f(y)|/\mathrm{d}(x, y)$ denotes the *local Lipschitz slope* and $|\mathrm{D}f|(x)$ denotes the minimal weak upper gradient at $x \in X$. We refer to [1] for details. As a part of the definition of RCD condition, $\mathcal{E}(\cdot)$ is a quadratic form. By polarization, this defines a quasi-regular, strongly local, conservative Dirichlet form admitting a carré du champ $\Gamma(f) := |\mathrm{D}f|^2$. We use the notations $W^{1,2}(X, \mathrm{d}, \mathfrak{m}) = \mathbb{V} = \mathrm{D}(\mathcal{E})$ and $S^2(X, \mathrm{d}, \mathfrak{m}) = \mathbb{V}^e$.

Definition 3.2. Given $w \in L^2_{\mathrm{loc}}(X, \mathfrak{m})$, the time-changed metric measure space is defined as $(X, \mathrm{d}^w, \mathfrak{m}^w)$ where $\mathfrak{m}^w := e^{2w}\mathfrak{m}$ and d^w is given by

$$\mathrm{d}^w(x, y) := \sup\big\{\phi(x) - \phi(y) : \phi \in \mathbb{V}_{\mathrm{loc}} \cap C(X), |\mathrm{D}\phi| \le e^w \ \mathfrak{m}\text{-a.e. in } X\big\} \qquad (3.1)$$

for any $x, y \in X$.

Remark 3.3. There are various alternative definitions for the distance function under time change. The first of them is

$$\mathrm{d}_w(x, y) := \sup\big\{\phi(x) - \phi(y) : \phi \in \operatorname{Lip}_{\mathrm{loc}}(X), \operatorname{lip}(\phi) \le e^w \ \mathfrak{m}\text{-a.e. in } X\big\}. \qquad (3.2)$$

Since $\operatorname{Lip}_{\mathrm{loc}}(X) \subset \mathbb{V}_{\mathrm{loc}} \cap C(X)$ and $|\mathrm{D}\phi| \le \operatorname{lip}(\phi)$ for $\phi \in \operatorname{Lip}_{\mathrm{loc}}(X)$, obviously $\mathrm{d}_w \le \mathrm{d}^w$. It is easy to see that in both of these definitions, the class of functions under consideration can equivalently be restricted to those with compact supports. In other words, $\mathrm{d}^w(x, y) = \sup\big\{\phi(x)-\phi(y) : \phi \in \mathbb{V}\cap C_c(X), |\mathrm{D}\phi| \le e^w \ \mathfrak{m}\text{-a.e. in } X\big\}$ and $\mathrm{d}_w(x, y) = \sup\big\{\phi(x) - \phi(y) : \phi \in \operatorname{Lip}_c(X), \operatorname{lip}(\phi) \le e^w \ \mathfrak{m}\text{-a.e. in } X\big\}$.

Moreover, we consider the metric $e^w \odot \mathrm{d}$ defined in a dual way by

$$\big(e^w \odot \mathrm{d}\big)(x,y) := \inf\Big\{ \int_0^1 e^w(\gamma_s)\,|\dot\gamma_s|\,\mathrm{d}s : \gamma \in \mathrm{AC}([0,1],X), \gamma_0 = x, \gamma_1 = y\Big\}. \quad (3.3)$$

Of particular interest is the metric $e^{\bar w}\odot\mathrm{d}$ with w replaced by its upper semicontinuous envelope $\bar w$ defined by

$$\bar w(x) := \limsup_{y\to x} w(y).$$

Lemma 3.4. *Assume that w is continuous a.e. on X. Then each of the metrics $\mathrm{d}^w, \mathrm{d}_w$, $e^w \odot \mathrm{d}$ and $e^{\bar w}\odot \mathrm{d}$ induces the same minimal weak upper gradient $|\mathrm{D}^w f| = e^{-w}|\mathrm{D}f|$ $\mathfrak{m}$-a.e. on X for each $f \in L^2_{\mathrm{loc}}(X)$. In particular,*

$$\Gamma^w = e^{-2w}\,\Gamma \quad \text{on } \mathbb{V}_{\mathrm{loc}} = \mathbb{V}^w_{\mathrm{loc}}$$

where here and henceforth Γ^w denotes the carré du champ operator induced by the metric measure space $(X, \mathrm{d}^w, \mathfrak{m}^w)$.

Proof. Assume that w is continuous at $x \in X$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|w(x) - w(y)| < \varepsilon$ for $y \in B_\delta(x)$. Hence, by using appropriate truncation arguments it is easy to see that for each $\mathrm{d}^* \in \big\{\mathrm{d}^w, \mathrm{d}_w, e^w \odot \mathrm{d}, e^{\bar w}\odot \mathrm{d}\big\}$ and all $y \in B_\delta(x)$

$$e^{w(z)-\varepsilon}\cdot \mathrm{d}(x,y) \quad\le\quad \mathrm{d}^*(x,y) \quad\le\quad e^{w(z)+\varepsilon}\cdot \mathrm{d}(x,y).$$

Hence, $\mathrm{lip}^*(f)(x) = e^{-w(x)}\,\mathrm{lip}(x)$ for the respective local Lipschitz constants associated with d^*.

To obtain the respective minimal weak upper gradient for $f \in L^2(X,\mathfrak{m})$ associated with d^*, one has to consider the relaxations of $\mathrm{lip}^*(f)$ w.r.t. the measure $\mathfrak{m}^w = e^{2w}\mathfrak{m}$. This, however, amounts to study the relaxations of the original $\mathrm{lip}(f)$ w.r.t. the measure $\mathfrak{m}$. Thus the claimed identify $\Gamma^w(f) = e^{-2w}\,\Gamma(f)$ $\mathfrak{m}$-a.e. on X follows. ☐

In the following lemma we show the coincidence of d^w and $e^{\bar w}\odot\mathrm{d}$, see [?] for related results.

Lemma 3.5. *Assume that w is continuous a.e. on X. Then*

$$\mathrm{d}^w = e^{\bar w}\odot \mathrm{d}.$$

In particular, d^w is a geodesic metric.

Proof. i) Let us first prove that d^w is a geodesic metric. Since X is locally compact w.r.t. the metric d and since the metrics d^w and d are locally equivalent, the space X is also locally compact w.r.t. the metric d^w. Therefore, it suffices to prove that d^w is a length metric. Assume this is not the case. Then there exist points $x \ne y$ with $\mathrm{d}^w(x,y) < 2r$ and $B^w_r(x)\cap B^w_r(y) = \emptyset$. Put

$$f = \mathrm{d}^w(.,X\setminus B_r(x)) - \mathrm{d}^w(.,X\setminus B_r(y)).$$

It is easy to verify that $\Gamma^w(f) \le 1$ and obviously f is continuous. Hence, by the very definition of d^w

$$\mathrm{d}^w(x, y) \ge f(x) - f(y) = 2r$$

which is in contradiction to our initial assumption.

ii) Now let us consider the particular case where w is continuous on all of X. Then $\mathrm{d}^w = e^w \odot \mathrm{d}$. Indeed, both metrics are geodesic metrics on X and coincide up to multiplicative pre-factors $e^{\pm\varepsilon}$ on suitable neighborhoods $B_\delta(x)$ of each point $z \in X$.

iii) To deal with the general case, let us choose a decreasing sequence of continuous functions w_n with $w_n \downarrow \bar{w}$ as $n \to \infty$. Then $\mathrm{d}^{w_n} = e^{w_n} \odot \mathrm{d}$ for each n by the preceding case ii) and thus by monotonicity for all x, y

$$\mathrm{d}^w(x, y) \le \inf_n \mathrm{d}^{w_n}(x, y) = \inf_n (e^{w_n} \odot \mathrm{d})(x, y) = (e^{\bar{w}} \odot \mathrm{d})(x, y).$$

iv) To prove the reverse estimate, for given $x \in X$ observe that $f = (e^{\bar{w}} \odot \mathrm{d})(x, .)$ is continuous and obviously $\mathrm{lip}^w(f)(y) \le 1$ in each point y of continuity of w. Thus, in particular, $\Gamma^w(f) \le 1$ $\mathfrak{m}^w$-a.e. on X. This indeed implies that $\mathrm{d}^w(x, z) \ge |f(x) - f(z)| = (e^{\bar{w}} \odot \mathrm{d})(x, z)$ for each $z \in X$. □

Lemma 3.6. *Assume that w is continuous a.e. on X. Then the metric measure space $(X, \mathrm{d}^w, \mathfrak{m}^w)$ has the Sobolev-to-Lipschitz property.*

Proof. Assume that $f \in \mathbb{V}_{\mathrm{loc}}$ is given with $\Gamma^w(f) \le 1$ $\mathfrak{m}^w$-a.e. on X. By truncation one can achieve on each bounded set B that $f = f_B$ a.e. on B for some f_B with bounded support and with $\Gamma^w(f_B) \le 1$ $\mathfrak{m}^w$-a.e. on X. Since $w \in L^\infty_{\mathrm{loc}}$, moreover, $\Gamma(f_B) \le C$ $\mathfrak{m}$-a.e. on X. By the Sobolev-to-Lipschitz property of the original metric measure space $(X, \mathrm{d}, \mathfrak{m})$ it follows that $f = \bar{f}_B$ a.e. on B for some $\bar{f}_B$ with $\mathrm{Lip}(\bar{f}_B) \le C$. In particular, $\bar{f}_B$ is continuous and $\Gamma^w(\bar{f}_B) \le 1$ $\mathfrak{m}^w$-a.e. on X. Hence,

$$\mathrm{d}^w(x, y) \ge |\bar{f}_B(x) - \bar{f}_B(y)|$$

for all $x, y \in X$ by the very definition of the metric d^w. In other words, $\bar{f}_B \in \mathrm{Lip}^w(X)$ with $\mathrm{Lip}^w(\bar{f}_B) \le 1$. Considering these constructions for an open covering of X by such sets B, it follows that there exists $\bar{f} \in \mathrm{Lip}^w(X)$ with $f = \bar{f}$ $\mathfrak{m}$-a.e. on X and $\mathrm{Lip}^w(\bar{f}) \le 1$. □

Finally we can prove the transformation formula for the $\mathrm{RCD}(K, N)$ condition under time change.

Theorem 3.7. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K, N)$ space and let $w \in \mathrm{D}_{\mathrm{loc}}(\mathbf{\Delta}) \cap L^\infty_{\mathrm{loc}}(X)$ be continuous $\mathfrak{m}$-a.e. with $\mathbf{\Delta} w = \mathbf{\Delta}_{sing} w + \Delta_{ac} w\, \mathfrak{m}$ and $\mathbf{\Delta}_{sing} w \le 0$. Then the time-changed metric measure space $(X, \mathrm{d}^w, \mathfrak{m}^w)$ satisfies the $\mathrm{RCD}(K', N')$ condition for any $N' \in (N, +\infty]$ and $K' \in \mathbb{R}$ such that $\mathfrak{m}$-a.e. on X*

$$K' \le e^{-2w} \Big[K - \frac{(N-2)(N'-2)}{N'-N} |\mathrm{D}w|^2 - \Delta_{ac} w \Big]. \tag{3.4}$$

A particular consequence of the Theorem is that the time-changed metric measure space $(X, \mathrm{d}^w, \mathfrak{m}^w)$ satisfies the squared exponential volume growth condition: $\exists C \in \mathbb{R}, z \in X$:

$$\mathfrak{m}^w(B^w_r(z)) \le C\, e^{Cr^2} \qquad (\forall r > 0). \tag{3.5}$$

Proof. From the work of [3, 9], we know that the curvature dimension condition $\mathrm{RCD}(K,N)$ implies the Bakry-Émery condition $\mathrm{BE}(K,N)$ for the Dirichlet form $\mathcal{E}$ on $L^2(X,\mathfrak{m})$ induced by the measure space $(X,\mathrm{d},\mathfrak{m})$. According to Theorem 2.17, this implies the Bakry-Émery condition $\mathrm{BE}(K',N')$ for the Dirichlet form $\mathcal{E}^w$ on $L^2(X,\mathfrak{m}^w)$. Due to Lemma 3.4, the latter indeed is the Dirichlet form induced by the metric measure space $(X,\mathrm{d}^w,\mathfrak{m}^w)$. Finally, again by [3, 9], $\mathrm{BE}(K',N')$ for the Dirichlet form induced by $(X,\mathrm{d}^w,\mathfrak{m}^w)$ will imply $\mathrm{RCD}^*(K',N')$ provided the volume-growth condition (3.6) is satisfied and the Sobolev-to-Lipschitz property holds. The latter was proven in Lemma 3.6. To deal with the former, we proceed in two steps.

i)) Let us first consider the case $w\in L^\infty(X)$. Then the volume-growth condition (3.6) for $(X,\mathrm{d}^w,\mathfrak{m}^w)$ obviously from that for $(X,\mathrm{d},\mathfrak{m})$ which in turn follows from the $\mathrm{RCD}(K,N)$ assumption.

ii) Now let general $w\in L^\infty_{\mathrm{loc}}(X)$ be given as well as K' and N' such that (3.4) is satisfied. Given $z\in X$, define $w_l = w\cdot\chi_\ell$ with suitable cut-off functions $(\chi_\ell)_{\ell\in\mathbb{N}}$ (cf. [4], Lemma 6.7) such that for all $\ell\in\mathbb{N}$

- $w_\ell = w$ on $B_\ell(z)$
- w_ℓ is bounded on X
- $e^{-2w_\ell}\Big[K-\frac{(N-2)(N'-2)}{N'-N}|\mathrm{D}w_\ell|^2-\Delta_{ac}w_\ell\Big]\ge K'-1.$

Then according to part i) of this proof, the metric measure space $(X,\mathrm{d}^{w_\ell},\mathfrak{m}^{w_\ell})$ satisfies $\mathrm{RCD}(K'-1,N')$. This in particular implies that there exists a constant C (which indeed can be chosen independent of ℓ) such that

$$\mathfrak{m}^{w_\ell}\big(B_r^{w_\ell}(z)\big)\le C\,e^{C\,r^2}$$

for all $r>0$. Since $\mathfrak{m}^{w_\ell}\big(B_r^{w_\ell}(z)\big)=\mathfrak{m}^{w}\big(B_r^{w}(z)\big)$ for all $r\le\ell$, this finally proves the requested volume growth condition. □

It might be of certain interest to analyze the validity of the volume growth condition (3.6) under time change without referring to curvature bounds.

Lemma 3.8. *Suppose there exist non-negative $p,q\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ with*

$$-q(\mathrm{d}(\cdot,x_0))\le w(.)\le p(\mathrm{d}(\cdot,x_0))\qquad \textit{on } X.$$

Then $(X,\mathrm{d}^w,\mathfrak{m}^w)$ satisfies the squared exponential volume growth condition: $\exists C\in\mathbb{R}, x_0\in X$:

$$\mathfrak{m}^w(B_r^w(x_0))\le C\,e^{Cr^2}\qquad(\forall r>0)\tag{3.6}$$

provided the function $f(r):=\int_0^r e^{-q(s)}ds$ satisfies

(i) $\liminf_{r\to\infty}\frac{1}{r}f(r)>0$ *and*

(ii) $\limsup_{r\to\infty}\frac{1}{r^2}p\big(f^{-1}(r)\big)<\infty$.

In particular, if q is bounded and $\overline{\lim}_{r\to\infty}\frac{p(r)}{r^2}<\infty$, then $(X,\mathrm{d}^w,\mathfrak{m}^w)$ satisfies the squared exponential volume growth condition.

Proof. From Lemma 3.5, we know

$$\mathrm{d}^w(x_0, y) \geq \int_0^{\mathrm{d}(x_0,y)} \exp\big(-q(r)\big)\,\mathrm{d}r = f(\mathrm{d}(x_0, y))$$

for any $y \in X$. Since f^{-1} is strictly increasing, this implies

$$B^w_R(x_0) \subset B_{f^{-1}(R)}(x_0), \qquad \forall\, R > 0.$$

Hence,

$$\mathfrak{m}^w\big(B^w_R(x_0)\big) \leq \exp\big(2p(f^{-1}(R))\big)\mathfrak{m}\big(B_{f^{-1}(R)}(x_0)\big).$$

Recall that the $\mathrm{RCD}(K, \infty)$ condition implies the squared exponential volume growth condition, so there exists $M, c > 0$ such that

$$\mathfrak{m}^w\big(B^w_R(x_0)\big) \leq M \exp\Big(2p(f^{-1}(R)) + c\big(f^{-1}(R)\big)^2\Big).$$

Note that (i) implies $\limsup_{r\to\infty} \frac{1}{r} f^{-1}(r) < \infty$. Hence, together with (ii), this implies the squared exponential volume growth condition for $(X, \mathrm{d}^w, \mathfrak{m}^w)$. □

3.2 Convexity transform

Firstly we introduce the notion of local ℓ-convexity in non-smooth setting. Such notion is derived from [15] by the second author and Lierl (see Definition 2.6 and Definition 2.9 therein).

Definition 3.9 (ℓ-convex functions, Definition 2.6 [15])**.** Given $\ell \in \mathbb{R}$, we say that a function V is ℓ-convex on a closed subset $Z \subset X$ if there exists a convex open covering $\cup_i X_i \supset Z$ such that each $V|_{\overline{X_i}} : \overline{X_i} \mapsto (-\infty, +\infty]$ is ℓ-geodesically convex, in the sense that for each $x_0, x_1 \in \overline{X_i}$, there exists a geodesic $\gamma : [0, 1] \mapsto X$ from x_0 to x_1 such that

$$V(\gamma(t)) \leq (1 - t)V(\gamma(0)) + tV(\gamma(1)) - \frac{\ell}{2} t(1 - t)|\dot{\gamma}_t|^2, \quad \forall t \in [0, 1].$$

Definition 3.10 (Locally ℓ-convex sets, Definition 2.9 [15])**.** Let $\Omega \subset X$ be an open subset and let $V := \mathrm{d}(., \Omega) - \mathrm{d}(., X \setminus \Omega)$ denote the signed distance from the boundary, in the sequel also briefly denoted by $\pm\mathrm{d}(., \partial\Omega)$.

We say that Ω is locally ℓ-convex if for each $\delta > 0$ there exists $r > 0$ such that V is $(\ell - \delta)$-convex on Ω^r_{-r} with $|\mathrm{D}V| \geq 1 - \delta$ where $\Omega^r_{-r} := \{-r < V < r\}$.

Remark 3.11. Assume that X is a smooth Riemannian manifold, and Ω is a bounded open subset of X with smooth boundary. It is proved in Proposition 2.10 [15], that the real-valued second fundamental form on $\partial\Omega$ is bounded from below by ℓ if and only if Ω is locally ℓ-convex.

Then we can convexify locally ℓ-convex sets using time change and the following convexification technique developed in [15] (see Theorem 2.17 therein).

Lemma 3.12 (Convexification Theorem)**.** *Let Ω be a locally ℓ-convex subset in X for some $\ell \leq 0$. Then Ω is locally geodesically convex in $(X, \mathrm{d}^{-\ell' V})$ for any $\ell' < \ell$.*

Next we recall some important results concerning L^1-optimal transport and measure decomposition. This theory has proven to be a powerful tool in studying the fine structure of metric measure spaces. We refer the readers to the lecture note [6] for an overview of this topic and the bibliography.

Lemma 3.13 (Localization for $\mathrm{RCD}(K,N)$ spaces, Theorem 3.8 and Theorem 5.1 [7])**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be an essentially non-branching metric measure space with* $\mathrm{supp}\,\mathfrak{m}=X$*, and satisfying* $\mathrm{RCD}(K,N)$ *condition for some $K\in\mathbb{R}$ and $N\in(1,\infty)$. Then for any 1-Lipschitz function u on X and the transport set T_u associated with u (up to $\mathfrak{m}$-measure zero set, T_u coincides with $\{|\nabla u|=1\}$), there is a disjoint family of unparameterized geodesics $\{X_q\}_{q\in\mathfrak{Q}}$ such that*

$$\mathfrak{m}(\mathsf{T}_u\setminus\cup X_q)=0, \tag{3.7}$$

and

$$\mathfrak{m}|_{\mathsf{T}_u}=\int_{\mathfrak{Q}}\mathfrak{m}_q\,\mathrm{d}\mathfrak{q}(q),\quad \mathfrak{q}(\mathfrak{Q})=1\quad\text{and}\quad \mathfrak{m}_q(X_q)=1\quad \mathfrak{q}-a.e.\ q\in\mathfrak{Q}. \tag{3.8}$$

Furthermore, for $\mathfrak{q}$-a.e. $q\in\mathfrak{Q}$, $\mathfrak{m}_q$ is a Radon measure with $\mathfrak{m}_q\ll\mathcal{H}^1|_{X_q}$ and $(X_q,\mathrm{d},\mathfrak{m}_q)$ satisfies $\mathrm{RCD}(K,N)$*. In particular, $\mathfrak{m}_q=h_q\mathcal{H}^1|_{X_q}$ for some* $\mathrm{CD}(K,N)$ *probability density h_q.*

Lemma 3.14. *Let Y be a domain in X with $\mathfrak{m}(\partial Y)=0$, and $V:=\mathrm{d}(.,Y)-\mathrm{d}(.,X\setminus Y)$ be the signed distance from the boundary. Then the transport set T_V associated with V has full measure in X. There is a disjoint family of unparameterized geodesics $\{X_q\}_{q\in\mathfrak{Q}}$ satisfying (3.7) and (3.8) in Theorem 3.13, and a constant $r_0>0$ such that*

$$V(a_q)\geq 0,\quad V(b_q)\leq -r_0\quad \mathfrak{q}-a.e.\ q\in\mathfrak{Q}$$

where $a_q=a_q(X_q),b_q=b_q(X_q)$ are the end points of X_q.

Proof. Firstly, recall that $\mathrm{RCD}(K,N)$ condition yields local compactness, so for any $x\in X\setminus\partial Y$, there is $z\in\partial Y$ such that $\mathrm{d}(x,z)=\mathrm{d}(x,\partial Y)$ and thus $x\in\mathsf{T}_V$. So $\mathfrak{m}(X\setminus\mathsf{T}_V)=0$.

Secondly, by Definition 3.10, V is semi-convex on $Y^{r_0}_{-r_0}$ for some $r_0>0$. By the main theorem of [22], for each $x_0\in Y^{r_0}_{-r_0}$ there exists a unique gradient flow for V starting in x_0. In particular, there is a maximal transport (geodesic) line $\gamma\subset\mathsf{T}_V$ satisfying $V(\gamma_1)-V(\gamma_0)=\mathrm{d}(\gamma_0,\gamma_1)$, $V(\gamma_1)\geq V(x_0)$ and $V(\gamma_0)\leq -r_0$.

By Theorem 3.13 there is a disjoint family of unparameterized geodesics $\{X_q\}_{q\in\mathfrak{Q}}$ such that $\mathfrak{m}(\mathsf{T}_V\setminus\cup X_q)=0$. In addition, $X_q\cap\{V\leq -r_0\}\neq\emptyset$ and $X_q\cap\{V\geq 0\}\neq\emptyset$ for any $q\in\mathfrak{Q}$. Therefore $\mathfrak{m}(X\setminus\cup X_q)=0$, $V(a_q)\geq 0$ and $V(b_q)\leq -r_0$. $\square$

Proposition 3.15 (Convexification)**.** *Let Ω be a locally ℓ-convex domain in (X,d) for some $\ell\leq 0$, and $\mathfrak{m}(\partial\Omega)=0$. Then for any $\ell'<\ell$, there exist $r_0>0$ and a Lipschitz function w such that Ω is locally geodesically convex in (X,d^w) and $w\in\mathrm{D}(\mathbf{\Delta},X\setminus\partial\Omega)$ with*

$$\mathbf{\Delta}w|_{X\setminus\partial\Omega}\leq -\ell'\Big(\cot_{K,N}(r_0/4)+\frac{2}{r_0}\Big)\mathfrak{m}|_{X\setminus\partial\Omega}.$$

where the function $\cot_{K,N} : [0, +\infty) \mapsto [0, +\infty)$ *is defined by*

$$\cot_{K,N}(x) := \begin{cases} \sqrt{K(N-1)}\cot(\sqrt{\frac{K}{N-1}}x), & \text{if } K > 0, \\ (N-1)/x, & \text{if } K = 0, \\ \sqrt{-K(N-1)}\coth(\sqrt{\frac{-K}{N-1}}x), & \text{if } K < 0. \end{cases}$$

Proof. Let $V := \pm\mathrm{d}(\cdot, \partial Y)$ be the signed distance from the boundary and $r_0 > 0$ be the constant in Proposition 3.14. Given $\ell' < \ell \le 0$, we can find a smooth cut-off function $\phi : \mathbb{R} \mapsto [0, r_0]$ satisfying

$$\phi(t) := \begin{cases} t, & \text{if } t \in [\frac{1}{4}\ell' r_0, -\frac{1}{4}\ell' r_0] \\ -\frac{1}{2}\ell' r_0, & \text{if } t \in [-\frac{3}{4}\ell' r_0, +\infty) \\ \frac{1}{2}\ell' r_0, & \text{if } t \in (-\infty, \frac{3}{4}\ell' r_0] \end{cases}$$

with $0 \le \phi' \le 1$, $|\phi''| \le -\frac{2}{\ell' r_0}$ on $\mathbb{R}$. Then we define $w := \phi(-\ell' V)$. By Convexification Theorem (c.f. Theorem 2.17 [15]) we know Ω is locally geodesically convex in (X, d^w).

By chain rule (c.f. Proposition 4.11 [11]) and Corollary 4.16 [8] we have $w \in \mathrm{D}(\mathbf{\Delta}, X \setminus \partial\Omega)$, and

$$\mathbf{\Delta} w|_{X\setminus\partial\Omega} = -\ell'\phi'(-\ell' V)\mathbf{\Delta} V|_{X\setminus\partial\Omega} + (\ell')^2\phi''(-\ell' V)|\mathrm{D}V|^2\,\mathfrak{m}|_{X\setminus\partial\Omega}. \tag{3.9}$$

In addition, by Corollary 4.16 [8] and the fact $\phi'' \le -\frac{2}{\ell' r_0}$, we obtain

$$\begin{aligned} &\mathbf{\Delta} w|_{X\setminus\partial\Omega} \\ \le\; & -\ell'\phi'(-\ell' V)\mathbf{\Delta} V|_{X\setminus\partial\Omega} - \frac{2\ell'}{r_0}|\mathrm{D}V|^2\,\mathfrak{m}|_{X\setminus\partial\Omega} \\ \le\; & -\ell'\phi'(-\ell' V)\Big(\cot_{K,N}(\mathrm{d}(x, b_q))\mathfrak{m}|_{X\setminus\partial\Omega} + \int_{\mathfrak{Q}} h_q \delta_{b_q}\,\mathrm{d}\mathfrak{q}(q)\Big) - \frac{2\ell'}{r_0}\mathfrak{m}|_{X\setminus\partial\Omega}. \end{aligned}$$

Furthermore, we know $\phi'(-\ell' V) = 0$ on $\{V \le -\frac{3}{4}r_0\} \cup \{V \ge \frac{3}{4}r_0\}$. So from Proposition 3.14 we can see that

$$\phi'(-\ell' V)\int_{\mathfrak{Q}} h_q \delta_{b_q} = 0.$$

Combining with the monotonicity of $\cot_{K,N}$ we obtain

$$\mathbf{\Delta} w|_{X\setminus\partial\Omega} \le -\ell'\Big(\cot_{K,N}(r_0/4) + \frac{2}{r_0}\Big)\mathfrak{m}|_{X\setminus\partial\Omega}$$

which is the thesis. □

Combining Lemma 3.7 and Proposition 3.15 we can prove the main theorem of this section. Recall that the Minkowski content of a set $Z \subset X$ is defined by

$$\mathfrak{m}^+(Z) := \liminf_{\epsilon\to 0} \frac{\mathfrak{m}(Z^\epsilon) - \mathfrak{m}(Z)}{\epsilon}$$

where $Z^\epsilon \subset X$ is the ϵ-neighbourhood of Z defined by $Z^\epsilon := \{x : \mathrm{d}(x, Z) < \epsilon\}$.

Theorem 3.16. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K, N)$ space and Ω be a bounded ℓ-convex domain in (X, d) with $\mathfrak{m}(\partial\Omega) = 0$ and $\mathfrak{m}^+(\partial\Omega) < \infty$. Then for any $N' \in (N, +\infty]$, there exists a Lipschitz function w such that $(\overline{\Omega}, \mathrm{d}^w, \mathfrak{m}^w)$ is a $\mathrm{RCD}(K', N')$ space for some $K' \in \mathbb{R}$.*

Proof. Let w be the reference function obtained in Proposition 3.15. Denote by μ the trivial extension of $\mathbf{\Delta} w|_{X \setminus \partial\Omega}$ on whole X. To apply Lemma 3.7 and Proposition 3.15, it suffices to show that $w \in \mathrm{D}(\mathbf{\Delta})$ and $\mathbf{\Delta} w \le \mu$.

Given an arbitrary non-negative Lipschitz function $\varphi \in \mathrm{Lip}(X, \mathrm{d})$ with bounded support. For any $\epsilon > 0$, there exists a Lipschitz function $\phi_\epsilon \in \mathrm{Lip}(\mathbb{R})$ satisfying

$$\phi_\epsilon(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{\epsilon}{2}] \\ \frac{2}{\epsilon}(t - \frac{\epsilon}{2}), & \text{if } t \in [\frac{\epsilon}{2}, \epsilon] \\ 1, & \text{if } t \in [\epsilon, +\infty) \end{cases}$$

Define $\bar{\varphi}_\epsilon := \phi_\epsilon(\mathrm{d}(\cdot, \partial\Omega))\varphi$. By Leibniz rule and chain rule we know $\bar{\varphi}_\epsilon \in \mathrm{Lip}(X, \mathrm{d})$, and $\operatorname{supp} \bar{\varphi}_\epsilon \subset X \setminus \partial\Omega$. Therefore by Lemma 3.15 and monotone convergence theorem we get

$$\begin{aligned} \int \varphi \, \mathrm{d}\mu &= \lim_{\epsilon \to 0} \int \bar{\varphi}_\epsilon \, \mathrm{d}\mathbf{\Delta} w|_{X \setminus \partial\Omega} \\ &= -\lim_{\epsilon \to 0} \int_{X \setminus \partial\Omega} \Gamma(\bar{\varphi}_\epsilon, w) \, \mathrm{d}\mathfrak{m} \\ &= -\lim_{\epsilon \to 0} \int \phi_\epsilon(\mathrm{d}(\cdot, \partial\Omega)) \Gamma(\varphi, w) \, \mathrm{d}\mathfrak{m} - \lim_{\epsilon \to 0} \int \varphi \Gamma(\phi_\epsilon(\mathrm{d}(\cdot, \partial\Omega)), w) \, \mathrm{d}\mathfrak{m} \\ &= -\int \Gamma(\varphi, w) \, \mathrm{d}\mathfrak{m} - \lim_{\epsilon \to 0} \int \varphi \Gamma(\phi_\epsilon(\mathrm{d}(\cdot, \partial\Omega)), w) \, \mathrm{d}\mathfrak{m}. \end{aligned}$$

By Lemma 3.13 we have a measure decomposition $\mathrm{d}\mathfrak{m} = \mathrm{d}\mathfrak{m}_q \, \mathrm{d}\mathfrak{q}(q)$ associated with the signed distance function $\pm \mathrm{d}(\cdot, \partial\Omega)$. Thus

$$\begin{aligned} &\lim_{\epsilon \to 0} \int \varphi \Gamma(\phi_\epsilon(\mathrm{d}(\cdot, \partial\Omega)), w) \, \mathrm{d}\mathfrak{m} \\ = \;& \lim_{\epsilon \to 0} \int_{\Omega_{-\epsilon}^{-\epsilon/2}} \varphi \Gamma(\phi_\epsilon(\mathrm{d}(\cdot, \partial\Omega)), w) \, \mathrm{d}\mathfrak{m} + \lim_{\epsilon \to 0} \int_{\Omega_{\epsilon/2}^{\epsilon}} \varphi \Gamma(\phi_\epsilon(\mathrm{d}(\cdot, \partial\Omega)), w) \, \mathrm{d}\mathfrak{m} \\ = \;& \lim_{\epsilon \to 0} \frac{2\ell'}{\epsilon} \left(\int_{\Omega_{-\epsilon}^{-\epsilon/2}} \varphi \, \mathrm{d}\mathfrak{m} - \int_{\Omega_{\epsilon/2}^{\epsilon}} \varphi \, \mathrm{d}\mathfrak{m} \right) \\ = \;& \lim_{\epsilon \to 0} \frac{2\ell'}{\epsilon} \int_{\mathfrak{Q}} \left(\int_{\Omega_{-\epsilon}^{-\epsilon/2} \cap X_q} \varphi \, \mathrm{d}\mathfrak{m}_q - \int_{\Omega_{\epsilon/2}^{\epsilon} \cap X_q} \varphi \, \mathrm{d}\mathfrak{m}_q \right) \mathrm{d}\mathfrak{q}(q). \end{aligned}$$

Notice that

$$\mathfrak{m}^+(\partial\Omega) = \liminf_{\epsilon \to 0} \frac{\mathfrak{m}\big((\partial\Omega)^\epsilon\big)}{\epsilon} = \liminf_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{\Omega_{-\epsilon}^{0}} 1 \, \mathrm{d}\mathfrak{m} + \int_{\Omega_0^{\epsilon}} 1 \, \mathrm{d}\mathfrak{m} \right).$$

Therefore we obtain

$$
\begin{aligned}
&\left|\int \Gamma(\varphi, w)\,\mathrm{d}\mathfrak{m}\right| \\
\leq\ & \left|\int \varphi\,\mathrm{d}\mu\right| + \left|\lim_{\epsilon\to 0}\int \varphi\Gamma(\phi_\epsilon(\mathrm{d}(\cdot,\partial\Omega)), w)\,\mathrm{d}\mathfrak{m}\right| \\
\leq\ & \max|\varphi|\left\{|\mu|(\operatorname{supp}\varphi) + \lim_{\epsilon\to 0}\frac{2|\ell'|}{\epsilon}\left(\int_{\Omega^0_{-\epsilon}} 1\,\mathrm{d}\mathfrak{m} + \int_{\Omega^\epsilon_0} 1\,\mathrm{d}\mathfrak{m}\right)\right\} \\
\leq\ & \max|\varphi|\left\{|\mu|(\operatorname{supp}\varphi) - 2\ell'\mathfrak{m}^+(\partial\Omega)\right\}.
\end{aligned}
$$

By Riesz-Markov-Kakutani Representation theorem we know $w \in \mathrm{D}(\boldsymbol{\Delta})$.

Since $\mathfrak{m}_q = h_q\mathcal{H}^1|_{X_q}$ for some $\mathrm{CD}(K,N)$ probability density h_q, we know h_q and $(\ln h_q)'$ are bounded. So for any X_q such that $\Omega^{-\epsilon/2}_{-\epsilon}\cap X_q \neq \emptyset$ and $\Omega^\epsilon_{\epsilon/2}\cap X_q\neq\emptyset$ for ϵ small enough, we have

$$
\left|\int_{\Omega^{-\epsilon/2}_{-\epsilon}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q - \int_{\Omega^\epsilon_{\epsilon/2}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q\right| \ \leq\ \mathrm{Lip}(\varphi h_q)\frac{3\epsilon}{2}\mathcal{H}^1(\Omega^\epsilon_{-\epsilon}\cap X_q) = O(\epsilon^2).
$$

Hence by Lemma 3.14 and Fatou's lemma, we obtain

$$
\begin{aligned}
& \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathfrak{Q}}\Big(\int_{\Omega^{-\epsilon/2}_{-\epsilon}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q - \int_{\Omega^\epsilon_{\epsilon/2}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q\Big)\,\mathrm{d}\mathfrak{q}(q) \\
\geq\ & \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{q\in\mathfrak{Q},\Omega^{r_0}_0\cap X_q\neq\emptyset}\Big(\int_{\Omega^{-\epsilon/2}_{-\epsilon}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q - \int_{\Omega^\epsilon_{\epsilon/2}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q\Big)\,\mathrm{d}\mathfrak{q}(q) \\
\geq\ & 0.
\end{aligned}
$$

In conclusion, we obtain

$$
\begin{aligned}
&\int\varphi\,\mathrm{d}\boldsymbol{\Delta}w = -\int\Gamma(\varphi,w)\,\mathrm{d}\mathfrak{m} \\
=\ & \int\varphi\,\mathrm{d}\mu + \lim_{\epsilon\to 0}\frac{2\ell'}{\epsilon}\int_{\mathfrak{Q}}\Big(\int_{\Omega^{-\epsilon/2}_{-\epsilon}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q - \int_{\Omega^\epsilon_{\epsilon/2}\cap X_q}\varphi\,\mathrm{d}\mathfrak{m}_q\Big)\,\mathrm{d}\mathfrak{q}(q) \leq \int\varphi\,\mathrm{d}\mu.
\end{aligned}
$$

Therefore $\boldsymbol{\Delta}w\leq\mu$, by Lemma 3.15 we know $\boldsymbol{\Delta}_{sing}w\leq 0$ and $\big(\Delta_{ac}w\big)^+\in L^\infty$. Then by Proposition 3.7 we know $(\overline{\Omega},\mathrm{d}^w,\mathfrak{m}^w)$ is a $\mathrm{RCD}(K',N')$ space. □

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