RESEARCH STATEMENT
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The primary areas of my research include Metric Geometry, Geometric Measure Theory and Geometric Analysis. My research has recently been focused on understanding the structure and geometry of smooth non-Riemannian (e.g. Finsler), non-smooth metric-measure spaces, as well as the relation between the topology and geometry of these spaces under various curvature or non-collapsing conditions.

This research statement is organized as follows. In section 1, I will describe my PhD dissertation work regarding smooth convergence of Riemannian metrics away from singular sets, its possible application to degeneration of Calabi-Yau manifolds, and more generally, degeneration of Kähler-Einstein manifolds. Section 2 is devoted to my ongoing research regarding flows of non-smooth spaces and of Finsler structures that generalize the smooth Ricci flow (in different senses). In section 3, I will briefly mention my recent work and work in progress regarding the topological implications of weak lower Ricci curvature bounds on metric spaces and of non-negative flag curvature in Finsler manifolds.

1. Smooth Convergence Away from Singular Sets

In the past few decades, understanding the structure of limit spaces of sequences of Riemannian manifolds with various kinds of curvature bounds has been a central theme in geometric analysis. There is a long list of contributors to the field but the corner stones were laid by the pioneering work of Cheeger and Gromoll in the 70’s, which was later followed by deeper results of Cheeger, Gromoll, Colding, Anderson and others.

My thesis research was to understand the relation between the Gromov-Hausdorff limit of a sequence of Riemannian metrics \( \{g_i\} \) on a manifold \( M \) that are converging away from a singular set, \( S \subset M \).

Since the convergence is required locally (i.e. on compact sets off the singular set), one can not expect the global geometry of the sequence to be well controlled under this notion of convergence. This can be caused by rather uncontrolled behavior of metrics near the singular sets (for example, the metrics might collapse or oscillate wildly near the singular set). Therefore, it is natural to impose some conditions on the metrics in order to tame the sequence to some extent. One may ask under what conditions the metric completion of the smooth limit away from the singular set coincides with the Gromov-Hausdorff (GH) or the Sormani-Wenger Intrinsic Flat (SWIF) limits of the sequence \( M_i = (M^n, g_i) \). Examples of cases in which these limits are different have been constructed and clearly explained in Lakzian-Sormani [34] and Lakzian [29].

This problem has been studied by many geometers and in particular has proven to be a key piece in the convergence theory of Kähler-Einstein and Calabi-Yau manifolds and orbifolds [3, 9, 19, 25, 48, 49, 56, 59].

Roughly speaking, the coincidence of the Gromov-Hausdorff limit and smooth limit (off the singular set) of resolutions and smoothings of Calabi-Yau varieties that are degenerating to the same Calabi-Yau variety, is the essential part in the proof of Candelas-de la Ossa’s conjecture by Rong-Zhang [46] [47] and Song [52], as well as, in the recent ground breaking results of Donaldson-Sun [18], Chen-Donaldson-Sun [14] [12] [13] [15] and Tian [57] in which they prove Tian’s conjecture.

1.1. General Theory of Smooth Convergence off Singular Sets. In Lakzian-Sormani [34], we provide conditions under which the above limits coincide provided that the singular set is a submanifold (sub-variety) of codimension at least 2.

Unlike other analytic approaches mentioned above, our methods are purely metric-geometric in nature and therefore stronger and consequently, can be applied to a wider class of converging manifolds. Our approach was to relate both the Gromov-Hausdorff and smooth limits with singular sets to Sormani-Wenger’s intrinsic flat convergence (which is weaker than smooth convergence and in some sense stronger than the Gromov-Hausdorff convergence).

In our approach, the restriction on the (size of) singular set comes from the well-embeddness of our sequence. Well-embeddedness controls the discrepancy between intrinsic and restricted distances:

**Definition 1.1** (well-embeddedness). Given an open subset, \( U \subset M \), a connected precompact exhaustion \( \{W_j\} \), of \( U \) is called uniformly well embedded if there exists a \( \lambda_0 \geq 0 \) such that

\[
\limsup_{j \to \infty} \limsup_{k \to \infty} \limsup_{i \to \infty} \lambda_{i,j,k} \leq \lambda_0, \tag{1.1}
\]
and,
\begin{equation}
\limsup_{k \to \infty} \lambda_{i,j,k} = \lambda_{i,j} \text{ where } \limsup_{i \to \infty} \lambda_{i,j} = \lambda_j \text{ and } \lim_{j \to \infty} \lambda_j = 0,
\end{equation}
where,
\begin{equation}
\lambda_{i,j,k} = \sup_{x,y \in W_j} |d(W_j,g_i)(x,y) - d(M,g_j)(x,y)|.
\end{equation}

The distortion, $\lambda_{i,j,k}$, is an important quantity to study in these types of convergence questions. Under stronger conditions, one can control $\lambda_{i,j,k}$ using analytic methods, such as, by applying Nash-Moser iteration to the solutions of a Monge-Ampère equation, as is done in [46].

In Lakzian-Sormani [34], we show that if $S$ is a sub-manifold of co-dimension 2 then, any exhaustion is uniformly well-embedded. Our first main result is the following.

**Theorem 1.2.** Suppose $M$ is compact, oriented and the singular set $S$ is closed. Assume there exists a uniformly well embedded exhaustion, $W_j$, of the regular set $M \setminus S$ such that $g_i$ converge smoothly to $g_\infty$ on each $W_j$ with
\begin{equation}
\text{diam}_{M_i}(W_j) \leq D_0 \quad \forall i \geq j, \text{ and } \text{Vol}_{g_i}(\partial W_j) \leq A_0 \quad \forall i,j
\end{equation}
and
\begin{equation}
\text{Vol}_{g_i}(M \setminus W_j) \leq V_j \quad \lim_{j \to \infty} V_j = 0.
\end{equation}
Then the sequence converges in the SWIF sense, i.e.
\begin{equation}
\lim_{j \to \infty} d_{\mathcal{F}}(M_j',N') = 0
\end{equation}
where $N'$ is the settled completion of $(M \setminus S,g_\infty)$, and that the settled completion is the collection of points in the metric completion that have positive density. It does not include cusp points but includes cone points.

Recall that Gromov’s compactness theorem states that a sequence $(M_i,g_i)$ with uniform lower Ricci curvature bounds and upper diameter bounds has a subsequence that converges in the Gromov-Hausdorff sense. By adding a Ricci curvature condition to our theorem, we obtain the following theorem concerning the Gromov-Hausdorff limits:

**Theorem 1.3.** Suppose $\text{diam}(M_i) \leq D_0$ and $\text{Ricci}_{g_i} \geq (n-1)H_{g_i} \quad (H \in \mathbb{R})$ and suppose the singular set $S$ is a closed sub-manifold of co-dimension 2. If there exists an exhaustion, $W_j$, of $M \setminus S$ (as in Theorem 1.2), then
\begin{equation}
\lim_{j \to \infty} d_{\mathcal{GH}}(M_j,N) = 0,
\end{equation}
where $N$ is the metric completion of $(M \setminus S,g_\infty)$.

While prior results demanded a diameter bound on the singular set, our stronger results only requires volume bounds on neighborhoods of this set. This is a consequence of the fact that the SWIF distance is measured with volumes and the fact that in this setting, we can show that the SWIF and GH limits agree.

Greene-Petersen [24] prove that if a sequence of Riemannian manifolds $(M_i,g_i)$ have $\text{Vol}(M_i) \leq V$ and if there exists a uniform contractibility function $\rho : [0,r_0] \to [0,\infty)$ for all the $M_i$, then, a subsequence is converging in the Gromov-Hausdorff sense. Using this, in Lakzian-Sormani [34], we also prove that one can replace the Ricci curvature bound condition in above theorem with existence of a uniform linear contractibility function.

**General Singular Sets.** In Lakzian [29], we build upon Lakzian-Sormani [34] and prove an easily tested criterion for uniform well-embeddedness. Using techniques from Geometric Measure Theory along with a variational argument, I prove that the Hausdorff measure condition $H^{n-1}(S) = 0$ is sufficient for the exhaustion, $W_j$, to be uniformly well-embedded and no regularity assumption on the singular set $S$ is needed. This implies the following much stronger result:

**Theorem 1.4.** Let $M$ be as before and suppose the singular set $S$ satisfies $H^{n-1}(S) = 0$. Suppose there exists an exhaustion, $W_j$, of $M \setminus S$ as before; Then, the sequence converges in the SWIF sense, i.e.
\begin{equation}
\lim_{i \to \infty} d_{\mathcal{F}}(M_i',N') = 0,
\end{equation}
where $M_i'$ and $N'$ are the settled completions of $(M,g_i)$ and $(M \setminus S,g_\infty)$ respectively.

In Lakzian [29], we also show that assuming uniform lower Ricci curvature bounds, we have
Theorem 1.5. Assume \((M, g_i)\) satisfies the hypotheses of Theorem 1.4 and furthermore, assume \(\text{Ricci}_{g_i} \geq (n - 1)H_{g_i}\) then
\[
\lim_{j \to \infty} d_{GH}(M_j, N) = 0,
\]
where \(N\) is the metric completion of \((M \setminus S, g_\infty)\).

We have constructed examples with \(\text{Ricci} \geq 0\) that demonstrate the necessity of each condition.

As a separate result in Lakzian [29], we also prove that one can replace the Hausdorff measure condition \(H^{n-1}(S) = 0\) on the singular set with a diameter bound on the boundaries of the exhaustion:

Theorem 1.6. Let \(M_i = (M, g_i)\) be as before with a general singular set \(S\). Suppose each connected component of \(M \setminus W_j\) has a connected boundary, and
\[
\limsup_{i \to \infty} \left\{ \sum_\beta \text{diam}(\Omega^\beta_j : \Omega^\beta_j \text{ connected component of } \partial W_j) \right\} \leq B_j,
\]
where \(\lim_{j \to \infty} B_j = 0\). Then
\[
\lim_{j \to \infty} d_F(M_j', N') = 0,
\]
where \(N'\) is the settled completion of \((M \setminus S, g_\infty)\).

Note that we do not require \(\text{diam}(M \setminus W_j) \to 0\) and that the example of sphere with many splines (see Lakzian-Sormani [34]) satisfies the conditions of this theorem without such diameter control.

We then use Theorem 1.6 as a platform and prove the Gromov-Hausdorff convergence results by adding the Ricci curvature bound. Similar to what we had done in Lakzian-Sormani [34], in Lakzian [29], we also show that one can replace the Ricci curvature bound with assuming the existence of a uniform linear contractibility function.

1.2. Application to the Degeneration of Kähler (-Einstein) Metrics. Proving the existence and finding Kähler-Einstein metrics have been in the spotlight in the past few decades. In many cases, Ricci flow on Kähler manifolds (or the corresponding Kähler-Ricci flow) has proven to be a proper means of deforming Kähler metrics to a Kähler-Einstein metric on Kähler-Einstein manifolds (for example see [58] and references therein). In particular, there is a notion of Kähler Ricci flow through singularities established by Song-Tian in [53]. There are many directions to be explored in this context especially regarding the degeneration of Kähler Ricci Flow or Chern-Ricci flow and the geometry of the resulting singular limits. I am very interested to work in these areas and benefit from collaboration, and ideas from experts in the field as a part of my future research. I have been working on adapting the results of Lakzian [29] to the following problems (which are more or less of the same in nature), in order to make the proofs simpler.

(a) Continuity of Extremal Transitions and Flops (Candelas- de la Ossa’s Conjecture):

Roughly speaking, Candelas- de la Ossa’s conjecture says that extremal transitions and flops of \(n\)-dimensional Calabi-Yau varieties with singular sets of codimension at least 2, are continuous in the Gromov-Hausdorff sense. Rong-Zhang [46], [47] along with Song [52] prove some versions of Candelas-de la Ossa’s conjecture and Donaldson [18] gives a rather complete proof.

The proof boils down to proving the coincidence of smooth limits away from singular sets and the Gromov-Hausdorff limits. I believe the results of Lakzian [29] can be used to get the desired results provided that we can prove the required volume bounds (which is easier to do than proving diameter bounds). This is one of the problems that I am currently working on.

(b) Continuity of Kähler-Ricci and Chern-Ricci Flows through Singularities:

Kähler-Ricci flows with natural algebraic geometric surgeries were first introduced in Tian-Song [53]. The surgery at the singular time is comprised of divisorial contractions and flips (for details see [53]). Again the Gromov-Hausdorff continuity of the flow through this surgery turns out to be of the same nature as the problem we discussed previously. Another interesting geometric flow is Chern-Ricci flow which has only recently been introduced by [23] and further developed by Gill, Weinkove, Tossati and others. It seems like one should be able to perform similar surgeries in Chern-Ricci flow (though, it has not been made precise yet). Providing an alternative and simpler metric-geometric proof for the Gromov-Hausdorff continuity through singularities in both Kähler-Ricci and Chern-Ricci flows, as well as, precisely defining Chern-Ricci flow with surgery is an ongoing project.
(c) Existence of Kähler-Einstein Metrics:

Existence of Kähler-Einstein metrics has arguably been one of the most important problems in Kähler geometry. Tian’s conjecture about the existence of Kähler-Einstein metrics on Fano manifolds under stability conditions was proven by the collective work of Chen-Donaldson-Sun [14][12][13][15][18] and separately by Tian [57]. The proofs involve proving the existence of conic Kähler-Einstein metrics with cone singularities along a divisor and then using a smoothing scheme. Again, the convergence parts of the proofs are of the type that we have discussed so far and might be simplified using our techniques. This is another question that I am working on.

2. Generalizing Ricci Flow

Another aspect of my research involves studying flows on non-smooth or Finsler spaces that can be considered as generalizations of Riemannian Ricci flows.


In Angenent-Knopf [6], rigorous examples of non-degenerate neckpins are constructed on $S^{n+1}$ for $n + 1 \geq 3$ in the presence of rotational symmetry. Angenent-Knopf [6] also obtain the precise asymptotic profile for the rotationally symmetric neckpinch and formal matched asymptotic profile for the general Ricci flow neckpinch (Angenent-Knopf [7]). They prove that the diameter stays bounded as $t \rightarrow T$ and as a result under rotational and reflection symmetry, the neckpinch happens at one point rather than an interval.

In Angenent-Caputo-Knopf [5], the authors construct a canonical Ricci flow with surgery out of the neckpinch singularity and call it a smooth forward evolution of Ricci flow, consisting of two disjoint smooth Ricci flows at post surgery times. We will refer to this as the ACK forward evolution of Ricci flow. Professor Knopf has suggested that one can similarly introduce the forward evolution out of a degenerate neckpinch singularity and that is one of my ongoing projects.

2.1.1. Intrinsic Flat Continuity of Ricci Flow through Neckpinch Singularities. Smooth forward evolution of Ricci flow could serve as a first example of a weak Ricci flow provided that we had a metric defined on the disjoint union of smooth Ricci flows at post surgery times. Unlike the Mean Curvature Flow, Ricci flow does not come with an apriori ambient space, so one needs to define a proper metric on the disjoint union of the smooth Ricci flows in post surgery times. As for the continuity of the flow, one needs to appeal to the intrinsic notions of distance (due to lack of an ambient space). Recall that the continuity of weak MCF with respect to the Flat distance was proven by B. White [61]. Following a suggestion of Knopf, in Lakzian [28], I consider a metric on the post surgery times which is produced by joining the two smooth Ricci flows at the post surgery time $t \geq T$ by a thread of continuous length $L(t)$ that connects the post surgery futures of the neckpinch singularity point. In Lakzian [28], I prove that the Ricci flow through a neckpinch singularity (in the sense of ACK forward evolution of Ricci flow) is continuous under SWIF convergence when considered as a flow of integral current spaces:

**Theorem 2.1.** Let $(M^{n+1}, g(t))$ be a rotationally symmetric and reflection symmetric Ricci flow on $S^{n+1}$ defined for $t \in [0, T)$ with initial metric $g_0$. Suppose the flow develops a neckpinch singularity in finite time $T$ and suppose that the smooth forward evolution does not develop any singularities in $t \in (T, T + \epsilon)$. Let $(X(t), D(t), T(t))$ be the corresponding integral current flow defined for $t \in [0, T + \epsilon)$ by joining the post surgery futures of the neckpinch singularity by a thread of continuous length $L(t)$ with

$$
\lim_{t \searrow T} L(t) = 0
$$

Then, the integral current flow $(X(t), D(t), T(t))$ is continuous with respect to the Sormani-Wenger Intrinsic Flat distance (SWIF) on the time interval $[0, T + \epsilon)$.

2.1.2. Intrinsic Flat Convergence into the General Neckpinch Singularity. For a general neckpinch singularity without rotational symmetry, a canonical Ricci flow with surgery is still missing. One can still talk about the continuity of the flow into the singular time. One of my projects in progress is to investigate the following questions:

**Question 1.** Suppose $(M, g(t))$ flows into a neckpinch singularity (not necessarily rotationally symmetric) that is a point or a finite interval, then is there intrinsic flat convergence into the singular time? And what happens if we remove the diameter bound?

Although it is not clear what the weak flow at post surgery times should be, it is natural to think that a canonical Ricci flow through singularities must be a limit of a sequence of Ricci flows with surgery at scale $s_i$ as $s_i \rightarrow 0$. One might ask the following question:
Question 2. Is the limit of this hypothetical sequence continuous with respect to the intrinsic flat distance?

2.2. Weak Ricci Flow and Optimal Transport. McCann-Topping define a super Ricci flow on a smooth manifold as a supersolution to the Ricci flow i.e., a family of metrics $g(\tau)$ that satisfy $\frac{\partial g}{\partial \tau} \leq 2 \text{Ric}(g(\tau))$ where, $\tau$ is the backward time parameter. They prove that $g(\tau)$ is a super Ricci flow if and only if the Lipschitz constant

$$\sup_{M} |\nabla f(., \tau)|,$$

is non-decreasing in $\tau$ for any $f$ that is a solution to the heat equation

$$- \frac{\partial}{\partial \tau} f = \Delta g(\tau) f.$$

McCann-Topping suggest that one can define a weak Ricci flow on more general spaces to be a weak super Ricci flow that at each time expands distances no faster (to the first order in time) than any other weak super Ricci flow which coincides with the given weak super Ricci flow at that time.

2.2.1. Weak Super Ricci Flow on Disjoint Unions. In Lakzian-Munn [33], we observed that in order to have a weak super Ricci flow on the disjoint union of two smooth Ricci flows $(M_1, g_1(t)) \sqcup (M_2, g_2(t))$, it is sufficient that the distance between a point $x \in M_1$ and $y \in M_2$ satisfies a heat type inequality. In fact, we prove the following theorem:

**Theorem 2.2.** Let $(M_1, g_1(t))$ and $(M_2(t), g_2(t))$ be two smooth super Ricci flows. Consider the family of metric spaces $(M_1 \sqcup M_2, d(t))$ and suppose that for $t \in (0, T)$ satisfying

$$\frac{\partial}{\partial t} d(t)(x, y) \geq \Delta_{M_1 \times M_2} d(t)(x, y) \text{ for } (x, y) \in M_1 \times M_2$$

where, $\Delta_{M_1 \times M_2}$ denotes the Laplacian on $(M_1, g_1(t)) \times (M, g_2(t))$. Then the family of metrics $d(t)$ is a weak super Ricci flow of $M_1 \sqcup M_2$ in the sense of Topping-McCann.

In Lakzian-Munn [33], we present an example where we consider the disjoint union of two flat tori, $\mathbb{T}_1 \sqcup \mathbb{T}_2$ with the metric

$$D^r(a, b) = \begin{cases} d_1(a, b), & \text{if } a, b \in \mathbb{T}_1; \\ \sqrt{L^2(t) + d^2_1(\phi(a), b)}, & \text{if } a \in \mathbb{T}_1, b \in \mathbb{T}_2, \end{cases}$$

where $\phi: \mathbb{T} \to \mathbb{T}$ is the identity map. We then show that any $L(t)$ with growth larger than $t^{1/2}$ gives us a weak super Ricci flow. This is interesting because it is consistent with the fact that in parabolic flows (such as mean curvature flow or Ricci flow), the distance scales like the square root of time.

It is easy to see that starting from some initial metric $d(0)$ (perhaps the ‘distance’ at the singular time), $d(t)$ stays a semi-metric but there are examples for which the solution to the heat equation $\frac{\partial}{\partial t} d(t)(x, y) = \Delta_{M_1 \times M_2} d(t)(x, y)$ starting from a metric does not remain a metric (fails to satisfy the triangle inequalities), it is natural to ask the following questions which are under investigation:

**Question 3.** What is a recipe for producing metrics that satisfy the inequality (2.4)?

**Question 4.** Is there a canonical weak Ricci flow on disjoint unions (in the McCann-Topping sense) and how could such flows on disjoint unions (if there is any) be characterized with an equation?

**Question 5.** What are the super solutions with initial metric $d(0)$ coming from the Angenent-Knopf’s rotationally symmetric neckpinch?

2.2.2. Size of the Neckpinch Singularity and Optimal Transport. McCann-Topping suggest that a weak Ricci flow must be a weak super Ricci flow that expands the distances the least. Based on the work of McCann-Topping [37], $g(\tau)$ is a super Ricci flow i.e. $\frac{\partial g}{\partial \tau} \leq \text{Ric}(g(\tau))$ if and only if for any two unit mass solutions to the diffusion equation $\frac{\partial u}{\partial \tau} = \Delta g(\tau) u$ (or $\frac{\partial u}{\partial \tau} = \Delta g(\tau) u - Ru$ if we assume $\omega(\tau) = u(\tau) \text{ dVol}$ where $R$ is the scalar curvature), the Wasserstein distance $W_i(\omega(\tau), \tilde{\omega}(\tau), \tau)$ non-increasing in $\tau$ for $i = 1, 2$. Hence, one can have a more clear picture of McCann-Topping’s weak super Ricci flow if one can make sense of the diffusion equation,

$$\frac{\partial u}{\partial \tau} = \Delta g(\tau) u - Ru,$$

on metric-measure spaces.

In Lakzian-Munn [32], we consider the diffusion equation under Ricci flow and generalize it to metric measure spaces using semigroups and a Trotter-Chernov product formula. In particular, this can be applied to the disjoint union of two smooth Ricci flows.
In [6] and [7], Angenent-Knopf prove that if \((M, g(t))\) is a rotationally and reflection symmetric Ricci flow with one neck that develops a neckpinch singularity in finite time, then the neckpinch happens at only one point. They first prove that if the diameter stays bounded, then there is only one point pinching and then they prove that the diameter actually stays bounded.

In Lakzian-Munn [32], we have explored the application of the Topping-McCann’s definition of weak Ricci flow to the rotationally symmetric neckpinch in order to generalize the only one point pinching result of Angenent-Knopf to the case in which we do not have reflection symmetry, or even to a general neckpinch.

2.2.3. Ricci Flow on Metric Measure Spaces as a Gradient Flow. Perelman [43] expresses Ricci flow (up to a family of diffeomorphisms) as a gradient flow of the \(F^-\) entropy functional on the space of weighted smooth metric measure spaces. Gradient flow approaches might also be useful in the context of non-smooth metric measure spaces.

(I) Ricci Flow via Conjugate Heat Kernel:

Gigli-Mantegazza [22] propose a flow starting from an \(RCD(K, N)\) space, \((X, d_0, m)\) as

\[
d_t(x, y) := W^2_2(\mathbf{H}_t(\delta_x), \mathbf{H}_t(\delta_y)),
\]

where, \(W_2\) is the Wasserstein distance and \(\mathbf{H}_t\) is the heat operator. Based on [2], the heat flow on \(RCD(K, N)\) spaces exists and is the gradient flow of the Cheeger energy (at the level of functions) and coincides with the gradient flow of Shannon’s relative entropy in the Wasserstein space (as measures).

The Gigli-Mantegazza’s flow, \((X, d_t, m)\), almost never coincides with the Ricci flow because it actually does not see the curvature of the space at positive times. In that sense, the GM flow is actually just the linearization of the Ricci flow and is tangent to the actual Ricci flow at the initial time. I am investigating possible ways to remedy this issue by using the conjugate heat kernel under Ricci flow (i.e. the fundamental solution to the diffusion equation \(\partial u/\partial \tau = \Delta g(\tau) u - R u\) under a smooth Ricci flow). The possible ways to capture this equation in the non-smooth setting that I am investigating are:

(a) Run the Gigli-Mantegazza’s flow for a short time and then modify the measure at the final time slice, i.e., considering \((X, d_e, e^{f(x, \cdot)} m)\) in such a way that the asymptotics of \(R\) are captured. We then run the Gigli-Mantegazza’s flow with the new initial data. Then we try to pass to a limit as the time increment, \(\epsilon\), goes to 0.

(b) Making sense of the diffusion equation by finding a family of time-dependent operators that generate the diffusion equation (similar to the approach that is used in Lakzian-Munn [32]).

(II) Generalizing the \(F^-\) Entropy Functional:

In the smooth setting, the Ricci flow (up to diffeomorphisms and rescaling) can be thought of as the gradient flow of the Perelman’s \(F^-\) functional which is given by:

\[
F(f, g) := \int_M (R + |\nabla f|^2 - f) e^{-f} dV.
\]

In this project, our goal is to define an entropy functional on the space of metric measure spaces (in the sense of Sturm [55]) that generalizes Perelman’s functional. There are a few possible ways to define \(F^-\) entropy on non-smooth spaces:

(a) It is straightforward to see that, under a smooth Ricci flow, the \(F\) functional is the time derivative of the Boltzman-Shannon entropy, \(\mathcal{N}\).

For a fixed integer \(n \geq 0\), Let \((X, d, m) \in \mathbf{X}\) and \(H^n\) be the Hausdorff measure; Define \(\mathcal{N}\) (the Shannon entropy) as follows:

\[
\mathcal{N}((X, d, m)) := \text{Ent}(m|H^n)
\]

In order to define \(F\) as the rate of change of \(\mathcal{N}\), we need a direction (to differentiate along) which means that \(F\) has to be a functional on the tangent space of the space of metric measure spaces (which is a Hilbert space after completion).

One of the questions that I am investigating is to see if \(F\) has a gradient flow in the sense of Sturm [55],
(b) Based on a recent second order calculus on $RCD(K,N)$ spaces (see [20]), the term $|\nabla f|^2$ in the expression of the $F$–functional is well understood, and the integral of the scalar curvature (which is the change of measure up to sign) can be captured as follows: take an $RCD$ space $(X,d,m)$, flow it under the Gigli-Mantegazza’s flow (which is tangent to Ricci flow) and compute the rate of change of total measure at time $t = 0$.

2.3. Finsler-Ricci Flow. The notion of scalar Finsler-Ricci flow was first introduced in Bao [10] to be
\[
\frac{\partial F^2}{\partial t} = -2 \text{Ric}
\]
where $F$ is a time-dependent family of Finsler metrics and $\text{Ric}$ is the scalar Ricci curvature in Finsler geometry. It is straightforward to see that this equation implies
\[
\frac{\partial g_{ij}}{\partial t}(x,y) = -2 \text{Ric}_{ij}(x,y)
\]
in which $\text{Ric}_{ij}$ are the coefficients of Akbarzadeh’s Ricci tensor.

Similar to the Riemannian case, this flow might also provide a tool for attacking some fundamental questions in Finsler geometry such as Professor Chern’s question regarding the existence of Finsler-Einstein metrics on every smooth manifold. There is still a lot of work to be done in this setting. For example, existence and uniqueness of this flow is not yet well understood in general.

We know that differential Harnack estimates for heat type equations under Ricci flow play an important role in singularity analysis of Ricci flow. In Lakzian [30], I prove a differential Harnack estimate for solutions of the nonlinear heat equation under Finsler-Ricci flow.

**Theorem 2.3.** Let $(M^n, F(t)), t \in [0,T]$ be a closed Finsler-Ricci flow such that for all $t \in [0,T],
\begin{enumerate}
  \item $\text{Ric}_N \geq K$ for some $N < 2n$ and, $K_1 \leq \text{Ric}_{ij} \leq K_2$ ,
  \item $F(t)$ is uniformly smooth and convex along the flow with constants $\lambda$ and $\Lambda$,
  \item $F(t)$ enjoys mild non-linearity (see [30] for a definition) , $C(x) \leq C$.
\end{enumerate}

Let $u(t,x) \in L^2([0,T], H^1(M)) \cap H^1([0,T], H^{-1}(M))$ be a positive global solution (in the sense of distributions) of the heat equation under $FRF$, i.e., for any test function, $\phi \in C^\infty(M)$, and for all $t \in [0,T]$,\[
\int_M \phi \partial_t u(t,.) \ dm = -\int_M D\phi(\nabla u(t,.)) \ dm dt;
\]
Then, $u$ satisfies,
\[
F(\nabla (\log u)(t,x))^2 - \theta \partial_t (\log u)(t,x) \leq \left( \frac{2}{N} - \frac{1}{n} \right)^{-1} \frac{\theta^2}{t} + \frac{n\theta^3 C_1}{\theta - 1} + n^{3/2}\theta^2 \sqrt{C_2},
\]
for any $\theta > 1$ and $C_1, C_2$ depending on the curvature bounds $K, K_1, K_2$, the non-linearity, $C$ and the constants $\lambda$ and $\Lambda$.

The key tool in proving Theorem 2.3 is the Bochner-Weitzenböck formula and inequality for Finsler manifolds which were recently proven by Ohta-Sturm [42].

3. Topological Implications of Non-negative (Ricci) Curvature

There is a great deal of literature regarding the topological implications of non-negative sectional or Ricci (or even scalar) curvatures on complete and non-compact Riemannian manifolds. Cheeger-Gromoll’s soul theorem, Milnor’s conjecture, Schoen-Yau’s minimal surface theory, etc. are among the most important ones. Below, I describe my ongoing research regarding these topological implications of different non-negative curvatures in non-smooth or Finsler settings.

3.1. Finite Generation of the Fundamental Group in $RCD(0,N)$ Spaces. In [38], Milnor conjectures that a complete non-compact Riemannian manifold, $M^n$, with non-negative Ricci curvature possesses a finitely generated fundamental group. The finite generation of fundamental group has been proven in the following cases:
\begin{enumerate}
  \item (i) If $M$ has non-negative sectional curvature (Cheeger-Gromoll [11]);
  \item (ii) When $M$ is three dimensional and $\text{Ric} \geq 0$ (Liu [36]);
  \item (iii) When $M$ has Euclidean volume growth (Anderson [4] and Li [35]);
  \item (iv) $M^n$ has small diameter growth ($O(\sqrt{r})$) and sectional curvature bounded below (Abresch-Gromoll [1]).
\end{enumerate}

As far as finite generation results in non-smooth spaces satisfying curvature-dimension bounds, Bacher-Sturm [8] prove the finite generation of the fundamental group for $CD(K, N)$ spaces with $K > 0$. In the smooth setting, this is an immediate result of the Myer’s Theorem.
Sormani [54] proves that a Riemannian manifold $M^n$ ($n \geq 3$) with $\text{Ric} \geq 0$ has a finitely generated fundamental group if it has small linear diameter growth,

$$\limsup \frac{\text{diam} \partial (B(p,r))}{r} < 4S_n,$$

where, the universal constant $S_n$ (coming from Abresch-Gromoll’s excess estimates) is

$$S_n := \left\{ \begin{array}{ll}
\left( \frac{9}{2} - \frac{N}{2} \right)^{-1} & \text{if } 1 < N < 2, \\
1 & \text{if } N = 2, \\
\left( 4 + 2 \cdot 3^N \left( \frac{N-1}{N-2} \right)^{N-1} \right)^{-1} & \text{if } N > 2;
\end{array} \right.$$  (3.2)

In Kittabepu-Lakzian [26], we generalize this result to metric spaces satisfying the curvature-dimension condition $CD(0,N)$ that are also infinitesimally Hilbertian (in short, $RCD(0,N)$ spaces). In the course of the proof, it becomes clear that we need to assume some other metric conditions on the space but the general approach is reminiscent of that in Sormani [54]. The key tool we have used is the Abresch-Gromoll’s excess estimates in the non-smooth setting which has been proven by Gigli [21]. The main theorem of Kittabepu-Lakzian [26] is the following:

**Theorem 3.1.** Let $(X,d_X,m)$ be a connected, locally contractible, and non-branching geodesic metric-measure space with $\text{supp}(m) = X$. Suppose $X$ satisfies the $CD(0,N)$ curvature-dimension conditions that is also infinitesimally Hilbertian (see [2] for the detailed definition). If $X$ has small linear diameter growth

$$\limsup \frac{\text{diam} \partial (B(p,r))}{r} < 4S_N,$$

where,

$$S_N = \left\{ \begin{array}{ll}
\left( \frac{9}{2} - \frac{N}{2} \right)^{-1} & \text{if } 1 < N < 2, \\
1 & \text{if } N = 2, \\
\left( 4 + 2 \cdot 3^N \left( \frac{N-1}{N-2} \right)^{N-1} \right)^{-1} & \text{if } N > 2;
\end{array} \right.$$  (3.3)

Then, $X$ has finitely generated fundamental group.

3.2. Other Questions Regarding the Structure of $RCD(K,N)$ Spaces. Rajala-Sturm [45] prove that $RCD(K,\infty)$ spaces are essentially non-branching. Also Mondino-Naber [39] prove that $RCD(K,N)$ spaces are rectifiable and hence almost everywhere have unique tangent cones that are Euclidean. There are also a few recent important unanswered conjectures/questions about these spaces which interest me.

**Question 6.** Are $RCD(K,N)$ spaces non-branching?

**Question 7.** Are $RCD(K,2)$ spaces Alexandrov spaces? (Conjectured by K. T. Sturm)

**Question 8.** Are $RCD(K,N)$ spaces of Hausdorff dimension $N$ topological manifolds? (suggested by N. Honda)

3.3. Non-negatively Curved Finsler Manifolds with Large Volume Growth. In the Riemannian setting, large volume growth and lower bound on sectional or Ricci curvatures has topological implications. Perelman [44] proves that non-negative Ricci curvature and large enough volume growth implies the contractibility of the underlying manifold and Munn [40][41] finds estimates for such thresholds both for Riemannian manifolds and Alexandrov spaces. Shen [50] proves that large volume growth and a lower bound on sectional (or with a lower bound on conjugate radius) implies that the underlying manifold is of finite topological type. In the same direction, Wan [60] proves that a non-negatively curved complete Riemannian manifold with large volume growth has no closed geodesics. The key tool he uses is the Toponogov’s comparison theorem. In the virtue of the celebrated Cheeger-Gromoll’s soul theorem, this implies that the soul is a point and hence a non-negatively curved manifold with large volume growth is diffeomorphic to $\mathbb{R}^n$.

A version of Toponogov’s comparison theorem has only recently been proven by Kondo-Ohta-Tanaka [27]. In Lakzian [31], I use this comparison theorem to generalize the result of Wan [60] to Finsler manifolds with large volume growth. The main theorem of Lakzian [31] goes as follows:

**Theorem 3.2.** A complete non-compact Berwaldian and reversible Finsler manifold with non-negative flag curvature and large volume growth has no closed geodesics:

There does not yet exist a soul theorem for Finsler manifolds but if the Riemannian Cheeger-Gromoll’s soul theorem were to hold in Finsler setting then we would have an affirmative answer for the following question.
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Question 9. A complete non-compact Berwaldian and reversible Finsler manifold with non-negative flag curvature and large volume growth is diffeomorphic to $\mathbb{R}^n$.

3.4. Comparison Theorems for Finsler Manifolds with Non-negative Ricci curvature. Comparison theorems for Finsler manifolds with a lower bound on their Ricci curvatures and two sided bounds on their $\mathbf{S}$-curvatures first appeared in Shen [51], followed by other authors (e.g. [62] and [17]). Recently, there has also been other developments in the geometry of Finsler manifolds such as a Bochner-Weitzenböck formula proven by Ohta-Sturm [42] and a Toponogov type comparison theorem proven by Kondo-Ohta-Tanaka [27].

One ongoing research theme in Finsler geometry is to see to what extent the results from Riemannian geometry can be generalized to Finsler structures. One question that I am currently working on is to extend, a small scale Toponogov type comparison theorem with lower bounds on Ricci curvature that geometry can be generalized to Finsler structures. One question that I am currently working on is to extend, a small scale Toponogov type comparison theorem with lower bounds on Ricci curvature that has been proven in Dai-Wei [16], to the Finsler setting.

REFERENCES


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