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Chapter 1

Introduction

1.1 Universality in Random Systems

The concept of universality arose in the context of statistical mechanics as a consequence of the study of critical phenomena. To understand what a critical phenomenon is, we consider a bar magnet. We describe its capacity of picking up thumbtacks with an order parameter $M$. As we heat the system, $M$ decreases and eventually, at a certain critical temperature $T_c$, it reaches zero, which means the bar loses its magnetization. The special value of the parameter at which the system changes its phase is the critical point. For systems that exhibit universality, the closer the parameter is to its critical value, the less sensitively the order parameter depends on the details of the system. The order parameter is well approximated by

$$M \sim |T - T_c|^{\alpha},$$

(1.1.1)

where the exponent $\alpha$ is the critical exponent. It was found empirically that one can partition critical systems into “universality classes”. In the last decade the concept of universality class has been investigated and exploited and now it plays a central role in probability and mathematical physics.

1.1.1 Gaussian Universality Class

If we move away from statistical mechanics, we realize that the concept of universality class is more familiar than we thought. The simplest and first historically example of universality is provided by the Central Limit Theorem.

Consider a sample (i.e. a collection of data) obtained by a large number of observations randomly and independently generated, and compute the average of the observed values. Repeating the computation many times, one will notice that the computed averages are distributed according to a normal distribution: this is the content of the Central Limit Theorem.

**Theorem 1.1.1.** Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables with mean $\mu$ and variance $\sigma^2 < \infty$. Let $S_n := \sum_{k=1}^{n} X_k$. Then

$$\lim_{n \to \infty} P\left( \frac{S_n - \mu n}{\sigma \sqrt{n}} \leq s \right) = \int_{-\infty}^{s} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

(1.1.2)

This means that, despite the model dependent features as the mean and the variance, any sum of i.i.d random variables with finite variance will show the same limiting behavior, described by the normal distribution, and fluctuations around the mean
value of order $n^{1/2}$. Physical and mathematical systems accurately described in terms of Gaussian statistics are said to be in the Gaussian universality class.

Theorem 1.1.1 illustrates well the universal and the non-universal quantities. The model-dependent and thus non-universal quantities are $\mu$ and $\sigma^2$, while the $1/2$ exponent in the normalization as well as the normal distribution are universal.

### 1.1.2 Random and Ballistic Deposition

In this lecture we will be dealing with models related to stochastically growing interfaces. Let us starts with two simple models, which however shows very different limiting behaviors.

![Figure 1.1: Illustration and simulation of the random deposition model](image)

In the random deposition model unit blocks fall independently and in parallel on $\mathbb{Z}$ after an exponentially distributed waiting time of parameter 1 (see Figure 1.1(a)). Due to the memoryless property of the exponential distribution, this model is a Markov process, its evolution depends only on the present and not on the history. Each column evolves independently and is a Poisson process. Denote by $h(x,t)$ the height function that gives the value of the height of the column on the site $x$ at time $t$. By the Law of Large Numbers and the Central Limit Theorem, for any $x \in \mathbb{Z}$,

\[
\lim_{t \to \infty} \frac{h(x,t)}{t} = 1,
\]

\[
\frac{h(x,t) - t}{\sqrt{t}} \Rightarrow N(x).
\]  

Figure 1.1: Illustration and simulation of the random deposition model.

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\[
\lim_{t \to \infty} \frac{h(x,t)}{t} = 1,
\]

\[
\frac{h(x,t) - t}{\sqrt{t}} \Rightarrow N(x).
\]  

The model shows linear growth speed and lack of spatial correlation (see Figure 1.1(b)) and the fluctuations belong to the Gaussian universality class.
1.1. **UNIVERSALITY IN RANDOM SYSTEMS**

It is enough to slightly modify the rules of the growth of the interface to lose the Gaussian behavior. Consider the same model, but now, instead of falling on the ground or the interface, a new block sticks to the first edge (see Figure 1.2(a)): this is known as the *ballistic deposition model*, introduced by Vold in 1959. This change in the evolution rules turns into large time effects: the interface grows faster than in the random deposition model (the value of the velocity is still unknown) and simulations (see Figure 1.2(b)) show that the height function has smaller fluctuations, on the scale $t^{1/3}$, and demonstrate nontrivial correlations on the transversal scale of $t^{2/3}$. Moreover the rescaled height does not converge anymore to the Gaussian distribution.

1.1.3 **KPZ Universality Class**

The ballistic deposition model is believed to belong to the *Kardar-Parisi-Zhang* (KPZ) *universality class*. A model is in this universality class if the following properties are satisfied:

- **Random local growth rules**: the evolution of the height function depends only the local (in space and time) configuration of the interface.

- **Smoothing**: there is a smoothing mechanism implying that deep holes are rapidly filled to smooth the interface such that there exists a non-random macroscopic limit shape $h_{\text{ma}}(\xi, \tau) = \lim_{t \to \infty} \frac{h(\xi t, \tau t)}{t}$.

- **Non-linearity of speed of growth**: Macroscopically the growth of the interface will evolves deterministically according to some PDE. Since the growth rules are local and locally one will see a given slope only, one will get $\partial_\tau h_{\text{ma}} = v(\nabla h_{\text{ma}})$. In order to be in the KPZ class, we need to have the condition $v''(u) \neq 0$. This property implies irreversibility of the model.
The KPZ universality class was introduced in the context of studying the motion of growing interfaces in 1986 in a paper of Kardar, Parisi and Zhang \[8\], where they studied a continuum stochastically growing height function \( h(x,t) \) given in term of a stochastic PDE, known as the KPZ equation:

\[
\frac{\partial h}{\partial t}(x,t) = \nu \frac{\partial^2 h}{\partial x^2}(x,t) + \frac{1}{2} \lambda \left( \frac{\partial h}{\partial x}(x,t) \right)^2 + \sqrt{D} \xi(x,t),
\]

where \( \xi(x,t) \) is Gaussian space-time white noise, \( \lambda, \nu \in \mathbb{R}, D > 0 \) (the stationary solution is formally a two-sided Brownian motion). This equation is ill-posed, since the nonlinearity does not make sense in this case. The equation contains of course the key features: local random growth (the noise \( \xi \)), the smoothing (from the Laplacian), the non-linearity (from the square of the gradient).

### 1.2 Integrable systems in the KPZ universality class

So-far, the ballistic deposition model could not be investigated analytically, but there is a class of systems that is integrable. For an integrable probabilistic system, it is possible to compute concise formulas for averages of a class of observables; furthermore, taking limits of the system, observables and formulas, it is possible to access detailed descriptions of universal classes. We will focus on a few examples in the Kardar-Parisi-Zhang class. Consider an interface modeled by a height function \( h(x,t), x, t \in \mathbb{Z} \), with \( h(x \pm 1, t) - h(x, t) \in \{-1, 1\} \). Wlog set \( h(0,0) = 0 \). The height function evolves according to the following dynamics: each local minimum turns into a local maximum after an exponentially distributed waiting time of parameter 1 (see Figure 1.3). Particularly interesting are the cases of two initial configurations: wedge initial condition, which means that \( h(x,0) = |x| \), and flat initial condition, which means that \( h(x,0) \) is a sawtooth function between 0 and 1.

The macroscopic limit shape for wedge initial condition is a parabola continued by two straight lines:

\[
h_{ma}(\xi) = \begin{cases} 
\frac{1}{2} (1 + \xi^2), & \text{for } |\xi| \leq 1 \\
|\xi|, & \text{for } |\xi| \geq 1.
\end{cases}
\]

Johansson proved in \[10\] large time results for the rescaled height function.

**Theorem 1.2.1.** For wedge initial condition, for any \( \xi \in (-1, 1) \),

\[
\lim_{t \to \infty} P \left( \frac{h(\xi t, t) - c_1(\xi) t^{1/3}}{-c_2(\xi) t^{1/3}} \leq s \right) = F_2(s),
\]

(1.2.1)
1.2. INTEGRABLE SYSTEMS IN THE KPZ UNIVERSALITY CLASS

Figure 1.4: A simulation of the height function fluctuations starting from wedge initial condition. The curve represents the limit shape (a parabola) while the piecewise linear line represents the height function. Fluctuations live on the $t^{1/3}$ scale and are correlated spatially in the $t^{2/3}$ scale.

with $c_1, c_2$ model-dependent constants.

The function $F_2$ is known as the GUE Tracy-Widom distribution, first discovered in random matrices \[12\] (see Section 1.3.1).

For flat initial condition the macroscopic shape is simply $h_{m_0}(\xi) = \frac{1}{2}$. An analogous result for the limit distribution stands also for this case \[14\].

**Theorem 1.2.2.** For flat initial condition, for any $\xi \in (-1, 1)$,

$$
\lim_{t \to \infty} \mathbb{P} \left( \frac{h(\xi t, t) - c_3(\xi)t^{1/3}}{c_4(\xi) t^{1/3}} \leq s \right) = F_1(2s),
$$

(1.2.2)

with $c_3, c_4$ model-dependent constants.

Also the function $F_1$ was first observed in the random matrix context and it is known as GOE Tracy-Widom distribution (see Section 1.3.2). An interesting fact to observe is that, although the scaling exponent are invariant, the limit distribution depends on the initial condition.

### 1.2.1 Directed polymers/Last Passage Percolation

Consider the model of growing interface just introduced starting from wedge initial configuration. There is an alternative way to describe the evolution of the height function, by taking the time at which a box appears on a local minimum, and call this time $T_{i,j}$ for $i,j \in \mathbb{N}$. Let indicate with $w_{i,j}$ the "waiting time" for the local valley at $(i,j)$ to become a local maximum. Since a box can appear only once the blocks $(i-1,j)$ and $(i,j-1)$ have appeared, $T_{i,j}$ must satisfy the recursive relation $T_{i,j} = T_{i-1,j} \vee T_{i,j-1} + w_{i,j}$, and iterating

$$
T_{i,j} = \max_{\pi : (1,1) \to (i,j)} \sum_{(k,l) \in \pi} w_{k,l},
$$

(1.2.3)

where $\pi$ are all steps made of consecutive steps of $(1,0)$ or $(0,1)$. This model is called *last passage percolation* with exponential weight. Another solvable model is last passage percolation with geometric weights, namely take $\mathbb{P}(w_{i,j} = k) = (1 - q)^{q^k}$ for $k \geq 0$. The exponential case can be recovered by the limit $q \to 1$ and setting
\[ k = t/(1 - q). \] Another limit is obtained by letting \( q \to 0 \) and choosing the unit of \( i, j \) to be \( \sqrt{2} \) instead of 1. If we draw one point each time that there is a \( w_{ij} = 1 \) (higher values will not occur in the limit), then we are left with a Poisson point process with density 1 on \( \mathbb{R}^2 \). The paths can then be taken to be directed paths which maximizes the number of Poisson points visited. Once can reformulate the problem as the one of finding the longest increasing subsequence of a random permutation, which was solved by Baik, Deift and Johansson [15]:

**Theorem 1.2.3.** [15] Let \( \sigma \) be a uniformly distributed permutation of \( \{1, \ldots, n\} \) and let \( \ell_n(\sigma) \) be the length of the longest increasing subsequence. Then

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{\ell_n(\sigma) - 2\sqrt{n}}{n^{1/6}} \leq s \right) = F_2(s). \tag{1.2.4}
\]

Here the macroscopic parameter is \( \sqrt{n} \) and the fluctuations are \( \sqrt{n^{1/3}} = n^{1/6} \).

In the next chapter we will consider this model with \( P(w_{ij} = k) = (1 - a_i b_j)(a_i b_j)^k, \quad k \geq 0 \), for arbitrary family of parameters \( \{a_i, b_i\} \) satisfying \( 0 < a_i, b_i < 1 \). The reason is that having the parameters, the mathematical structure in the background is much more visible.

### 1.3 Random matrices

In Thm [1.2.1] and [1.2.2] we introduced the distributions \( F_1 \) and \( F_2 \) and said that they were first observe in random matrices. In this section we explain in which framework they appear.

#### 1.3.1 Gaussian Unitary Ensemble

The Gaussian Unitary Ensemble (GUE) of random matrices consists of Hermitian matrices \( H \) of size \( N \times N \) distributed according to the probability measure

\[
p^{\text{GUE}}(H) dH = \frac{1}{Z_N} \exp \left( -\frac{1}{2N} \text{Tr}(H^2) \right) dH, \tag{1.3.1}
\]

where \( dH = \Pi_{i=1}^N dH_i, \Pi_{1 \leq i < j \leq N} \text{Re}(H_{i,j}) \text{Im}(H_{i,j}) \) is the reference measure and \( Z_N \) the normalization constant. Denote by \( \lambda_{N,\text{max}}^{\text{GUE}} \) the largest eigenvalue of a \( N \times N \) GUE.
1.3. RANDOM MATRICES

1.3.1 Tracy-Widom

Then Tracy and Widom \[12\] proved that the asymptotic distribution of the (properly rescaled) largest eigenvalue is \( F_2 \) (see Figure 1.6):

\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{\lambda_{\text{GUE}, \text{max}} - 2N}{N^{1/3}} \leq s \right) = F_2(s). \tag{1.3.2}
\]

Figure 1.6: Dashed line: the densities of the GUE Tracy-Widom distribution; solid line: the GOE Tracy-Widom distribution density.

1.3.2 Gaussian Orthogonal Ensemble

The Gaussian Orthogonal Ensemble (GOE) of random matrices consists of symmetric matrices \( H \) of size \( N \times N \) distributed according to the probability measure

\[
p^{\text{GOE}}(H) dH = \frac{1}{Z_N} \exp \left( -\frac{1}{4N} \text{Tr}(H^2) \right) dH,
\]

where \( dH = \prod_{1 \leq i < j \leq N} dH_{ij} \) is the reference measure and \( Z_N \) the normalization constant. Denote by \( \lambda_{\text{GOE}, \text{max}} \) the largest eigenvalue of a \( N \times N \) GOE matrix. The asymptotic distribution of the (properly rescaled) largest eigenvalue is \( F_1 \) \[13\] (see Figure 1.6):

\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{\lambda_{\text{GOE}, \text{max}} - 2N}{N^{1/3}} \leq s \right) = F_1(s). \tag{1.3.4}
\]
Chapter 2

The Polynuclear Growth Model

2.1 The discrete polynuclear growth model

The discrete polynuclear growth model (PNG) is a growth model with discrete space, \( x \in \mathbb{Z} \), and discrete time, \( t \in \mathbb{Z}_+ \). The height function \( h \) is integer-valued, \( h(x,t) \in \mathbb{Z} \) for any \( x \in \mathbb{Z}, t \in \mathbb{Z}_+ \). We consider the PNG model in the so-called "droplet geometry", which correspond to the wedge initial configuration for the growth model presented at page 10. This is also known in literature as the corner growth model. Fix the initial condition \( h(x,0) = 0 \), for any \( x \in \mathbb{Z} \) and define the dynamics as

\[
h(x,t+1) = \max\{h(x-1,t), h(x,t), h(x+1,t)\} + \omega(x,t+1), \tag{2.1.1}
\]

where \( \omega(x,t) \) are independent random variables with the following distribution:

\[
\begin{align*}
\omega(x,t) &= 0 \text{ if } t - x \text{ is even or } |x| > t \\
\omega(i-j, i+j-1) &= w(i,j), i, j \in \mathbb{Z}_+
\end{align*}
\]  

(2.1.2)

with \( P(w(i,j) = k) = (1 - a_i b_j)(a_i b_j)^k, k \geq 0 \), with \( a_i, b_j \in (0,1) \forall i, j \in \mathbb{Z}_+ \). We decided to use the set of parameters \( \{a_i, b_j\} \) in order to observe better the structure; later, when we will study large time asymptotics, we will fix \( a_i = b_j = \sqrt{q} \in (0,1) \), so that the parameter in the dynamics is \( q \) at all times.

Figure 2.1: Setting for the PNG model. Representation of the \( \omega(x,t) \): a black dot is drawn on sites where \( \omega(x,t) \neq 0 \).

To understand how the PNG height function evolves, let us consider an explicit case until time \( t = 3 \), see Figure 2.2. For a better graphical visualization, instead of a
single point at position \((x,h(x,t))\), we draw a segment of size 1 centered at \(x\) at the vertical position \(h(x,t)\), and finally we join the end of the horizontal lines by vertical ones.

Observe that the max function in (2.1.1) can be implemented as follows: each vertical upwards (resp. downwards) line its left (resp. right) by one unit. Then we add the new random variables \(\omega(x,t)\) on top of it. As you can see from the example, until time \(t = 2\) we can recover from the graph of the height function the values of the \(w(i,j)\) used so-far (i.e., for \(i + j - 1 \leq t\)). However this no longer possible for times \(t \geq 3\) and the mapping from \(\{w(i,j), i + j - 1 \leq t\}\) to \(\{h(x,t), |x| \leq t\}\) is not anymore a bijection. Due to a lost of information when there is an overlap, given \(\{h(x,t), |x| \leq t\}\) we cannot recover the values \(\{w(i,j), i + j - 1 \leq t\}\). Thus even nice measure on the \(w(i,j)\) will not translate to a simple measure on \(\{h(x,t), |x| \leq t\}\).

In order to maintain a bijection between the growth model and the underlying random variables, the \(w(i,j)\)'s, we need to extend the model. This extension is known as the multilayer PNG model.

![Figure 2.2: Evolution of the PNG model.](image)

Figure 2.2: Evolution of the PNG model. In figure (a) the values of the random variables \(w\) are represented on the plane \((i,j)\). From (b) to (e) the first three iterations of the height function: we start from \(h(x,0) = 0\), then we add two ”boxes” on the site 0, since \(w(1,1)=2\); at the next step we move all the points of the previous configuration of one step to the left and to the right, corresponding to taking the maximum (2.1.1), and we add a number of boxes on the sites 1,-1 indicated by the values of \(w(1,2), w(2,1)\), and so on. At the third step two ”island” meet and we have the formation of an overlap. Note that the height function is represented with a continuous line, but it is defined only on \(\mathbb{Z} + \frac{1}{2}\).
2.1. THE DISCRETE POLYNUCLEAR GROWTH MODEL

2.1.1 The multilayer PNG

The multilayer PNG consists is a set of height function, which in the line representation used above, by construction will consists in a set of non-intersecting lines \( \{ h_\ell, \ell \geq 1 \} \) where \( h_\ell = \{ h_\ell(x,t), x \in \mathbb{Z}, t \in \mathbb{Z}_+ \} \) is the \( \ell \)th line of the model. It is defined as follows: first of all, \( h_1(x,t) := h(x,t) \) (this the function we actually want to study). For the other lines, the initial condition is \( h_\ell(x,0) = -\ell + 1 \), for any \( x \in \mathbb{Z}, \ell \geq 1 \). The dynamics of \( h_1(x,t) \) is the same as in the discrete PNG, as for \( \ell \geq 2 \), \( h_\ell(x,t+1) = \max \{ h_\ell(x-1,t), h_\ell(x,t), h_\ell(x+1,t) \} + w_\ell(x,t+1) \), where \( w_\ell(x,t+1) \) is given by the overlap on the site \( x \) at level \( \ell-1 \) (see Figure 2.3). The lines \( \{ h_\ell(x,t), x \in \mathbb{Z}, t \in \mathbb{Z}_+, \ell \geq 1 \} \) are non-intersecting by construction. Now we want to understand how the geometric distribution on the \( \{ \omega(x,t) \} \) is transported to a measure on the non-intersecting paths. For \( t - x \) odd we define the jumps,

\[
\eta^+(x,t) = h(x,t) - h(x-1,t), \\
\eta^-(x,t) = h(x,t) - h(x+1,t). 
\]

(2.1.3)

\( \eta^+, \eta^- \geq 0 \) and we should think of \( \eta^+(x,t) \) as a positive jump at \( x \) at time \( t \), and \( \eta^-(x,t) \) as a the size of a negative jump at \( x+1 \) at time \( t \). Define also

\[
T\omega(x,t) = \min \{ \eta^+(x+1,t-1), \eta^-(x-1,t-1) \} \quad \text{if } t-x \text{ is odd} \\
T\omega(x,t) = 0 \quad \text{if } t-x \text{ is even.} 
\]

(2.1.4)

and \( T\omega(i,j) = T\omega(i-j,i+j-1) \). Now we assign weights to the jumps. Let \( a_i \) and \( b_j \) be given variables. The jumps are assigned weights as follows: \( \eta^+(x,t) \) has weight \( a_i \eta^+(x,t) \), \( i = (t + x + 1)/2 \) and \( \eta^-(x,t) \) has weight \( b_j \eta^-(x,t) \), \( j = (t - x + 1)/2 \). Also, to \( T^k\omega(x,t) \) we assign the weight \( (a_i b_j) T^{k\omega(x,t)} \) with the same correspondence between \( (i,j) \) and \( (x,t) \), \( k \geq 0 \). The measure is then given by the product of all the weights of all the jumps in the multilayer PNG, \( h_1, \ldots, h_n \), at time \( t = 2n - 1 \), that equals

\[
\prod_{i+j \leq 2n} (a_i b_j)^{w(i,j)}. 
\]

(2.1.5)

Details can be found in [16] and in Figure 2.4 an example of the construction of such a measure is represented.

Consider the multilayer PNG model with droplet geometry at time \( t \). At each \( x \in \mathbb{Z}, |x| < t \) the height functions are random variables. Thus a natural question is to
CHAPTER 2. THE POLYNUCLEAR GROWTH MODEL

\[ t = 1 \]
\[ (a_1 b_1)^2 \]

\[ t = 2 \]
\[ (a_1 b_1)^2 (a_2 b_1^3) \]

\[ t = 3 \]
\[ (a_1 b_1)^2 (a_2 b_1^4) (a_1 b_2)^4 (a_2 b_2)^1 (a_3 b_1)^2 \]

Figure 2.4: Construction of the measure on \( \{ \omega(x,t), |x| \leq t \} \) on the multilayer PNG for the example in Figure 2.2 up to time \( t = 3 \).

study the joint distributions \( \mathbb{P}(h_1(x_1,t) \leq a_1, \ldots, h_1(x_m,t) \leq a_m) \), or, as first simpler question just \( \mathbb{P}(h_1(0,t) \leq a) \). For that purpose, we will first study in more details the structure of the line ensemble \( \{ h_\ell(x,t), \ell \geq 1, t \in \mathbb{Z}_+ \} \) and we first ask

\[ \mathbb{P}\left( \bigcap_{\ell \geq 1} (h_\ell(0,t) = x_\ell) \right) = ? \quad (2.1.6) \]

The answer to this question can be obtained as special care of a result from graph theory presented in the next section.

2.1.2 The LGV Theorem

Consider a graph \((V,E)\) directed, loop-free (without cycles), where \(V\) is set of vertices and \(E\) the set of edges. A path \(\pi\) is a sequence of vertices connected by directed edges. Given two points \(u,v \in V\), we define \(\mathcal{P}(u,v)\) as the set of paths from \(u\) to \(v\). We say that two paths \(\pi, \pi'\) intersect if they have a common vertex.

To every edge \(e \in E\) we associated a positive weight \(w(e) > 0\), the weight of a path \(\pi\) by

\[ w(\pi) = \prod_{e \in \pi} w(e) \quad (2.1.7) \]

and let

\[ W(u,v) := \sum_{\pi \in \mathcal{P}(u,v)} w(\pi) \quad (2.1.8) \]

be the total weight of paths from \(u\) to \(v\).
2.1. THE DISCRETE POLYNUCLEAR GROWTH MODEL

Finally, take \( m \) initial points \( \vec{u} = (u_1, \ldots, u_m) \) and \( m \) end points \( \vec{v} = (v_1, \ldots, v_m) \). Here comes an important assumption, which is satisfied for the underlying graph of our PNG multilayer model.

**Assumption A:** There exists at most one permutation \( \sigma \in S_m \) such that we can connect \( u_i \) to \( v_{\sigma(i)} \), \( i = 1, \ldots, m \) by a set of \( m \) non-intersecting paths.

If this assumption is satisfied, then we can label the end points such that the assumption is verified with the identity permutation only. Now we are ready to state the Lindström–Gessel–Viennot (LGV) theorem.

**Theorem 2.1.1.** (LGV Theorem) Let Assumption A holds. Denote by \( \mathcal{P}_{\text{non}}^{-\cap}(\vec{u}, \vec{v}) \) be the set of all \( m \)-tuples of non-intersecting paths from \( \vec{u} \) to \( \vec{v} \). Then

\[
\sum_{(\pi_1, \ldots, \pi_m) \in \mathcal{P}_{\text{non}}^{-\cap}(\vec{u}, \vec{v})} w(\pi_1) \cdots w(\pi_m) = \det [W(u_i, v_j)]_{1 \leq i, j \leq m}. \tag{2.1.9}
\]

**Proof.** Writing down explicitly the expression of the determinant

\[
\det [W(u_i, v_j)]_{1 \leq i, j \leq m} \overset{\text{def}}{=} \sum_{\sigma \in S_m} (-1)^{|\sigma|} \prod_{k=1}^{m} h(W_{k}, v_{\sigma(k)}) = \sum_{\sigma \in S_m} (-1)^{|\sigma|} \sum_{\pi \in \mathcal{P}_{\text{non}}^{-\cap}(u_k, v_{\sigma(k)})} w(\pi) \tag{2.1.10}
\]

First notice that Assumption A implies in the first term or the last equation exists only for the identity permutation and in the case where the paths do not intersect. But this is exactly the r.h.s. of (2.1.9). Thus to conclude the proof of (2.1.9) we need to show that the second term in the last equation is zero.

Let \( I \) be the first intersection point between two consecutive lines, say \( \pi_k, \pi_{k+1} \). Notice that the definition of \( I \) is not unique, one can look e.g. at the smallest \( k \) such that \( \pi_k, \pi_{k+1} \) intersect and choose \( I \) as the intersection point closest to the origin. Then consider the map \( \Phi \) acting on the paths as \( \Phi(\pi_1, \ldots, \pi_m) = (\Phi_1(\vec{\pi}), \ldots, \Phi_m(\vec{\pi})) \), that exchanges the two paths which intersect at \( I \) after the point \( I \). Notice that whatever the definition of first intersection you use, \( \Phi \circ \Phi = \mathbb{1} \), so that we have a bijection.

Since the weights are product of the edges visited by the paths, the these are unchanged under the action of \( \Phi \), then we have \( w(\pi_1) \cdots w(\pi_m) = \)
Figure 2.5: The map $\Phi$ acts on the paths $\pi_k$ (in blue) and $\pi_{k+1}$ (in red) exchanging the parts of the paths after the intersection point $I$ and giving the paths $\Phi_k(\vec{\pi})$ (in blue) and $\Phi_{k+1}(\vec{\pi})$ (in red).

$$w(\Phi_1(\vec{\pi})) \cdots w(\Phi_m(\vec{\pi})).$$ Thus we have

$$\sum_{\sigma \in S_m} (-1)^{\sigma} \sum_{\pi_1: u_1 \rightarrow v_{\sigma(1)}} \cdots \sum_{\pi_m: u_m \rightarrow v_{\sigma(m)}} w(\pi_1) \cdots w(\pi_m) =$$

$$= \sum_{\sigma \in S_m} (-1)^{\sigma} \sum_{\Phi_1(\vec{\pi}): u_1 \rightarrow v_{\sigma(1)}} \cdots \sum_{\Phi_m(\vec{\pi}): u_m \rightarrow v_{\sigma(m)}} w(\Phi_1(\vec{\pi})) \cdots w(\Phi_m(\vec{\pi})), \quad (2.1.11)$$

where $\tilde{\sigma}$ is the permutation such that $\Phi_k(\vec{\pi}) : u_k \rightarrow v_{\tilde{\sigma}k}$. The sum can only be zero, since $(-1)^{\tilde{\sigma}} = -(1)^{\sigma}$. By relabeling the paths, we have that r.h.s of (2.1.11) is equal minus 1 times l.h.s. of (2.1.11), which thus implies (2.1.11) = 0.

Now we want to apply the LGV theorem to the non-intersecting line ensemble of the multilayer PNG. Consider the square graph $\frac{1}{2}Z \times Z$, with horizontal edges directed to the right with weight 1, the vertical edges at horizontal position $-t+\frac{1}{2}+2k$, $k = 1, \ldots, t$ are directed upwards with weights $a_{k+1}$, and the vertical edges at horizontal position $t-\frac{1}{2}-2k$, $k = 1, \ldots, t$ are directed downwards with weights $b_{k+1}$. Then consider a set of $N \geq t$ non-intersecting lines starting from $u_i = (-t, -i+1)$ and ending at $v_i = (t, -i+1)$, $1 \leq i \leq N$ and restrict the graph to $V = \frac{1}{2}Z \times \{-N+1, -N+2, \ldots\}$. This setting fits the hypothesis of the LGV theorem, so if $P_{(t_1, t_2)}(x, y) := \sum_{\pi : (t_1, x) \rightarrow (t_2, y)} w(\pi)$, then

$$P(h_1(0, t) = x_1, \ldots, h_N(0, t) = x_N)$$

$$= \frac{1}{Z_N} \cdot [P_{(-t, 0)}(-i + 1, x_j)]_{1 \leq j \leq N} \cdot [P_{(0, t)}(x_i, -j + 1)]_{1 \leq i \leq N} \quad (2.1.12)$$

for some normalization constant $Z_N$. The next step is to find an expression for the transition weight $P_{(t_1, t_2)}$ in a form more explicit than the sum over all possible paths.

### 2.2 Partitions and Young diagrams

In the line ensemble of the multilayer PNG at any position $x \in Z$ the lines have heights $H_1 > H_2 > H_2 > \ldots$ with $H_i \in \mathbb{Z}$ for any $i \geq 1$ and $H_n + n - 1 \rightarrow 0$ for $n \rightarrow \infty$. Equivalently, setting

$$\lambda_i := H_i + i - 1, \quad (2.2.1)$$

we get a weakly decreasing sequence in $\mathbb{N}$ converging to zero: $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ with $\lambda_i \rightarrow 0$ for $i \rightarrow \infty$. 

2.2. PARTITIONS AND YOUNG DIAGRAMS

Definition 2.2.1. A partition is a sequence of non-negative integer \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( \lambda_1 \geq \lambda_2 \geq \ldots \) and containing only finitely many non-zero elements.

Remark 2.2.2. We regard \((3, 1), (3, 1, 0), (3, 1, 0, 0), \ldots\) as the same partition.

Different quantities related with partitions needs a definition.

Definition 2.2.3. Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition.

- The non-zero entries \( \lambda_i \) of \( \lambda \) are called the parts of \( \lambda \).
- The number of parts is called the length of \( \lambda \). We denote it by \( \ell(\lambda) \).
- The sum of the parts \( |\lambda| := \sum_i \lambda_i \) is called the weight of \( \lambda \).
- If \( |\lambda| = n \) we say that \( \lambda \) is a partition of \( n \).

We denote by \( \mathcal{P}_n \) the set of all partitions of \( n \), by \( \mathcal{P} \) the set of all partitions and by \( \mathcal{P}_0 \) the set containing only the empty partition \( \emptyset \). A concise notation for a partition is given by \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \) with \( m_i = \text{Card}\{ j : \lambda_j = i \} \).

There is a nice graphical representation of the partition in terms of Young diagrams.

Definition 2.2.4. A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order representing the parts of a partition. An example is represented in Figure 2.6. We denote by \( \mathcal{Y} \) the set of all Young diagrams.

![Figure 2.6: The Young diagram associated to the partition \( \lambda = (7, 5, 2, 1, 1) \).](image)

Listing the number of boxes of a Young diagram in each column gives another partition, the conjugate partition of \( \lambda \); one obtains a Young diagram of that shape by reflecting the original diagram along its main diagonal (see Figure 2.7). In formulas,

\[
\lambda'_i = |\{ j : \lambda_j \geq i \}|
\]

from which one has the identity \( m_i(\lambda) = \lambda'_i - \lambda'_{i+1} \).

![Figure 2.7: The conjugate \( \lambda' \) of the Young diagram in Figure 2.6. \( \lambda' \) is a Young diagram associated to the partition (5, 3, 2, 2, 1, 1).](image)

Definition 2.2.5. We define now a partial ordering on partitions: given two partitions \( \lambda, \mu \), we say that \( \lambda \subset \mu \) if \( \lambda_i \leq \mu_i \) for any \( i \geq 1 \). If \( \lambda \subset \mu \) we can define the skew diagram, the set-theoretic difference \( \theta = \mu - \lambda \). We set \( |\theta| := |\mu| - |\lambda| = \sum_i (\mu_i - \lambda_i) \).
An example of skew diagram is represented in Figure 2.8.

Particular cases of skew diagram which occurs in our PNG multilayer model are horizontal strips.

**Definition 2.2.6.** A skew diagram $\theta$ is a horizontal $m$-strip (resp. vertical $m$-strip) if the total number of extra boxes is exactly $m$ and $\theta_i \leq 1$ (resp. $\theta_i \leq 1$) for any $i$. A horizontal (resp. vertical) strip has at most one square in each column (resp. row).

**Remark 2.2.7.** If $\theta = \mu - \lambda$, then $\theta$ is an horizontal (resp. vertical) strip iff

$$
\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \ldots
$$

In this case we say that $\lambda$ and $\mu$ interlace and we use the notation $\lambda \prec \mu$ or $\mu \succ \lambda$.

Now we come back to the multilayer PNG model and try to clarify the connection with Young diagrams using an example illustrated in Figure 2.9.

Taking Figure 2.9 as reference, we define the following partitions obtained from the heights at different positions:

- $\lambda(x = -3) = \emptyset$
- $\lambda(x = -2) = (6)$
- $\lambda(x = -1) = (4)$
- $\lambda(x = 0) = (9, 3)$
- $\lambda(x = 1) = (3)$
- $\lambda(x = 2) = (7)$
- $\lambda(x = 3) = \emptyset$.

Then the following interlacing relations hold: $\emptyset \prec (6) \prec (4) \prec (9, 3) \prec (3) \prec (7) \succ \emptyset$. 

---

Figure 2.8: Skew diagram (blue area) $\theta = (1, 1, 2, 1)$ for the partition $\mu = (5, 4, 4, 1)$ and $\lambda = (4, 3, 2)$.

Figure 2.9: Multilayer PNG at time $t = 3$ and representation in terms of Young diagrams of the partitions. Notice that $\lambda(x + 1) - \lambda(x)$ or $\lambda(x) - \lambda(x + 1)$ are horizontal strips.
The transition weights in this example are
\[
\begin{align*}
  a_1^{[\lambda(-2)] - [\lambda(-3)]} &= a_1^6 & \text{from } x = -3 \text{ to } x = -2 \\
  b_3^{[\lambda(-2)] - [\lambda(-1)]} &= b_3^2 & \text{from } x = -2 \text{ to } x = -1 \\
  a_2^{[\lambda(0)] - [\lambda(-1)]} &= a_2^8 & \text{from } x = -1 \text{ to } x = 0 \\
  b_2^{[\lambda(0)] - [\lambda(1)]} &= b_2^9 & \text{from } x = 0 \text{ to } x = 1 \\
  a_3^{[\lambda(2)] - [\lambda(1)]} &= a_3^4 & \text{from } x = 1 \text{ to } x = 2 \\
  b_1^{[\lambda(2)] - [\lambda(3)]} &= b_1^7 & \text{from } x = 2 \text{ to } x = 3
\end{align*}
\]

In general, a transition from \( x = -t + 2k \) to \( y = -t + 2k + 1 \) has a weight \( a_k^{[\lambda(x+1)] - [\lambda(x)]} \) and one from \( x = t - 2k - 1 \) to \( y = t - 2k \) has a weight \( b_k^{[\lambda(x)] - [\lambda(x+1)]} \). These transitions can be described using Schur functions, are a particular type of symmetric functions.

### 2.3 Symmetric functions

#### 2.3.1 Basic notions

In this section we briefly present the algebra \( \Lambda \) of symmetric functions in infinitely many variables\(^3\). Let \( \Lambda_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N} \) be the space of polynomials in \( x_1, \ldots, x_N \) which are symmetric under the action of the semigroup \( S_N \), which means with respect to permutations of the \( x_i \)'s. \( \Lambda_N \) is a graded ring, in fact

\[
\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k,
\]

where \( \Lambda_N^k \) consists of homogeneous symmetric polynomials of degree \( k \), together with the zero polynomial. Constant are in the 0th graded component.

Consider the homomorphism \( \pi_{N+1} : \mathbb{C}[x_1, \ldots, x_{N+1}] \to \mathbb{C}[x_1, \ldots, x_N] \), which sends \( x_{N+1} \) to zero and the other \( x_i \)'s to themselves. This maps elements of \( \Lambda_{N+1} \) to elements of \( \Lambda_N \) and it generates a so-called tower of graded algebras:

\[
\mathbb{C} \xleftarrow{\pi_1} \Lambda_1 \xleftarrow{\pi_2} \Lambda_2 \xleftarrow{\pi_3} \ldots
\]

Finally one sets

\[
\Lambda := \lim_{N \to \infty} \Lambda_N = \{(f_1, f_2, \ldots) | f_i \in \Lambda_i, \pi_i f_i = f_{i-1}, \deg(f_i) < \infty\}
\]

**Example 2.3.1.** Here are a few examples.

- \( p_1 = x_1 + x_2 + \ldots \) belongs to \( \Lambda \),
- \( p_2 = (1 + x_1)(1 + x_2) \ldots \) does not belong to \( \Lambda \), since it has infinite degree,
- \( p_3 = x_1x_2 + x_1x_3 + \ldots + x_2x_3 + \ldots \) belongs to \( \Lambda \),
- \( p_4 = 1 \) belongs to \( \Lambda \),
- \( p_5 = x_1 + x_2^2 + x_3 + x_4^2 + \ldots \) does not belong to \( \Lambda \), since it is not symmetric.

**Remark 2.3.2.** One can think also of element of \( \Lambda \) as formal power series in infinitely many variables \( x_1, x_2, \ldots \) of bounded degree which are invariant under permutations of the \( x_i \)'s.

---

\(^3\)For more details, see Chapter 1 of [17].
In order to work with symmetric functions, it is useful to know some basis of \( \Lambda \). Let us define three useful symmetric functions.

**Definition 2.3.3.** (a) For \( n \geq 1 \), the \( n \)th elementary symmetric function \( e_n \) is defined by

\[
e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} \cdots x_{i_n}.
\]

(2.3.4)

We will also use the convention \( e_0 = 1 \).

(b) For \( n \geq 1 \), the \( n \)th complete homogeneous function \( h_n \) is defined by

\[
h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}.
\]

(2.3.5)

We will also use the convention \( h_0 = 1 \).

(c) For \( n \geq 1 \), the \( n \)th power sum \( p_n \) is defined by

\[
p_n = \sum_{i \geq 1} x_i^n.
\]

(2.3.6)

It can be proven that each of these set are basis of \( \Lambda \).

**Theorem 2.3.4.** The systems \( \{e_n\}, \{h_n\}, \{p_n\} \) are algebraically independent generators of \( \Lambda \), thus \( \Lambda = \mathbb{C}[e_1, e_2, \ldots] = \mathbb{C}[h_1, h_2, \ldots] = \mathbb{C}[p_1, p_2, \ldots] \).

**Proof.** See [17], Chap. 1, Sect. 2, (2.3), (2.4), (2.8), (2.12).

Generating functions are very often useful, thus let us compute define and compute them for our family of polynomials.

**Definition 2.3.5.** Define the following generating functions,

\[
H(z) = \sum_{k \geq 0} h_k z^k \quad E(z) = \sum_{k \geq 0} e_k z^k \quad P(z) = \sum_{k \geq 1} p_k z^{k-1},
\]

(2.3.7)

with \( e_0 = h_0 = 1 \).

**Lemma 2.3.6.** The following identities hold:

\[
H(z) = \prod_{i \geq 1} \frac{1}{1 - x_i z}, \quad E(z) = \prod_{i \geq 1} (1 + x_i z),
\]

(2.3.8)

\[
P(z) = \frac{d}{dz} \sum_{i \geq 1} \ln \left( \frac{1}{1 - x_i z} \right), \quad H(z) = \frac{1}{E(-z)} = \exp \left( \sum_{k \geq 1} \frac{z^k}{k} p_k \right).
\]

Proof. A series expansion of \( \frac{1}{1 - x_i z} = 1 + x_i z + (x_i z)^2 + \ldots \) leads to

\[
(1 + x_1 z + x_1^2 z^2 + \ldots) \cdot (1 + x_2 z + x_2^2 z + \ldots) \cdots = 1 + z \sum_{i \geq 1} x_i + z^2 \left( \sum_{i \geq 1} x_i x_j + \sum_{i \geq 1} x_i^2 \right) + \ldots
\]

(2.3.9)

\[
= 1 + zh_1 + z^2 h_2 + \cdots = H(z).
\]
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For $E(z)$ we have

\[
\prod_{i \geq 1} (1 + x_i z) = (1 + x_1 z)(1 + x_2 z) \ldots
\]

\[
= 1 + z \sum_{i \geq 1} x_i + z^2 \sum_{1 \leq i < j} x_i x_j + \ldots
\]

(2.3.10)

\[
= 1 + \sum_{k \geq 1} z^k e_k = E(z).
\]

A series expansion of $\ln \left( \frac{1}{1 - x_i z} \right) = - \ln(1 - x_i z) = x_i z + \left( \frac{(x_i z)^2}{2} \right) + \left( \frac{(x_i z)^3}{3} \right) + \ldots$ leads to

\[
\sum_{i \geq 1} \frac{d}{dz} \ln \left( \frac{1}{1 - x_i z} \right) = \sum_{i \geq 1} (x_i + x_i^2 z + x_i^3 z^2 + \ldots)
\]

(2.3.11)

\[
=p_1(x_1, x_2, \ldots) + z p_2(x_1, x_2, \ldots) + z^2 p_3(x_1, x_2, \ldots) + \ldots = P(z).
\]

Finally, we have

\[
P(z) = \sum_{i \geq 1} p_i z^{i-1} = \frac{d}{dz} \sum_{i \geq 1} \ln \left( \frac{1}{1 - x_i z} \right)
\]

(2.3.12)

\[
= \frac{d}{dz} \ln \left( \prod_{i \geq 1} \frac{1}{1 - x_i z} \right) = \frac{d}{dz} \ln H(z),
\]

which after integrating in $z$ gives the claimed relation. 

\[\square\]

2.3.2 Schur functions

There exists another basis of $A$ which is more useful for our goals and it is defined in terms of Schur functions. We start by defining the Schur polynomials.

Definition 2.3.7. The Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ is a symmetric polynomial in $N$ variables $x_1, \ldots, x_N$ parametrized by $\lambda \in \mathcal{Y}$ with $\ell(\lambda) \leq N$, defined by

\[
s_\lambda(x_1, \ldots, x_N) = \frac{\det \left[ x_i^{\lambda_j + N - j} \right]_{1 \leq i, j \leq N}}{\prod_{1 \leq i < j \leq N} (x_i - x_j)}.
\]

(2.3.13)

Remark 2.3.8. The following identity holds

\[
\prod_{1 \leq i < j \leq N} (x_i - x_j) = \det \left[ x_i^{N-j} \right]_{1 \leq i, j \leq N}.
\]

(2.3.14)

This is called Vandermonde determinant and is denoted by $\Delta_N(x)$.

Let us verify that (2.3.13) has the claimed properties. First of all, $\det \left[ x_i^{\lambda_j + N - j} \right]_{1 \leq i, j \leq N}$ is a polynomials with zeroes whenever $x_i = x_j$ for $i \neq j$. Thus this determinant is divisible by $x_i - x_j$ for all $i \neq j$. Therefore $s_\lambda(x_1, \ldots, x_N)$ is a polynomial. Furthermore, both numerator and denominator are antisymmetric over transpositions of the $x_i$'s. Thus the Schur polynomial are symmetric as claimed.

Here is a first property of the Schur polynomials.

Lemma 2.3.9. Let $\ell(\lambda) \leq N$. Then

\[
\pi_{N+1} s_\lambda(x_1, \ldots, x_{N+1}) := s_\lambda(x_1, \ldots, x_N, 0) = s_\lambda(x_1, \ldots, x_N)
\]

and

\[
\pi_{\ell(\lambda)} s_\lambda(x_1, \ldots, x_{\ell(\lambda)}) = 0.
\]

(2.3.15)
Proof. If \( \ell(\lambda) \leq N \), then \( \lambda_{N+1} = 0 \), so

\[
\det \begin{bmatrix}
  x_1^{\lambda_1+N} & \cdots & x_1^{\lambda_N+1} \\
  \vdots & \ddots & \vdots \\
  x_N^{\lambda_1+N} & \cdots & x_N^{\lambda_N+1} \\
  0 & \cdots & 0
\end{bmatrix} = s_\lambda(x_1, \ldots, x_N, 0) = \det \begin{bmatrix}
  x_1^{N} & \cdots & x_1^{1} \\
  \vdots & \ddots & \vdots \\
  x_N^{N} & \cdots & x_N^{1} \\
  0 & \cdots & 0
\end{bmatrix}
\]

This proves (2.3.15).

If \( \ell(\lambda) = N \), then \( \lambda_N \geq 1 \), \( \lambda_{N+1} = 0 \), so

\[
\pi_N s_\lambda(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_{N-1}, 0) = \Delta_N(x)^{-1} \det \begin{bmatrix}
  x_1^{\lambda_1+N-1} & \cdots & x_1^{\lambda_N} \\
  \vdots & \ddots & \vdots \\
  x_N^{\lambda_1+N-1} & \cdots & x_N^{\lambda_N}
\end{bmatrix}
\]

which proves (2.3.16). \( \square \)

As a consequence of this theorem, the sequence of symmetric polynomial \( s_\lambda(x_1, \ldots, x_N) \) with \( \lambda \) fixed and varying number of variables \( N \geq \ell(\lambda) \), defines an element of \( \Lambda \). This element is called Schur symmetric function \( s_\lambda \), where we set \( s_\emptyset(x) = 1 \).

An important property is that \( \{ s_\lambda, \lambda \in \mathbb{Y} \} \) forms a basis of \( \Lambda \).

**Proposition 2.3.10.** The Schur functions \( \{ s_\lambda, \lambda \in \mathbb{Y} \} \) forms a basis of \( \Lambda \). Their relation with the generators \( e_k, h_k \) are given by the Jacobi-Trudi formulas

\[
s_\lambda = \det [h_{\lambda_i-i+j}]_{1 \leq i, j \leq \ell(\lambda)} = \det [e_{\lambda_i'-i+j}]_{1 \leq i, j \leq \ell(\lambda')},
\]

where by definition \( h_k \equiv e_k \equiv 0 \), for \( k < 0 \).

**Proof.** The proof of the fact that Schur functions form a linear basis of \( \Lambda \) can be found in [17], Chap. 1, Sect. 3.

Let us prove only the first identity, namely \( s_\lambda = \det [h_{\lambda_i-i+j}] \). Let \( n = \ell(\lambda) \) and define \( e_r^{(k)} = e_r(x_1, \ldots, \hat{x}_k, \ldots, x_n) \) for \( 1 \leq k, r \leq n \). Then

\[
E^{(k)}(z) := \sum_{r=0}^{n-1} e_r^{(k)} z^r = \prod_{r \neq k} (1 + x_r z). \quad (2.3.20)
\]
Thus we have
\[ H(z)E^{(k)}(-z) = 1/(1 - x_kz) = 1 + x_kz + x_k^2z^2 + \ldots. \] (2.3.21)

Comparing the coefficients of \( z^n \) we have the identity
\[ \sum_{j=1}^{n} (-1)^{n-j} h_{m-n+j}^{(k)} e_n^{(k)} = x_k^n. \] (2.3.22)

For given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we define
\[ H_\alpha = [h_{\alpha_i-n+j}]_{1 \leq i, j \leq n}, \quad M = [(-1)^{n-j} e_{n-j}^{(k)}]_{1 \leq i, j \leq n}. \] (2.3.23)

Then, \( H_\alpha M = [x_\alpha^n]_{1 \leq i, j \leq n} \).

By choosing \( \alpha = \delta := (n-1, n-2, \ldots, 0) \) we get
\[ \det[x_\alpha^n] = \Delta_n(x) = \det[M] \det[h_{-1}] = \det[M]. \] (2.3.24)

By choosing \( \alpha = \delta + \lambda = (\lambda_1 + n - 1, \lambda_2 + n-2, \ldots, \lambda_n) \) we obtain
\[ s_\lambda = \frac{\det[x_\alpha^n]}{\det[x_\delta^n]} = \det[H_{\lambda+\delta}] = \det[h_{\lambda_i+i-j}] \] (2.3.25)
as claimed.

The proof of the second identity can be found in [17], Chap. 1, Sect. 2 (pages 22-23). The idea is the following. Define matrices \( H = [h_{i-j}]_{0 \leq i, j \leq N} \) and \( E = [(-1)^{n-j} e_{i-j}]_{0 \leq i, j \leq N} \), with \( N = \ell(\lambda) + \ell(\lambda') - 1 \). Using the identity obtained by comparing the coefficients of \( z^n \) in the relation \( H(z)E(-z) = 1 \) one gets that \( HE = 1 \) and \( \det[E] = \det[H] = 1 \). Then the determinant in the Schur polynomials is determinant of a submatrix, which is equal to the determinant of the complementary cofactor of \( E^T \). Some computations shows that the latter is given by the last formula in (2.3.10).

Now consider a measure giving to a configuration \( \lambda \in \mathcal{Y} \) a weight \( s_\lambda(x) \cdot s_\lambda(y) \), where \( x = (x_1, x_2, \ldots, y = (y_1, y_2, \ldots) \) are two sets of variables. If we want to turn this weight into a probability measure, we need

1. positive weights, \( s_\lambda(x_1, x_2, \ldots) \geq 0 \), for any \( \lambda \in \mathcal{Y} \);
2. a normalization constant, i.e. to compute \( \sum_{\lambda \in \mathcal{Y}} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) \).

Theorem 2.3.11 (Cauchy identities). The following identities hold:

(a)\[
\sum_{\lambda \in \mathcal{Y}} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i, j \geq 1} \frac{1}{1 - x_i y_j};
\] (2.3.26)

(b)\[
\sum_{\lambda \in \mathcal{Y}} \frac{p_\lambda(x_1, x_2, \ldots) p_\lambda(y_1, y_2, \ldots) }{Z_\lambda} = \exp \left( \sum_{k \geq 1} \frac{p_k(x_1, x_2, \ldots) p_k(y_1, y_2, \ldots) }{k} \right)
= \prod_{i, j \geq 1} \frac{1}{1 - x_i y_j},
\] (2.3.27)

where \( p_\lambda := p_{\lambda_1} \cdots p_{\lambda_m} \) and \( Z_\lambda := \prod_{i \geq 1} (i^{m_i} m_i !) \).
Remark 2.3.12. The RHS of (2.3.27) should be viewed as formal power series using
\( \frac{1}{1 - x_i y_j} = 1 + x_i y_j + (x_i y_j)^2 + \ldots \)
In order to prove the Cauchy Identity, we have to antepone the proof of another
identity, from which (2.3.27) will easily follow:

Theorem 2.3.13 (Cauchy-Binet Identity).

\[
\det \left[ \int_{\Omega} d\omega(x) \Phi_i(x) \Psi_j(x) \right]_{1 \leq i,j \leq N} = \frac{1}{N!} \int_{\Omega^N} d\omega(x_1) \ldots d\omega(x_n) \det [\Phi_i(x_j)]_{1 \leq i,j \leq N} \det [\Psi_i(x_j)]_{1 \leq i,j \leq N}. \quad (2.3.29)
\]

Proof. By multi-linearity of the determinant we get

\[
\det \left[ \int_{\Omega} d\omega(x) \Phi_i(x) \Psi_j(x) \right]_{1 \leq i,j \leq N} = \int_{\Omega^N} d\omega(x_1) \ldots d\omega(x_n) \det [\Phi_i(x)]_{1 \leq i \leq N} \det [\Psi_j(x)]_{1 \leq j \leq N} \quad (2.3.30)
\]

\[
= \int_{\Omega^N} d\omega(x_1) \ldots d\omega(x_N) \prod_{i=1}^n \Phi_i(x_i) \det [\Psi_j(x_i)]_{1 \leq i,j \leq N}.
\]

Renaming \( x_i = y_{\sigma(i)}, i = 1, \ldots, n, \)

\[
(2.3.31) = \int_{\Omega^N} d\omega(y_{\sigma(1)}) \ldots d\omega(y_{\sigma(N)}) \prod_{i=1}^N \Phi_i(y_{\sigma(i)}) \det [\Psi_j(y_{\sigma(i)})]_{1 \leq i,j \leq N}
\]

\[
= \int_{\Omega^N} d\omega(y_{\sigma(1)}) \ldots d\omega(y_{\sigma(N)}) \prod_{i=1}^N \Phi_i(y_{\sigma(i)})(-1)^{|\sigma|} \det [\Psi_j(y_i)]_{1 \leq i,j \leq N}.
\]

Since the integral does not depend on \( \sigma, \)

\[
(2.3.32) = \frac{1}{N!} \sum_{\sigma \in S_N} \int_{\Omega^N} d\omega(y_1) \ldots d\omega(y_N) \prod_{i=1}^N \Phi_i(y_{\sigma(i)})(-1)^{|\sigma|} \det [\Psi_j(y_i)]_{1 \leq i,j \leq N}
\]

\[
= \frac{1}{N!} \int_{\Omega^N} d\omega(y_1) \ldots d\omega(y_N) \sum_{\sigma \in S_N} (-1)^{|\sigma|} \prod_{i=1}^N \Phi_i(y_{\sigma(i)}) \det [\Psi_j(y_i)]_{1 \leq i,j \leq N}.
\]

We have finished since \( \sum_{\sigma \in S_N} (-1)^{|\sigma|} \prod_{i=1}^N \Phi_i(y_{\sigma(i)}) = \det [\Phi_i(y_i)]_{1 \leq i,j \leq N}. \)

Proof of Theorem 2.3.13(a). Consider \( \vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n). \) Then

\[
\Delta_N(\vec{x}) \Delta_N(\vec{y}) \sum_{\lambda \in \mathbb{Y}} s_\lambda(\vec{x}) s_\lambda(\vec{y}) \defeq \sum_{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0} \det \left[ x_i^{\lambda_j + N - j} \right]_{1 \leq i \leq N} \det \left[ y_i^{\lambda_j + N - j} \right]_{1 \leq i,j \leq N}
\]

\[
= \frac{1}{N!} \sum_{\xi \geq 0} \det \left[ x_i^{\xi_j} \right]_{1 \leq i \leq N} \det \left[ y_i^{\xi_j} \right]_{1 \leq i,j \leq N} \quad (2.3.33)
\]
where the last equality follows because of the summand is zero whenever \( \xi_i = \xi_j \) for \( i \neq j \) and it is symmetric.

Applying the Cauchy-Binet identity (2.3.29) we obtain

\[
\text{(2.3.33)} \quad \Delta = \det \left[ \sum_{i,j=1}^{N} \frac{1}{1-x_i y_j} \right] = \det \left[ \frac{1}{1-x_i y_j} \right] \quad 1 \leq i,j \leq N
\]

(2.3.34)

where the last step is proved in the following lemma.

\[\square\]

**Lemma 2.3.14.** It holds

\[
\left( \prod_{1 \leq i < j \leq N} (x_i - x_j)(y_i - y_j) \right)^{-1} \det \left[ \frac{1}{1-x_i y_j} \right] = \prod_{i,j=1}^{N} \frac{1}{1-x_i y_j}
\]

(2.3.35)

**Proof.** We won’t give a complete proof of this identity, but we will illustrate a direct computation for \( N = 3 \). The same ideas can be generalized for any \( N \).

\[
\det \begin{bmatrix}
\frac{1}{1-x_1 y_1} & \frac{1}{1-x_1 y_2} & \frac{1}{1-x_1 y_3} \\
\frac{1}{1-x_2 y_1} & \frac{1}{1-x_2 y_2} & \frac{1}{1-x_2 y_3} \\
\frac{1}{1-x_3 y_1} & \frac{1}{1-x_3 y_2} & \frac{1}{1-x_3 y_3}
\end{bmatrix} = \prod_{i,j=1}^{3} \frac{1}{1-x_i y_j}
\]

(2.3.36)

We want to prove that the determinant in the last line of \( \text{(2.3.34)} \) coincides with the Vandermonde determinant. Let \( C_i \) indicate the \( i \)-th column of the matrix. If we subtract \( C_1 \) to \( C_2 \) and \( C_3 \),

\[
\text{det} \begin{bmatrix}
(1-x_1 y_2)(1-x_1 y_3) & (1-x_1 y_1)(1-x_1 y_3) & (1-x_1 y_1)(1-x_1 y_2) \\
(1-x_2 y_1)(1-x_2 y_3) & (1-x_2 y_1)(1-x_2 y_3) & (1-x_2 y_1)(1-x_2 y_2) \\
(1-x_3 y_1)(1-x_3 y_3) & (1-x_3 y_1)(1-x_3 y_3) & (1-x_3 y_1)(1-x_3 y_2)
\end{bmatrix}
\]

\[
\text{det} \begin{bmatrix}
(1-x_1 y_2)(1-x_1 y_3) & (1-x_1 y_1)(1-x_1 y_3) & (1-x_1 y_1)(1-x_1 y_2) \\
(1-x_2 y_1)(1-x_2 y_3) & (1-x_2 y_1)(1-x_2 y_3) & (1-x_2 y_1)(1-x_2 y_2) \\
(1-x_3 y_1)(1-x_3 y_3) & (1-x_3 y_1)(1-x_3 y_3) & (1-x_3 y_1)(1-x_3 y_2)
\end{bmatrix}
\]

(2.3.37)

Now subtracting \( C_2 \) to \( C_3 \),

\[
\text{det} \begin{bmatrix}
(1-x_1 y_2)(1-x_1 y_3) & (1-x_1 y_1)(1-x_1 y_3) & (1-x_1 y_1)(1-x_1 y_2) \\
(1-x_2 y_1)(1-x_2 y_3) & (1-x_2 y_1)(1-x_2 y_3) & (1-x_2 y_1)(1-x_2 y_2) \\
(1-x_3 y_1)(1-x_3 y_3) & (1-x_3 y_1)(1-x_3 y_3) & (1-x_3 y_1)(1-x_3 y_2)
\end{bmatrix}
\]

(2.3.38)
2.3.3 Skew Schur functions

Our next goal is to introduce an object that will give the transition from \( \lambda(x) \) to \( \lambda(x + 1) \) for the PNG droplet.

Consider two sets of variable \( x = (x_1, x_2, \ldots) \), \( y = (y_1, y_2, \ldots) \) and let \((x,y)\) be the union of the sets. Consider a function \( f(x,y) \) symmetric w.r.t all the permutations of the \( x_i, y_j \)'s. Since \( f(x,y) \) is symmetric in the \( x_i \)'s and in the \( y_j \)'s it must be expressed in the form \( f(x,y) = \sum_j f_j(x)g_j(y) \), for some symmetric functions \( f_j, g_j \). For instance, if one takes for \( f(x,y) \) the power sum (2.3.6). Then

\[
p_k(x,y) = \sum_{i \geq 1} x_i^k + \sum_{i \geq 1} y_i^k = p_k(x) + p_k(y). \tag{2.3.39}
\]

We want to understand how the Schur function on the union of two sets of variables, \( s_\lambda(x,y) \), decompose.

**Definition 2.3.15.** Let \( \lambda \) be a Young tableau. Then

\[
s_\lambda(x,y) = \sum_{\mu \in \mathcal{Y}} s_{\lambda/\mu}(x)s_\mu(y),
\tag{2.3.40}
\]

where the coefficients \( s_{\lambda/\mu}(x) \) are called skew Schur functions and are symmetric functions in \( x \).

We study some properties of the skew Schur functions.

**Proposition 2.3.16.** (Consistency) Let \( x, y \) be two sets of variables. Then for any \( \lambda, \mu \in \mathcal{Y} \)

\[
s_{\lambda/\mu}(x,y) = \sum_{\nu \in \mathcal{Y}} s_{\lambda/\nu}(x)s_{\nu/\mu}(y). \tag{2.3.41}
\]

**Proof.** Consider three sets of variables \( x, y, z \). Then, by definition

\[
s_\lambda(x,y,z) = \sum_{\mu \in \mathcal{Y}} s_{\lambda/\mu}(x,y)s_\mu(z) \tag{2.3.42}
\]

but also

\[
s_\lambda(x,y,z) = \sum_{\nu \in \mathcal{Y}} s_{\lambda/\nu}(x)s_{\nu}(y,z)
= \sum_{\mu \in \mathcal{Y}} \sum_{\nu \in \mathcal{Y}} s_{\lambda/\nu}(x)s_{\nu/\mu}(y)s_\mu(z) \tag{2.3.43}
\]

Since the equality holds for any choice of variables \( z \), we need to have the equivalence of the coefficients of \( s_\mu(z) \), which is (2.3.41).

**Lemma 2.3.17.** The following properties hold:

(a)

\[
s_{\lambda/\emptyset}(x) = s_\lambda(x) \tag{2.3.44}
\]

(b)

\[
s_{\emptyset/\nu}(x) = \begin{cases} 1 & \text{if } \nu = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.45}
\]

**Proof.** We can verify the properties on the Schur polynomial. Consider \( x = (x_1, \ldots, x_N) \). Remind that \( s_\lambda(x_1, \ldots, x_N, 0) = s_\lambda(x_1, \ldots, x_N) \) for any \( \ell(\lambda) \leq N \).
2.3. SYMMETRIC FUNCTIONS

(a) Thus we have

\[ s_\lambda(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_N, 0) = \sum_{\mu \in \mathcal{Y}} s_{\lambda/\mu}(x_1, \ldots, x_N) s_\mu(0) \]

\[ = s_{\lambda/\emptyset}(x_1, \ldots, x_N) \cdot 1, \]  

(2.3.46)

since \( s_\mu(0) = \begin{cases} 1 & \text{if } \mu = \emptyset, \\ 0 & \text{if } \mu \neq \emptyset. \end{cases} \)

(b) By definition

\[ 1 = s_\emptyset(x_1, \ldots, x_N; y_1, \ldots, y_N) \]

\[ = \sum_{\mu \in \mathcal{Y}} s_{\emptyset/\mu}(x_1, \ldots, x_N) s_\mu(y_1, \ldots, y_N) \]

\[ = \sum_{\mu \in \mathcal{Y}} s_{\emptyset/\mu}(x_1, \ldots, x_N) s_\mu(y_1, \ldots, y_N) + s_\emptyset(x_1, \ldots, x_N) s_\emptyset(y_1, \ldots, y_N). \]

(2.3.47)

But since \( s_\emptyset(y) = 1 \) for any choice of \( y \), then \( s_{\emptyset/\mu}(x_1, \ldots, x_N) = 0. \)

Remark 2.3.18. The consistency can be generalized to \( n \) sets of variables \( x^{(1)}, \ldots, x^{(n)} \) as

\[ s_{\lambda/\mu}(x^{(1)}, \ldots, x^{(n)}) = \sum_{\mu = \mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n)} = \lambda} \prod_{i=1}^n s_{\mu^{(i)}/\mu^{(i-1)}}(x^{(i)}) \]

\[ = \sum_{\mu = \mu^{(0)} \subseteq \mu^{(1)} \subseteq \ldots \subseteq \mu^{(n)} = \lambda} \prod_{i=1}^n s_{\mu^{(i)}/\mu^{(i-1)}}(x^{(i)}), \]  

(2.3.48)

since \( s_{\lambda/\mu} = 0 \) if \( \mu \not\subseteq \lambda \) (see Corollary 2.3.21 below). Furthermore, taking \( \mu = \emptyset \), then

\[ s_\lambda(x^{(1)}, \ldots, x^{(n)}) = \sum_{\emptyset = \emptyset^{(0)} \subseteq \emptyset^{(1)} \subseteq \ldots \subseteq \emptyset^{(n)} = \lambda} s_{\emptyset^{(1)}(x^{(1)})/\emptyset^{(1)}}(x^{(2)}) \ldots s_{\emptyset^{(n)}/\emptyset^{(n-1)}}(x^{(n)}). \]

(2.3.49)

Remark 2.3.19. The notations are somehow unusual from a probabilistic point of view. If one would call \( \pi_0(\mu) = s_\mu(x) \) the “probability” measure parameterized by \( x \) of \( \mu \) (to be normalized), and do a one-step in a Markov chain with transition probability \( P(\mu, \lambda) = s_{\lambda/\mu}(y) \) from state \( \lambda \) to state \( \mu \), given in terms of \( y \), then the distribution of \( \lambda \) is given by \( \pi_1(\lambda) = \sum_{\mu \in \mathcal{Y}} \pi_0(\mu) P(\mu, \lambda) = \sum_{\mu \in \mathcal{Y}} s_\mu(x) s_{\lambda/\mu}(y) = s_\lambda(x, y). \) In particular, \( s_\lambda(x^{(1)}, \ldots, x^{(n)}) \) is going to be the probability measure of \( \lambda \in \mathcal{Y} \) of a Markov chain started from the empty partition after \( n \) time steps, where the transition probability from time \( \nu^{(n)} \) at time \( n \) to \( \nu^{(n+1)} \) at time \( n+1 \) is given by \( s_{\nu^{(n+1)}/\nu^{(n)}}(x^{(n+1)}). \)

The skew-Schur functions can also be expressed in terms of the complete homogeneous functions 2.3.35:

Proposition 2.3.20. We define \( h_k = 0 \) for \( k < 0 \). Then

\[ s_{\lambda/\mu} = \det \left[ h_{\lambda_i - \mu_j - i + j} \right]_{1 \leq i, j \leq \max \{\ell(\lambda), \ell(\mu)\}}. \]  

(2.3.50)

Proof. The proof is an extension of one used to show the first equality in Proposition 2.3.10 see [17], Chap. 1, Sect. 5. for further details. \( \square \)
A consequence of this formula is the following

**Corollary 2.3.21.**

(a) \( s_{\lambda / \mu} = 0 \) unless \( \mu \subseteq \lambda \), i.e. \( \mu_i \leq \lambda_i \) for all \( i \geq 1 \).

(b) If \( \mu \subseteq \lambda \), let \( \theta = \lambda - \mu \) be the skew diagram. Then

\[
 s_{\lambda / \mu} = \prod_{i \geq 1} s_{\theta^i}. \tag{2.3.51}
\]

**Proof.**

(a) If \( \mu \not\subseteq \lambda \), for some \( k \) then \( \lambda_k \leq \mu_k \). From this follows that for all \( j \leq k \leq i \) we have \( \lambda_i \leq \lambda_k \leq \mu_k \leq \mu_j \), which implies \( \lambda_i - \mu_j - i + j < 0 \). So \( s_{\lambda / \mu} \) is the determinant of a matrix of the form

\[
 s_{\lambda / \mu} = \det \begin{bmatrix} A & B \\ O_{(n-k+1)\times k} & C \end{bmatrix} \tag{2.3.52}
\]

where \( A, B \) and \( C \) are non-zero blocks and \( O_{(n-k+1)\times k} \) is a zero block of size \( \frac{n}{2} \times \frac{n}{2} \)-matrices. However this is not possible since then \( k \) should be equal to \( \frac{n}{2} \) and \( \frac{n+1}{2} \) simultaneously.

(b) If \( \mu_k \geq \lambda_{k+1} \) for some \( k < n = \ell(\lambda) \), then \( s_{\lambda / \mu} \) is of the form

\[
 s_{\lambda / \mu} = \det \begin{bmatrix} A_{k \times k} \\ O_{(n-k)\times(n-k)} & B \\ C_{(n-k)\times(n-k)} \end{bmatrix} = \det [A] \det [C]. \tag{2.3.53}
\]

Thus if \( \lambda - \mu = (\theta^1, \theta^2) \), then \( \det [h_{\lambda_i-\mu_j-i+j}] = s_{\theta^1} \cdot s_{\theta^2} \). The extension to a general \( \mu \subseteq \lambda \) follows quite easily.

We give an example to illustrate Corollary 2.3.21. Consider \( \lambda = (9, 5, 3, 2) \), \( \mu = (4, 4, 2, 1) \). The skew diagram is given by \( \theta^1 = (5, 1) \), \( \theta^2 = (1) \), \( \theta^3 = (1) \). The table of \( h_{\lambda_i-\mu_j-i+j} \) is

<table>
<thead>
<tr>
<th>( i \setminus j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td>-2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>-5</td>
<td>-4</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

and \( s_{\lambda / \mu} \) is given by

\[
 s_{\lambda / \mu} = \det \begin{bmatrix} h_5 & h_6 & h_9 & h_{11} \\ h_0 & h_3 & h_4 & h_6 \\ 0 & 0 & h_1 & h_3 \\ 0 & 0 & 0 & h_1 \end{bmatrix} \tag{2.3.54}
\]

\[
 = \det \begin{bmatrix} h_5 & h_6 \\ h_0 & h_1 \end{bmatrix} \cdot \det [h_1] \cdot \det [h_1].
\]

We recall that \( s_\lambda = \det [h_{\lambda_i-i+j}]_{1 \leq i,j \leq \ell(\lambda)} \). So \( s_{\theta^1} = \det \begin{bmatrix} h_5 & h_6 \\ h_0 & h_1 \end{bmatrix} \), \( s_{\theta^2} = s_{\theta^3} = h_1 \) and we have \( s_{\lambda / \mu} = s_{\theta^1} \cdot s_{\theta^2} \cdot s_{\theta^3} \).
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Let us specialize the skew Schur functions to a special case, \( x = (\alpha, 0, 0, \ldots) = (\alpha) \). From Corollary 2.3.21 (b),

\[
sl/\mu = \det \left[ h_{\lambda_i - \mu_j - 1 + 1} \right]_{1 \leq i, j \leq \max(\ell(\lambda), \ell(\mu))} = \prod_{i \geq 1} s_{\theta_i}.
\] (2.3.55)

**Proposition 2.3.22.** For \( x = (\alpha) \), \( s_\lambda(x) = 0 \) unless \( \lambda \) is a one-row Young diagram, i.e. \( \ell(\lambda) = 1 \). More generally, \( s_{\lambda/\mu}(x) = 0 \) unless \( \lambda/\mu \) is a horizontal strip; in the latter case, \( s_{\lambda/\mu}(x) = a(\lambda|-|\mu|) \).

**Proof.** We know that, for \( \mu \subset \lambda \), \( s_{\lambda/\mu} = \prod_{i \geq 1} s_{\theta_i} \).

Assume first that \( \theta \) is not a one-row diagram. Then

\[
s_{\theta}(\alpha) = \det \begin{bmatrix} h_{\theta_1} & h_{\theta_2-1} & \cdots \\ h_{\theta_1+1} & h_{\theta_2} & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & & \ddots \\ \vdots & & & h_{\theta_n} \end{bmatrix}
\] (2.3.56)

Since \( h_k(\alpha) = \alpha^k \),

\[
s_{\theta}(\alpha) = \det \begin{bmatrix} \alpha^{\theta_1} & \alpha^{\theta_2-1} & \cdots \\ \alpha^{\theta_1+1} & \alpha^{\theta_2} & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & & \ddots \\ \vdots & & & \alpha^{\theta_n} \end{bmatrix},
\] (2.3.57)

which is zero for \( n \geq 2 \).

If instead \( \theta \) is a one-row diagram, then \( s_{\theta}(\alpha) = \alpha^\theta \). If \( \lambda/\mu \) is a horizontal strip, then \( s_{\lambda/\mu}(\alpha) = \prod_{i \geq 1} (\alpha^{\lambda_i - \mu_i} = \alpha^{\lambda|-|\mu|}; \) otherwise \( s_{\lambda/\mu}(\alpha) = 0 \).

The skew Cauchy function of Proposition 2.3.22 is exactly the transition weight in our PNG model. For the PNG model at, say time \( t = n = 3 \) in terms of partition we have the following picture

![Diagram](image)

and the weight of a configuration is given by

\[
s_{\lambda(1)}(a_1)s_{\lambda(1)/\mu(1)}(b_3)s_{\lambda(2)/\mu(1)}(a_2)s_{\lambda(2)/\mu(1)}(a_2)s_{\lambda(3)/\mu(2)}(b_2)s_{\lambda(3)/\mu(2)}(a_3)s_{\lambda(3)}(b_1).
\]

If we want to determine the distribution of \( \lambda(2) \), then we need to compute

\[
\frac{1}{Z} \sum_{\lambda(1), \mu(1), \mu(2), \lambda(3)} s_{\lambda(1)}(a_1)s_{\lambda(1)/\mu(1)}(b_3)s_{\lambda(2)/\mu(1)}(a_2)s_{\lambda(2)/\mu(1)}(a_2)s_{\lambda(3)/\mu(2)}(b_2)s_{\lambda(3)/\mu(2)}(a_3)s_{\lambda(3)}(b_1)
\] (2.3.58)

where

\[
Z = \sum_{\lambda(1), \mu(1), \lambda(3)} s_{\lambda(1)}(a_1)s_{\lambda(1)/\mu(1)}(b_3)s_{\lambda(2)/\mu(1)}(a_2)s_{\lambda(2)/\mu(2)}(b_2)s_{\lambda(3)/\mu(2)}(a_3)s_{\lambda(3)}(b_1).
\] (2.3.59)

To achieve this, we use a generalization of the Cauchy identity for the skew-Schur functions that will be a key identity as well.
CHAPTER 2. THE POLYNUCLEAR GROWTH MODEL

**Proposition 2.3.23** (Skew Cauchy Identity). For any $\lambda, \nu \in Y$,

$$
\sum_{\mu \in Y} s_{\mu/\lambda}(x)s_{\mu/\nu}(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} \sum_{\kappa \in Y} s_{\lambda/\kappa}(y)s_{\nu/\kappa}(x).
$$

(2.3.60)

To prove it, first we need another relation.

**Lemma 2.3.24.** Let us set the coefficients $c^{\lambda}_{\mu \nu}$ s.t.

$$
s_{\lambda/\mu}(x) = \sum_{\nu \in Y} c^{\lambda}_{\mu \nu} s_{\nu}(x).
$$

(2.3.61)

Then

$$
s_{\mu}(x)s_{\nu}(x) = \sum_{\lambda \in Y} c^{\lambda}_{\mu \nu} s_{\lambda}(x),
$$

(2.3.62)

which also gives $c^{\lambda}_{\mu \nu} = c^{\lambda}_{\nu \mu}$.

**Proof.** Let

$$
P(x,y) := \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}.
$$

(2.3.63)

Let $d^{\lambda}_{\mu \nu}$ s.t. $s_{\mu}(z)s_{\nu}(z) = \sum_{\lambda \in Y} d^{\lambda}_{\mu \nu} s_{\lambda}(z)$.

Using the Cauchy Identity (2.3.27), we can write, for $x, y, z$ three sequences of variables,

$$
P(x,z)P(y,z) = P((x,y),z) = \sum_{\lambda \in Y} s_{\lambda}(x,y)s_{\lambda}(z)
$$

$$
= \sum_{\lambda, \mu \in Y} s_{\lambda/\mu}(x)s_{\mu}(y)s_{\lambda}(z)
$$

$$
= \sum_{\lambda, \mu, \nu \in Y} c^{\lambda}_{\mu \nu} s_{\lambda}(z)s_{\mu}(x)s_{\nu}(y)
$$

(2.3.64)

But also

$$
P(x,z)P(y,z) = \sum_{\mu, \nu \in Y} s_{\mu}(y)s_{\mu}(z)s_{\nu}(x)s_{\nu}(z)
$$

$$
= \sum_{\mu, \nu \in Y} s_{\mu}(y)s_{\mu}(x) \sum_{\lambda \in Y} d^{\lambda}_{\mu \nu} s_{\lambda}(z).
$$

(2.3.65)

Since the relations (2.3.64), (2.3.65) are valid for any $x, y, z$, we have $d^{\lambda}_{\mu \nu} = c^{\lambda}_{\mu \nu}$.

**Proof of Theorem 2.3.23.** Let $P(x,y)$ as (2.3.63). Applying twice the Cauchy identity, see Theorem 2.3.11 and Lemma 2.3.24, we get

$$
P(x,y)P(x,u)P(z,y)P(z,u) = P((x,z),y)P((x,z),u)
$$

$$
= P((x,z),(y,u)) = \sum_{\rho \in Y} s_{\rho}(x,z)s_{\rho}(y,u)
$$

(2.3.66)

$$
= \sum_{\rho, \lambda, \mu \in Y} s_{\rho/\lambda}(x)s_{\lambda}(z)s_{\rho/\mu}(y)s_{\mu}(u),
$$
by definition of the skew Schur function \((2.3.40)\). But \((2.3.66)\) is also equal to
\[
(2.3.66) = P(x, y) \sum_{\sigma, \mu, \tau \in \mathcal{Y}} s_{\sigma}(x)s_{\sigma}(u)s_{\nu}(y)s_{\nu}(z)s_{\tau}(z)s_{\tau}(u)
\]
\[
= P(x, y) \sum_{\sigma, \mu, \tau \in \mathcal{Y}} s_{\sigma}(x)s_{\nu}(y)c_{\sigma, \tau}^\lambda s_{\mu}(u)c_{\tau, \lambda}(z)
\]
\[
= P(x, y) \sum_{\tau, \lambda, \mu \in \mathcal{Y}} s_{\mu/\tau}(x)s_{\lambda/\tau}(y)s_{\mu}(u)s_{\lambda}(z),
\]
where in the last passage we have summed over \(\sigma\) and \(\nu\) using \((2.3.61)\).

Since the identities \((2.3.66)\) and \((2.3.67)\) are valid \(\forall u, z\), combining them we obtained
\[
P(x, y) \sum_{\tau \in \mathcal{Y}} s_{\mu/\tau}(x)s_{\lambda/\tau}(y) = \sum_{\rho \in \mathcal{Y}} s_{\rho/\lambda}(x)s_{\rho/\lambda}(y).
\]

\(\square\)

**Proposition 2.3.25.** The normalization constant in the PNG model at time \(t = n\), defined by
\[
Z(\vec{a}, \vec{b}) = \sum_{\lambda(1), \ldots, \lambda(n)} s_{\lambda(1)}(a_1)s_{\lambda(2)/\mu(1)}(b_n)s_{\lambda(2)/\mu(1)}(a_2)s_{\lambda(3)/\mu(2)}(b_{n-1}) \ldots s_{\lambda(n)/\mu(n-1)}(a_n)s_{\lambda(n)}(b_1),
\]
is given by
\[
Z(\vec{a}, \vec{b}) = \prod_{1 \leq i \leq j \leq n} \frac{1}{1 - a_i b_{n+1-j}}.
\]

**Proof.** By the skew-Cauchy identity \((2.3.60)\)
\[
\sum_{\lambda(1)} s_{\lambda(1)}(a_1)s_{\lambda(1)/\mu(1)}(b_n) = H(a_1; b_n)s_{\mu(1)}(a_1),
\]
with \(H(\vec{a}; \vec{b}) = \prod_{i \geq 1} \frac{1}{1 - a_i b_i}\). So
\[
(2.3.69) = \sum_{\lambda(1), \ldots, \lambda(n)} H(a_1; b_n)s_{\mu(1)}(a_1)s_{\lambda(2)/\mu(1)}(a_2)s_{\lambda(3)/\mu(2)}(b_{n-1}) \ldots s_{\lambda(n)/\mu(n-1)}(a_n)s_{\lambda(n)}(b_1)
\]
\[
= H(a_1; b_n) \sum_{\lambda(2), \ldots, \lambda(n)} s_{\lambda(2)}(a_1, a_2)s_{\lambda(3)/\mu(2)}(b_{n-1})s_{\lambda(3)/\mu(2)}(a_3) \ldots s_{\lambda(n)/\mu(n-1)}(a_n)s_{\lambda(n)}(b_1)
\]
(2.3.71)

since \(\sum_{\mu(1)} s_{\mu(1)}(a_1)s_{\lambda(2)/\mu(1)}(a_2) = s_{\lambda(2)}(a_1, a_2)\). Summing over \(\lambda(2)\),
\[
(2.3.69) = H(a_1; b_n) \sum_{\lambda(3), \ldots, \lambda(n)} H(a_1, a_2; b_{n-1})s_{\mu(2)}(a_1, a_2)s_{\lambda(3)/\mu(2)}(a_3) \ldots s_{\lambda(n)/\mu(n-1)}(a_n)s_{\lambda(n)}(b_1).
\]
(2.3.72)

Then, summing over \(\mu(2)\),
\[
(2.3.69) = H(a_1; b_n)H(a_1, a_2; b_{n-1}) \sum_{\lambda(3), \ldots, \lambda(n)} s_{\lambda(3)}(a_1, a_2, a_3) \ldots s_{\lambda(n)/\mu(n-1)}(a_n)s_{\lambda(n)}(b_1).
\]
(2.3.73)
Iterating, we obtain

\[ Z(\vec{a}, \vec{b}) = \prod_{1 \leq i \leq j \leq n} H(a_i; b_{n+1-j}). \] (2.3.74)

Now we compute the measure at a given point, for example at the origin at time \( n = 2m + 1 \).

**Proposition 2.3.26.** The measure at \( x = 0 \) of the multilayer PNG at time \( n = 2m + 1 \) is given by

\[ s_\lambda(a_1, a_2, \ldots, a_{m+1})s_\lambda(b_1, b_2, \ldots, b_{m+1}) \prod_{i,j=1}^{m+1} \frac{1}{1-a_i b_j} \] (2.3.75)

**Proof.** The measure is given by

\[
\frac{1}{Z(\vec{a}, \vec{b})} \sum_{\lambda^{(1)}, \ldots, \lambda^{(m-1)}; \lambda^{(m)}, \lambda^{(m+1)}; \mu^{(1)}, \ldots, \mu^{(n-1)}} s_{\lambda^{(1)}}(a_1) s_{\lambda^{(1)}}/\mu^{(1)}(b_n) s_{\lambda^{(2)}}/\mu^{(2)}(a_2) \ldots s_{\lambda^{(m)}}(a_{m+1}) \\
\ldots s_{\lambda^{(m)}}/\mu^{(m+1)}(b_{m+1}) \ldots s_{\lambda^{(n)}}/\mu^{(n)}(a_n) s_{\lambda^{(n)}}(b_1). \] (2.3.76)

Repeating the same computation as in the proof of Proposition 2.3.25 and we obtain (2.3.75).

The Schur functions can be defined in a more general algebraic setting: in the next section we introduce the notion of specialization and describe its relation with the Schur functions and the PNG multilayer.

### 2.3.4 Specialization

**Definition 2.3.27.** A specialization is an algebra homomorphism \( \rho : \Lambda \to \mathbb{C} \), \( f \mapsto f(\rho) \), i.e. it has to satisfy:

i. \( (f + g)(\rho) = f(\rho) + g(\rho) \),

ii. \( (fg)(\rho) = f(\rho)g(\rho) \),

iii. \( (\theta f)(\rho) = \theta f(\rho), \forall \theta \in \mathbb{C} \).

**Example 2.3.28.** Given any sequence of complex numbers \( u_1, u_2, \ldots \) s.t. \( \sum_i |u_i| < \infty \), the substitution map \( \Lambda \to \mathbb{C}, x_i \mapsto u_i \) is a specialization.

In general, any specialization is uniquely determined by its values on any set of generators of \( \Lambda \). If the generators are algebraically independent, these values can be any number. This means that defining \( \rho \) is equivalent to setting the numbers \( \{p_1(\rho), p_2(\rho), \ldots\} \), \( \{e_1(\rho), e_2(\rho), \ldots\} \) or \( \{h_1(\rho), h_2(\rho), \ldots\} \).

**Example 2.3.29.** If \( \rho \) is the substitution map in Example 2.3.28 then the specialization

\[ p_k \mapsto p_k(\rho) = \sum_i (u_i)^k \] (2.3.77)

is well defined by the condition \( \sum_i |u_i| < \infty \).

As in the PNG model, to have a probability measure it is important to know which specializations are positive in a certain sense.
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Definition 2.3.30. If for every Young diagram $\lambda$,
\[ s_\lambda(\rho) \geq 0, \]  
(2.3.78)
we call $\rho$ a Schur-positive specialization.

One can show that if $\rho$ is a Schur-positive specialization, then also $s_{\lambda/\mu}(\rho) \geq 0$ for any $\lambda, \mu \in \mathcal{Y}$.

There is an explicit characterization of the Schur-positive specializations.

Theorem 2.3.31. The Schur-positive specializations are parametrized by a parameter $\gamma \geq 0$ and two sequences of non-negative real numbers $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0)$ and $\beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0)$ s.t. $\sum_{i \geq 1} (\alpha_i + \beta_i) < \infty$.
The specialization with parameters $(\alpha; \beta; \gamma)$ can be described by its values on power sums
\[ p_1 \mapsto p_1(\alpha; \beta; \gamma) = \gamma + \sum_{i \geq 1} (\alpha_i + \beta_i), \]  
(2.3.79)
\[ p_k \mapsto p_k(\alpha; \beta; \gamma) = \sum_{i \geq 2} (\alpha_i^k + (-1)^{k-1}\beta_i^k), k \geq 2; \]  
(2.3.80)
or equivalently by the generating functions
\[ \sum_{k \geq 0} h_k(\alpha; \beta; \gamma) z^k = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}, \]  
(2.3.81)

Proof. We verify only the equivalence of the two characterizations.

\[ \sum_{k \geq 0} h_k(\alpha; \beta; \gamma) z^k = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z} \]  
(2.3.82)
since $\sum_{k \geq 1} \frac{z^k}{k} \alpha_i - \sum_{k \geq 1} \frac{z^k}{k} (-\beta_i)^k = -\ln(1 - \alpha_i z) + \ln(1 + \beta_i z)$. \hfill $\square$

Remark 2.3.32. The case $(0; 0; \gamma)$ can be seen as a limit of $\alpha$’s or $\beta$’s; indeed, if $\beta_M = (\gamma_M, \gamma_M, \ldots, \gamma_M, 0, 0, \ldots)$, then
\[ \sum_{k \geq 0} h_k(0; \beta_M; 0) z^k \stackrel{M \to \infty}{\to} \sum_{k \geq 0} h_k(0; 0; \gamma) z^k, \]  
(2.3.83)
because $(1 + \frac{\gamma_M}{M})^M \to e^{\gamma}$ for $M \to \infty$.

We have already seen a special case, the one with $H(z) = \frac{1}{1 - \alpha z}$ which appears in the PNG model, see Proposition 2.3.22.

Proposition 2.3.33. Suppose $\rho = ((a, 0, 0, \ldots); 0; 0)$. Then $s_{\lambda/\mu}(\rho) = 0$ unless $\lambda/\mu$ is a horizontal strip; in which case $s_{\lambda/\mu}(\rho) = a^{(|\lambda| - |\mu|)}$.

Now we want to see what happens if $\beta = (b, 0, 0, \ldots)$ and the other parameters are zero.
Proposition 2.3.34. Suppose \( \rho = (0; (b, 0, 0, \ldots); 0) \). Then \( s_{\lambda/\mu}(\rho) = 0 \) unless \( \alpha/\beta \) is a vertical strip; in which case \( s_{\lambda/\mu}(\rho) = b^{\lambda - |\mu|} \).

Proof. We have

\[
H(z) = \sum_{k \geq 0} h_k z^k = 1 + \beta z,
\]
that is \( h_1(\rho) = \beta \) and \( h_k(\rho) = 0 \), for \( k \geq 2 \). By definition

\[
s_{\lambda/\mu}(\rho) = \det [h_{\lambda_i - \mu_j - 1}(\rho)]_{1 \leq i, j \leq \max(\ell(\lambda), \ell(\mu))}
\]

\[
= \det \begin{bmatrix}
h_{\lambda_1 - \mu_1 - 1}(\rho) & h_{\lambda_1 - \mu_2 + 1}(\rho) & h_{\lambda_1 - \mu_3 + 2}(\rho) & \cdots \\
h_{\lambda_2 - \mu_1 - 1}(\rho) & h_{\lambda_2 - \mu_2}(\rho) & h_{\lambda_2 - \mu_3 + 1}(\rho) & \cdots \\
h_{\lambda_3 - \mu_1 - 2}(\rho) & h_{\lambda_3 - \mu_2 - 1}(\rho) & h_{\lambda_3 - \mu_3}(\rho) & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \cdots & \ddots
\end{bmatrix} = 0,
\]

since the block of zero is for top-right from \((i, i)\). Similarly one shows that if \( \lambda_i - \mu_i < 0 \) for some \( i \), then \( s_{\lambda/\mu}(\rho) = 0 \). Thus \( s_{\lambda/\mu}(\rho) \) can be different from 0 only if \( \lambda_i - \mu_i \in \{0, 1\}, \forall i \).

Now, if \( \lambda_i - \mu_i = 1 \), then the column above \((i, i)\) and the row to the right have \( h_k(\rho) = 0 \) for \( k \geq 2 \). If \( \lambda_i - \mu_i = 0 \), then the row to the left of \((i, i)\) and the column below have \( h_k(\rho) = 0 \) for \( k < 0 \). So

\[
s_{\lambda/\mu}(\rho) = \det \begin{bmatrix}
h_{\lambda_1 - \mu_1}(\rho) & * & 0 & \cdots & 0 \\
0 & h_{\lambda_2 - \mu_2}(\rho) & * & \cdots & 0 \\
0 & 0 & h_{\lambda_3 - \mu_3}(\rho) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & h_{\lambda_n - \mu_n}(\rho)
\end{bmatrix}
\]

which means that \( h_{\lambda_i - \mu_i}(\rho) \) has zeros either on its right or below it \( \forall i \). Then

\[
s_{\lambda/\mu}(\rho) = h_{\lambda_1 - \mu_1}(\rho) \cdot h_{\lambda_2 - \mu_2}(\rho) \cdots h_{\lambda_n - \mu_n}(\rho)
\]

\[
= \prod_{i=1}^{n} k_{\lambda_i - \mu_i} = b^{\lambda - |\mu|},
\]

provided that \( \lambda/\mu \) is a vertical strip. \( \square \)

Finally we want to understand what it means to do a composition of two Schur-positive specializations.

Definition 2.3.35. Given two specializations \( \rho_1, \rho_2 \), their union \( (\rho_1, \rho_2) \) is defined through its values on the power sums \( p_k \):

\[
p_k(\rho_1, \rho_2) := p_k(\rho_1) + p_k(\rho_2).
\]
Proposition 2.3.36. If $\rho^{(1)} = (\alpha^{(1)}; \beta^{(1)}; \gamma^{(1)})$ and $\rho^{(2)} = (\alpha^{(2)}; \beta^{(2)}; \gamma^{(2)})$ are Schur-positive specializations, then $(\rho^{(1)}, \rho^{(2)})$ is a Schur-positive specialization with parameters $(\alpha^{(1)} \cup \alpha^{(2)}; \beta^{(1)} \cup \beta^{(2)}; \gamma^{(1)} + \gamma^{(2)})$.

Proof. The proof is a direct consequence of Theorem 2.3.31. 

Remark 2.3.37. If $\rho_1$ and $\rho_2$ specialize symmetric functions by substituting sets of variables $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$, then $(\rho_1, \rho_2)$ substitutes all the variables $(x_1, x_2, \ldots, y_1, y_2, \ldots)$.

To conclude the section about specializations, we explain the relation with the transition weights in the line ensembles.

We have already seen that an $\alpha$-specialization, i.e., $s_{\lambda/\mu}((a, 0, 0, \ldots); 0; 0)$ correspond to the LGV picture represented in Figure 2.10. It describes the evolution of a particle performing geometric jumps.

$$\begin{align*}
\lambda_1 \\
\mu_1 \\
\lambda_2 - 1 \\
\mu_2 - 1 \\
\mu_3 - 2 \\
\lambda_3 - 2
\end{align*}$$

Figure 2.10: LGV diagram corresponding to the specialization $s_{\lambda/\mu}((a, 0, 0, \ldots); 0; 0)$. The vertical segments have weight $a$, the horizontal ones weight 1.

For a $\beta$-specialization, i.e., $s_{\lambda/\mu}(0; (b, 0, 0, \ldots); 0)$, the corresponding LGV diagram is illustrated in Figure 2.11. It describes a particle performing a one-sided random walk.

$$\begin{align*}
\mu_1 \\
\lambda_1 \\
\lambda_2 - 1 \\
\mu_2 - 1 \\
\lambda_3 - 2 \\
\mu_3 - 2
\end{align*}$$

Figure 2.11: LGV diagram corresponding to the specialization $s_{\lambda/\mu}(0; (b, 0, 0, \ldots); 0)$. The diagonal segments have weight $b$, the horizontal ones weight 1.

But what happens for a composition? Consider for example $\rho = ((a_1, a_2, 0, \ldots); 0; 0)$. By definition

$$s_{\lambda/\mu}((a_1, a_2, 0, \ldots); 0; 0)) = \sum_{\nu \in Y} s_{\lambda/\nu}((a_2, 0, \ldots); 0; 0)) a_{\nu/\mu}((a_1, 0, \ldots); 0; 0)). \quad (2.3.90)$$

The LGV diagram is illustrated in Figure 2.12.
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Figure 2.12: LGV diagram corresponding to the composition of two \( \alpha \)-specialization, \(((a_1, 0, \ldots); 0; 0)\) and \(((a_2, 0, \ldots); 0; 0)\). The vertical weights between \( \mu \) and \( \nu \) have weight \( a_1 \), while between \( \nu \) and \( \lambda \) have weight \( a_2 \).

Remark 2.3.38. The positivity of \( \alpha, \beta, \gamma \) specializations are immediate from the interpretation in terms of non-intersecting line ensembles. Also, one can think at the \( \beta \) specialization as the "particle-hole" transformation of a \( \alpha \) specialization. Finally, a \( \gamma \)-specialization can be seen as the continuous limit of a \( \beta \)-specialization (or \( \alpha \)). Let the one-particles do one-sided simple random walks with rate 1, then \( s_{\lambda/\mu}(0; 0; \gamma) \) gives the transition weights for non-intersecting one-sided random walks during the time \([0, \gamma]\).

For the \( \gamma \) specialization there is also a combinatorial interpretation, to which we might come back later.

Proposition 2.3.39. We have

\[
s_{\lambda}(0; 0; \gamma) = \frac{\gamma^{\lambda}}{|\lambda|!} \dim(\lambda), \lambda \in \mathcal{Y}, \tag{2.3.91}
\]

where \( \dim(\lambda) \) is the number of standard Young tableaux of shape \( \lambda \), i.e. the number of ways of putting numbers \( \{1, 2, \ldots, N = |\lambda|\} \) inside the Young diagram \( \lambda \) s.t. the numbers strictly increase both along columns and rows.
Chapter 3

Point Processes

3.1 General point processes

In short, point processes are random point measures. Consider a state space or a one particle space $\Lambda$, in general a separable space, but we will usually choose Euclidean or discrete spaces as $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{Z}^d$ or $\mathbb{R} \times \{1, \ldots, N\}$. A point configuration $x = (x_i)_{i \in I}$, $x_i \in \Lambda$, $I \subset \mathbb{N}$ is called locally finite if for every bounded set $B \subset \Lambda$, $\xi(B) = \# \{x_i \in B\} < \infty$. Let $\Omega$ be the set of locally finite particle configurations, $F$ be the $\sigma$-algebra generated by the cylinder sets $C^B_n = \{ \xi \in \Omega | \xi(B) = n \}$, $n \geq 0$, $B \subset \Lambda$ bounded. Finally let $P$ be a probability measure on $(\Omega, \Lambda)$. In this way we have a probability space $(\Omega, F, P)$.

Now we define the space of point measures. Let $B(\Lambda)$ be the Borel $\sigma$-algebra of $\Lambda$. A point measure on $\Lambda$ is a positive measure $\nu$ on $(\Lambda, B(\Lambda))$, which is a locally finite sum of Dirac measures, i.e., $\nu = \sum_{i \in I} \delta_{x_i}$, $I \subset \mathbb{N}$, and for every bounded set $B \subset \Lambda$, $\nu(\mathbb{I}_B) = \# \{x_i \in B\} < \infty$. Denote with $M_p(\Lambda)$ the set of all positive measures on $\Lambda$ and with $M_p(\Lambda)$ the $\sigma$-algebra generated by the mappings from $M_p(\Lambda)$ to $\mathbb{N} \cup \{\infty\}$, $\nu \mapsto \nu(f)$, when $f$ spans $B(\Lambda)$.

**Definition 3.1.1.** A point process $\eta$ in $\Lambda$ is a measurable mapping from $(\Omega, F, P)$ to $(M_p(\Lambda), M_p(\Lambda))$. The law of $\eta$ is $P \circ \eta^{-1}$.

We say that $\eta$ is a simple point process if $P(\eta(\{x\}) \leq 1, \forall x \in \Lambda) = 1$. Given a set $A \subset \Lambda$, we denote the measure of the set as $\eta(A) := \eta(\mathbb{1}_A)$.

**Example 3.1.2.** The most known example of a point process is the homogeneous Poisson point process (PPP) $\eta$ on $\Lambda = \mathbb{R}^d$ with intensity $\rho$. A configuration $x = (x_i)_i$ is a set of points in $\Lambda$. For any bounded subsets $B \in \Lambda$ and $n \geq 0$,

$$P(C^B_n) = \frac{(\rho|B|)^n}{n!} e^{-\rho|B|}. \quad (3.1.1)$$

Moreover for any disjoint sets $B, \tilde{B}$, and for any $n, \tilde{n} \geq 0$,

$$P(C^B_n \cap C^{\tilde{B}}_{\tilde{n}}) = P(C^B_n)P(C^{\tilde{B}}_{\tilde{n}}). \quad (3.1.2)$$

Then, the Poisson point process is given by $\eta = \sum_i \delta_{x_i}$.

**Example 3.1.3.** Take $\Lambda = \mathbb{Z}$ and the measure of the multilayer PNG in the origin (2.3.75) and let $H_i = \lambda_i - i + 1$ with $\lambda = (\lambda_1, \lambda_2, \ldots)$ being distributed as const $\times s_\lambda(p_1)s_\lambda(p_2)$. Then the point process associated with the random partitions is the random measure

$$\eta = \sum_{i \geq 1} \delta_{H_i}. \quad (3.1.3)$$
3.1.1 The \( n \)-point correlation functions

In order to describe the point process in some region, we do not need to necessarily know all the details everywhere, but we need to know local information. These are encoded in the so-called correlation functions, which we are going to define below.

Let \( \eta \) be a point process. For any \( A_1, \ldots, A_n \) disjoint Borel sets of \( \Lambda \), define

\[
M_n(A_1, \ldots, A_n) = \mathbb{E} \left[ \prod_{i=1}^{n} \eta(\mathbb{1}_{A_i}) \right],
\]

while for generic \( A_1, \ldots, A_n \in \mathcal{B}(\Lambda) \),

\[
M_n(A_1, \ldots, A_n) = \mathbb{E} \left[ \sum_{(x_1, \ldots, x_n) \in \text{supp} \eta} \prod_{i=1}^{n} \mathbb{1}_{A_i}(x_i) \right].
\]

**Definition 3.1.4.** If \( M_n \) is absolutely continuous with respect to a reference measure \( \mu \otimes n \) on \( \Lambda \), i.e., if for any \( A_1, \ldots, A_n \in \mathcal{B}(\Lambda) \),

\[
M_n(A_1, \ldots, A_n) = \int_{A_1 \times \cdots \times A_n} d\mu(x_1) \cdots d\mu(x_n) \rho^{(n)}(x_1, \ldots, x_n),
\]

for some function \( \rho^{(n)} \), then \( \rho^{(n)} \) is called the \( n \)-point correlation function (with respect to the measure \( \mu \)).

We list a few properties of the correlation function:

(a) \( \rho^{(n)} \) is symmetric with respect to permutations of the \( x_i \)'s, i.e.

\[
\rho^{(n)}(x_1, \ldots, x_n) = \rho^{(n)}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \forall \sigma \in S_n.
\]

(b) If there exists a constant \( C \in (0, \infty) \) s.t.

\[
\rho^{(n)}(x_1, \ldots, x_n) \leq n^{2n} C^n,
\]

then \( \{ \rho^{(n)}, n \geq 1 \} \) defines uniquely a point process.

(c) For simple point processes, it is possible to give a probabilistic interpretation:

- If \( \Lambda = \mathbb{R} \) and \( \mu \) is the Lebesgue measure, then

\[
\rho^{(n)}(x_1, \ldots, x_n) = \lim_{\delta_i \to 0} \frac{\mathbb{P} \left( \text{there is a point in } [x_i, x_i + \delta_i], i = 1, \ldots, n \right)}{\delta_1 \cdots \delta_n}.
\]

- If \( \Lambda = \mathbb{Z} \) and \( \mu \) is the counting measure, then

\[
\rho^{(n)}(x_1, \ldots, x_n) = \mathbb{P} \left( \text{there is a point in each } x_1, \ldots, x_n \right).
\]

Notice that \( \rho^{(n)} \) is not a probability density, since it is not normalized to 1.

Another important property of the correlation measures is obtained by considering the test functions \( f \) given by products of characteristic functions of a set \( A \subset \Lambda \): this gives a relation with factorial moments.

**Lemma 3.1.5.** Let \( A \subset \Lambda \). Then

\[
\int_{A^n} d\mu(x_1) \cdots d\mu(x_n) \rho^{(n)}(x_1, \ldots, x_n) = \mathbb{E} \left[ \frac{\eta(\mathbb{1}_A)!}{(\eta(\mathbb{1}_A) - n)!} \right]
\]

(3.1.11)
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Proof. We prove the lemma for $\Lambda$ discrete and $\mu$ the counting measure. If $n = 1$,

$$E[\eta(1_A)] = E\left[\sum_i 1_{A_i}(x_i)\right] = E\left[\sum_{y \in A} 1_{A_y}(x_1)\right] = \sum_{y \in A} \rho^{(1)}(y). \tag{3.1.12}$$

If $n = 2$,

$$\eta(1_A)(\eta(1_A) - 1) = \sum_i 1_{x_i \in A} \sum_{i \neq j} 1_{x_j \in A}, \tag{3.1.13}$$

since if $x_i \in A$, then $\sum_{i \neq j} 1_{x_i \in A} = \eta(1_A) - 1$, while when $x_i \not\in A$, then the sum is zero. So

$$E[\eta(1_A)(\eta(1_A) - 1)] = E\left[\sum_{i \neq j} 1_{x_i \in A} 1_{x_j \in A}\right] = \sum_{y_1, y_2 \in A} E\left[\sum_{i \neq j} 1_{y_i}(x_1) 1_{y_2}(x_j)\right] = \sum_{y_1, y_2 \in A} \rho^{(2)}(y_1, y_2). \tag{3.1.14}$$

This computation can be easily generalized for any $n > 2$. \hfill $\square$

For a random point process with fixed number of particles, say $N$, described by a symmetric joint probability distribution $P_N(dx_1, ..., dx_N)$, then correlation measures are given as follows.

Lemma 3.1.6. Let $P_N$ be a symmetric density on $\Lambda^N$ w.r.t. $\mu^{\otimes N}$, i.e., the probability measure on $\Lambda^N$ is given by $P_N(x_1, \cdots, x_N) d\mu(x_1) \cdots d\mu(x_N)$. Then

$$\rho^{(n)}(x_1, \ldots, x_n) = \begin{cases} \binom{N}{n} \int_{\Lambda^{N-n}} P_N(x_1, \cdots, x_n) d\mu(x_{n+1}) \cdots d\mu(x_N), & n \leq N \\ 0, & n > N. \end{cases} \tag{3.1.15}$$

Proof. For Borel sets $A_1, \ldots, A_n \subset B(\Lambda)$ and a point process $\eta = \sum_i \delta_{z_i}$ on $\Lambda$,

$$M_n(A_1, \ldots, A_n) = E\left[\sum_{z_1, \ldots, z_n} \prod_{i=1}^n 1_{A_i}(z_i)\right] = \int_{\Lambda^N} P_N(x_1, \ldots, x_N) \sum_{z_1, \ldots, z_n} \prod_{i=1}^n 1_{A_i}(z_i) d\mu(x_1) \cdots d\mu(x_N). \tag{3.1.16}$$

Since there are $\binom{N}{n}$ ways of choosing $n$ out of the $x_1, \ldots, x_N$ and $n!$ ways of placing one at each $z_i$,

$$\rho^{(n)} = \frac{N!}{(N-n)!} \int_{\Lambda^N} P_N(x_1, \ldots, x_N) \prod_{i=1}^n 1_{A_i}(z_i) d\mu(x_1) \cdots d\mu(x_N) \tag{3.1.16}\]

$$= \frac{N!}{(N-n)!} \int_{A_1 \times \cdots \times A_n} d\mu(x_1) \cdots d\mu(x_n) \int_{\Lambda^{N-n}} P_N(x_1, \ldots, x_N) d\mu(x_{n+1}) \cdots d\mu(x_N). \tag{3.1.17}$$

This implies (3.1.15). \hfill $\square$
It is interesting to characterize the measures of "empty sets", since, for example, in the PNG droplet the distribution of the top line, i.e. the probability that the top line is under a certain level \(a\), is formulated in terms of a gap probability,

\[
P(H_0 \leq a) = P(\eta^{PNG}(\mathbb{1}_{(a,\infty)} = 0),
\]

which is the probability that there are no particles above \(a\).

**Proposition 3.1.7.** Let \(B \in \mathcal{B}(\Lambda)\). Then

\[
P(\eta(\mathbb{1}_B) = 0) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{B^n} d\mu(x_1) \cdots d\mu(x_n) \rho^{(n)}(x_1, \ldots, x_n). \tag{3.1.19}
\]

**Proof.** We prove it in the discrete setting. Let \(\eta = \sum_i \delta_{x_i}\). Then,

\[
P(\eta(\mathbb{1}_B) = 0) = \mathbb{E} \left[ \prod_{1 \leq i \leq n} (1 - \mathbb{1}_B(x_i)) \right]
= 1 - \sum_i \mathbb{E} [\mathbb{1}_B(x_i)] + \sum_{i < j} \mathbb{E} [\mathbb{1}_B(x_i) \mathbb{1}_B(x_j)] - \ldots
= \sum_{n \geq 0} (-1)^n \sum_{i_1 < i_2 < \cdots < i_n} \mathbb{E} \left[ \prod_{1 \leq k \leq n} \mathbb{1}_B(x_{i_k}) \right]
= \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{i_1, i_2, \ldots, i_n \text{ all different}} \mathbb{E} \left[ \prod_{1 \leq k \leq n} \mathbb{1}_B(x_{i_k}) \right]
= \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{y_1, \ldots, y_n \in B} \rho^{(n)}(y_1, \ldots, y_n). \tag{3.1.20}
\]

In the continuous setting we simply replace the sum over \(y_1, \ldots, y_n \in B\) with an integral. \(\square\)

### 3.2 Determinantal point processes

The processes like the one of the multilayer PNG have the nice property that the correlation functions are given in a very compact way.

**Definition 3.2.1.** A point process in \(\Lambda\) is said to be determinantal if there exists a function \(K(x,y)\) on \(\Lambda \times \Lambda\) such that the correlation functions (with respect to some reference measure \(\mu\)) are given by the determinantal formula

\[
\rho^{(n)}(x_1, \ldots, x_n) = \text{det} [K(x_i, x_j)]_{1 \leq i, j \leq n}, \tag{3.2.1}
\]

for \(n = 1, 2, \ldots\). The function \(K\) is called correlation kernel (and it can be though as the kernel of an integral operator \(K : L^2(\Lambda, d\mu) \rightarrow L^2(\Lambda, d\mu)\)).

Consequently, for determinantal point processes, we have

\[
\rho^{(1)}(x) = K(x, x),
\]

\[
\rho^{(2)}(x_1, x_2) = \text{det} \begin{bmatrix} K(x_1, x_1) & K(x_1, x_2) \\ K(x_2, x_1) & K(x_2, x_2) \end{bmatrix},
\]

\[
\ldots
\]
The correlation kernel is a single function of two variables while the correlation functions form an infinite sequence of functions of growing number of variables. Thus, if a point process happens to be determinantal, it can be described by a substantially reduced amount of data. This is somehow similar to centered Gaussian processes, for which all the information about process is encoded in a single covariance function.

As a consequence of Proposition 3.1.7,

**Proposition 3.2.2.** For a determinantal point process with correlation kernel $K(x, y)$,

$$\mathbb{P}(\eta(B) = 0) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{B^n} d\mu(x_1) \cdots d\mu(x_n) \det [K(x_i, x_j)]_{1 \leq i, j \leq n}$$

$$=: \det [\mathbb{1} - K]_{L^2(B, d\mu)}.$$  \hspace{1cm} (3.2.2)

The determinant in (3.2.2) is called the **Fredholm determinant** of the operator $K$ on the space $L^2(B, d\mu)$.

Note that the correlation kernel is not unique, indeed the process is invariant for gauge transformation of the kernel of the form

$$K(x, x) \mapsto f(x) f(y) K(x, y)$$  \hspace{1cm} (3.2.3)

for non-vanishing $f : \Lambda \to \mathbb{C}$.

**Remark 3.2.3.** The following property holds.

1. $\mathbb{P}(\# \{x_i \in \Lambda \} = N) = 1$ iff $K$ is the orthogonal projector with rank $N$.

2. $\mathbb{P}(\# \{x_i \in \Lambda \} < \infty$ iff $\text{Tr}(K) < \infty$.

3. A determinantal point process is a simple point process.

### 3.2.1 Biorthogonal ensembles

An important class of determinantal point processes are the so-called **biorthogonal ensembles**.

**Definition 3.2.4.** A $N$-point biorthogonal ensemble on a state space $\Lambda^N$ (with reference measure $\mu$) is a probability measure on $N$-point subsets $\{x_1, \ldots, x_N\} \subset \Lambda^N$ of the form

$$\frac{1}{Z_N} \det [\phi_i(x_j)]_{1 \leq i, j \leq N} \det [\psi_i(x_j)]_{1 \leq i, j \leq N} d\mu(x_1) \cdots d\mu(x_N)$$

$$=: \det [\mathbb{1} - K]_{L^2(\Lambda^N, d\mu)}.$$  \hspace{1cm} (3.2.4)

with $Z_N \neq 0$ a normalization constant and $\{\phi_i\}_{1 \leq i \leq N}$, $\{\psi_i\}_{1 \leq i \leq N}$ functions on $\Lambda^N$ such that $(\psi_j, \phi_i) = \int_{\Lambda} \phi_i(x) \psi_j(x) dx < \infty$.

One example that we have already seen is the measure of Proposition 2.3.26 since the functions $\phi_i$ and $\psi_i$ are given through the $h_i$ using Proposition 2.3.10. This will be generalized in the Section 3.2.2 below. Many other examples of biorthogonal ensemble can be found in the context of random matrices [18]. One case is the measure on $N$-particle configurations on the unit circle with density proportional to $\prod_{i<j} |x_i - x_j|^2$, which can be identified with the distribution of eigenvalues of the $N \times N$ random unitary matrices. Another is the measure on $N$-particle configurations on $\mathbb{R}$ with density proportional to $\prod_{i<j} (x_i - x_j)^2 \prod_i e^{-x_i^2}$, which is the distribution of eigenvalues of a random Hermitian matrix from Gaussian Unitary Ensemble.
Theorem 3.2.5. A biorthogonal ensemble defines a determinantal point process with correlation kernel (with respect to the reference measure $\mu$)

$$K(x, y) = \sum_{1 \leq i,j \leq N} \psi_i(x) \left[ A^{-1} \right]_{i,j} \phi_j(y), \quad (3.2.5)$$

where $A = [A_{i,j}]_{1 \leq i,j \leq N}$, $A_{i,j} = (\phi_i, \psi_j) = \int_A \mu(x) \phi_i(x) \psi_j(x)$ is called the Gram matrix.

Remark 3.2.6. If $\phi_i(x)$, $\psi_j(x)$ are biorthogonal, the matrix $A$ is diagonal and it can be easily inverted.

Lemma 3.2.7. Let $\{\eta_k\}_{k=1,\ldots,N}$, $\{\xi_\ell\}_{\ell=1,\ldots,N}$ two families of functions such that

$$\int_A \mu(x) \xi_\ell(x) \eta_k(x) = \delta_{k\ell}. \quad (3.2.6)$$

Define

$$K_N(x, y) = \sum_{k=1}^{N} \eta_k(x) \xi_k(y). \quad (3.2.7)$$

Then

$$\int_A \mu(x) K_N(x, x) = N, \quad (3.2.8a)$$

$$\int_A \mu(x) K_N(x, z) K_N(z, y) = K_N(x, y) \quad (3.2.8b)$$

Proof. The proof is straightforward. By simply using the definition of $K_N$:

$$\int_A \mu(x) \sum_{k=1}^{N} \eta_k(x) \xi_k(x) = N, \quad (3.2.9)$$

and

$$\sum_{k,\ell=1}^{N} \int_A \mu(x) \eta_k(x) \xi_k(z) \eta_\ell(z) \xi_\ell(y) = \sum_{k=1}^{N} \eta_k(x) \xi_k(y). \quad (3.2.10)$$

Proof of Theorem 3.2.5 First, let $\{\eta_k, k = 1, \ldots, N\}$, $\{\xi_\ell, k = 1, \ldots, n\}$ such that $\text{span}([\xi_k]_{1 \leq k \leq N}) = \text{span}([\phi_k]_{1 \leq k \leq N})$, $\text{span}([\eta_k]_{1 \leq k \leq N}) = \text{span}([\psi_k]_{1 \leq k \leq N})$ and $(\eta_k, \eta_\ell) = \delta_{k\ell}$ (this can be realized by Gram-Schmidt orthogonalization). Then

$$\begin{align*}
\det [\phi_i(x_j)] \det [\psi_i(x_j)] & = c \cdot \det [\xi_i(x_j)] \det [\eta_i(x_j)] \\
& = c \cdot \det \left[ \sum_{k=1}^{N} \eta_k(x_i) \xi_k(x_j) \right].
\end{align*} \quad (3.2.11)$$

By Lemma 3.2.6, for $n \leq N$

$$\begin{align*}
\rho^{(n)}(x_1, \ldots, x_n) & = \frac{N!}{(N-n)!} \int_{A_{N-n}} \mu(x_{n+1}) \cdots \mu(x_N) \det \left[ \sum_{k=1}^{N} \eta_k(x_i) \xi_k(x_j) \right]_{1 \leq i,j \leq N} \\
& = \frac{N!}{(N-n)!} \int_{A_{N-n}} \mu(x_{n+1}) \cdots \mu(x_N) \det [K(x_i, x_j)]_{1 \leq i,j \leq N},
\end{align*} \quad (3.2.12)$$
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with $\bar{c} = c/Z_N$. Suppose that we have already obtained a $m \times m$ determinant (true for $m = N$) and let us see what happens when we integrate out the variable $x_m$. We develop the determinant w.r.t. the last column,

$$\int_A d\mu(x_m) \det [K_N(x_i, x_j)]_{1 \leq i, j \leq m} = K_N(x_m, x_m) \det [K_N(x_i, x_j)]_{1 \leq i, j \leq m}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} K_N(x_k, x_m) \det \left[ [K_N(x_i, x_j)]_{1 \leq i, j \leq m-1, i \neq k} \right]. \quad (3.2.13)$$

By linearity and Lemma 3.2.7

$$\rho^{(n)}(x_1, \ldots, x_n) = \bar{C} N! \det [K_N(x_i, x_j)]$$

Applying this computation for $m = N, N - 1, \ldots, n + 1$ gives

$$\rho^{(n)}(x_1) = \bar{C} \cdot N! K_N(x_1, x_1)$$

$$= (N - (m - 1)) \det [K_N(x_i, x_j)]. \quad (3.2.14)$$

Now we want to find the value of $\bar{C}$. Since it is independent from $n$, we can take $n = 1$: for this case, $\rho^{(1)}(x_1) = \bar{C} \cdot N! K_N(x_1, x_1)$ and

$$\int_A d\mu(x_1) \rho^{(1)}(x_1) = N, \quad (3.2.16)$$

but also

$$\int_A d\mu(x_1) \rho^{(1)}(x_1) = \bar{C} N! \int_A K_N(x, x) d\mu(x) = \bar{C} N! \cdot N, \quad (3.2.17)$$

which implies $\bar{C} \cdot N! = 1$.

What’s left it to write $K_N$ in terms of the original functions $\phi_i, \psi_j$’s. Let $S, T$ be the matrices of the change of basis $\{\xi_i, \eta_i\} \rightarrow \{\phi_i, \psi_i\}$:

$$\phi_i = \sum_j S_{ij} \xi_j, \quad \psi_i = \sum_j T_{ij} \eta_j. \quad (3.2.18)$$

Then

$$K_N(x, y) = \sum_k \eta_k(x) \xi_k(y)$$

$$= \sum_k \left( \sum_i (T^{-1})_{ki} \psi_i(x) \right) \left( \sum_j (S^{-1})_{kj} \phi_j(y) \right)$$

$$= \sum_{ij} \psi_i(x) \phi_j(y) \sum_k (T^{-1})_{ki} (S^{-1})_{kj} \quad (3.2.19)$$

$$= \sum_{ij} \psi_i(x) \phi_j(y) \sum_k (T^{-1})_{ik} (S^{-1})_{kj}$$

$$= \sum_{ij} \psi_i(x) \phi_j(y) (S \cdot T^t)_{ij}^{-1}.$$

We have to see that $A = S \cdot T^t$, but

$$A_{ji} = (\psi_i, \phi_j) = \sum_{k \ell} T_{ik} S_{j\ell} (\eta_k, \xi_\ell)$$

$$= \sum_k T_{ik} \cdot S_{jk} = (S \cdot T^t)_{ji}. \quad (3.2.20)$$
3.2.2 Schur measures

Let us define a generalization of the measures seen in Proposition 2.3.26.

**Definition 3.2.8.** A Schur measure $S_{\rho_1;\rho_2}$ is a probability measure on the set of all Young diagrams defined through

$$\mathbb{P}_{\rho_1;\rho_2}(\lambda) = \frac{s_\lambda(\rho_1)s_\lambda(\rho_2)}{H(\rho_1;\rho_2)},$$

where $H(\rho_1;\rho_2) = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\rho_1)p_k(\rho_2)}{k}\right)$. This definition makes sense only for Schur-positive specializations $\rho_1,\rho_2$ satisfying $\sum_{\lambda \in \mathbb{Y}} s_\lambda(\rho_1)s_\lambda(\rho_2) < \infty$.

Since the Schur measures are essentially biorthogonal ensembles, they are determinantal with the correlation kernel given in the following theorem.

**Theorem 3.2.9.** Let $\lambda \in \mathbb{Y}$ be distributed according to $\mathbb{P}_{\rho_1;\rho_2}(\lambda)$. Let

$$X = \sum_{i \geq 1} \delta_{\lambda_i - i + 1}.$$  

Then $X$ is a determinantal point process on $\mathbb{Z}$ with correlation kernel $K(x,y)$ given by

$$K(x,y) = \frac{1}{(2\pi i)^2} \int dw \int dv \frac{H(\rho_1;v)H(\rho_2;w^{-1})}{H(\rho_2;v^{-1})H(\rho_1;w)} \frac{1}{v-w}v^{-x}w^{-y},$$

where $H(\rho;z) = \sum_{k=0}^{\infty} h_k(\rho)z^k = \exp\left(\sum_{k=1}^{\infty} p_k(\rho)\frac{z^k}{k}\right)$.

The integration paths for $w, v$ can be chosen as follows: take $|w| = R_1 < R_2 = |v|$ such that the functions $H(\rho_1;u), H(\rho_2;u^{-1})$ are holomorphic in the annulus $R_1 - \varepsilon < |u| < R_2 + \varepsilon$. In particular, if $|p_k(\rho_1)| < Cr^k$ and $|p_k(\rho_2)| < Cr^k$ for some constants $C > 0$, $0 < r < 1$, then any $r < R_1 < R_2 < r^{-1}$ are suitable.

**Proof.** Consider any finite set $A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}$. Let $X(\lambda) = (\lambda_i - i + 1)_{i \geq 1}$. What we need to prove is

$$\sum_{\lambda \in \mathbb{Y}} \sum_{A \subset N(\lambda)} s_\lambda(\rho_1)s_\lambda(\rho_2) \frac{1}{H(\rho_1;\rho_2)} = \det [K(a_i, a_j)]_{1 \leq i, j \leq m}.$$  

We show (3.2.24) for any two sets of variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, that is

$$\sum_{\lambda \in \mathbb{Y}} \sum_{A \subset N(\lambda)} s_\lambda(x)s_\lambda(y) \frac{1}{\prod_{i < j} 1-x_iy_j} = \det [\hat{K}(a_i, a_j)]_{1 \leq i, j \leq m},$$

where $\hat{K}(a,b)$ is just $K(a,b)$ with $x, y$ in place of $\rho_1, \rho_2$.

First we take finite sets of variables, $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ with $x_i \neq x_j$ and $y_i \neq y_j$ for all $i \neq j$. The general case is the $N \to \infty$ limit and/or analytic continuation in the $x_i$'s. Since $s_\lambda(x_1, \ldots, x_N) = 0$ if $\ell(\lambda) > N$, we consider $\hat{X}_N(\lambda) = \{\lambda_i - i + N, 1 \leq i \leq N\}$, a cut-off of $X(\lambda)$ shifted by $N - 1$, so that all elements of $\hat{X}_N(\lambda)$ are positive.

With respect to $\hat{X}_N(\lambda)$, $X(\lambda)$ has additional deterministically located points which go to $-\infty$ as $N \to \infty$ and do not affect the correlation functions. Thus the measure we need to consider is

$$c \cdot \det \left[ y_i^{\lambda_j+N-j} \right]_{1 \leq i,j \leq N} \cdot \det \left[ y_i^{\lambda_j+N-j} \right]_{1 \leq i,j \leq N},$$

(3.2.26)
with \( x_i \neq x_j \) and \( y_i \neq y_j \), \( \forall i \neq j \), and \( 0 \leq x_i y_j \leq 1 \), \( \forall i,j \) (required by the condition that \( \sum_{\lambda \in \mathbb{N}} s_\lambda(p_1)s_\lambda(p_2) < \infty \)). So we have a biorthogonal ensemble with kernel

\[
\tilde{K}_N(\ell_1, \ell_2) = \sum_{i,j=1}^{N} x_i^{\ell_1} y_j^{\ell_2} [A^{-1}]_{ij}, \tag{3.2.27}
\]

where \( A_{ij} = \sum_{\ell \geq 0} x_j^{\ell} y_i^{\ell} = \frac{1}{1 - x_i y_j} \). From Lemma 2.3.14 we know

\[
\det[A_{ij}]_{1 \leq i,j \leq N} = \det \left[ \frac{1}{1 - x_i y_j} \right] = \prod_{i<j}(x_i - y_j)(y_i - y_j) / \prod_{i,j}(1 - x_i y_j). \tag{3.2.28}
\]

Thus

\[
[A^{-1}]_{km} = \frac{(-1)^{k+m} \det[A_{ij}^*]_{i \neq k, j \neq m}}{\det[A_{ij}]_{1 \leq i,j \leq N}} = \frac{\prod_{j=1}^{N} (1 - x_j y_m)(1 - x_k y_j)}{(1 - x_k y_m) \prod_{j \neq k}(x_k - x_j) \prod_{j \neq m}(y_m - y_j)}, \tag{3.2.29}
\]

and

\[
\tilde{K}_N(\ell_1, \ell_2) = \sum_{k,m=1}^{N} x_k^{\ell_1} y_m^{\ell_2} [A^{-1}]_{km}
= \sum_{k,m=1}^{N} x_k^{\ell_1} y_m^{\ell_2} \frac{\prod_{j=1}^{N} (1 - x_j y_m)(1 - x_k y_j)}{(1 - x_k y_m) \prod_{j \neq k}(x_k - x_j) \prod_{j \neq m}(y_m - y_j)}
= \sum_{k,m=1}^{N} \frac{1}{(2\pi i)^2} \oint_{\Gamma_{x_k}} dz \oint_{\Gamma_{y_m}} dw \int_{\Gamma_{y_i}} \int_{\Gamma_{y_j}} \frac{z^{\ell_1} w^{\ell_2}}{1 - z w} \prod_{j=1}^{N} \frac{(1 - z y_j)(1 - w x_j)}{(z - x_j)(w - y_j)}
= \frac{1}{(2\pi i)^2} \oint_{\Gamma_{y_1} \cdots \Gamma_{y_N}} dz \oint_{\Gamma_{x_1} \cdots \Gamma_{x_N}} dw \int_{\Gamma_{y_i} \cdots \Gamma_{y_j}} \frac{z^{\ell_1} w^{\ell_2}}{1 - z w} \prod_{j=1}^{N} \frac{(1 - z y_j)(1 - w x_j)}{(z - x_j)(w - y_j)}.
\tag{3.2.30}
\]

Here we used Cauchy residue’s theorem and the notation \( \frac{1}{2\pi i} \oint_{z} dz f(z) \) meaning that the integration path is encircles anticlockwise the poles in the set \( I \), i.e.,

\( \frac{1}{2\pi i} \oint_{z} dz f(z) = \sum_{z \in I} \text{Residue}(f(z); z = x) \) for some set \( I \).

Remind that in order to have the Schur measure normalizable, we need \( x_i y_j < 1 \), \( \forall 1 \leq i, j \leq N \), then we can choose the contours of integration such that \( |zw| < 1 \). Then after the change of variable \( z = 1/v \ (dz = -\frac{1}{v^2} dv) \) we obtain the situation in Figure 3.1(b). Since we want both the contours to be anti-clockwise, we change direction to \( v \) multiplying by \(-1\). This leads to

\[
\tilde{K}(\ell_1, \ell_2) = \frac{1}{(2\pi i)^2} \oint_{\Gamma_{y_1} \cdots \Gamma_{y_N}} dv \oint_{\Gamma_{w}} dw \int_{\Gamma_{w}} \frac{w^{\ell_2} dv}{v^{\ell_1} - v \prod_{j=1}^{N} \frac{1 - y_j/v}{r_j(x_j)(w - y_j)}}
= \frac{1}{(2\pi i)^2} \oint_{\Gamma_{y_1} \cdots \Gamma_{y_N}} dv \oint_{\Gamma_{w}} dw \int_{\Gamma_{w}} \frac{w^{\ell_2} dv}{v^{\ell_1} - v \prod_{j=1}^{N} \frac{1 - y_j/v}{(1 - x_j)(w - y_j)/v}}.
\tag{3.2.31}
\]

For \( (x_1, \ldots, x_N) \) and \( (y_1, \ldots, y_N) \) for \( p_1 \) and \( p_2 \), \( H(p; v) = \prod_{j=1}^{N} \frac{1}{1 - x_j v} \) and

\[
\frac{H(p_1; v)H(p_2; v^{-1})}{H(p_1; w)H(p_2; v^{-1})} = \prod_{j=1}^{N} \frac{(1 - x_j v)(1 - x_j w)}{(1 - x_j v)(w - y_j)/v}. \tag{3.2.32}
\]
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Figure 3.1: In Figure (a) the contours of integration for the kernel \( \tilde{K}(\ell_1, \ell_2) \) in (3.2.30), e.g., \( w = re^{i\theta}, r > \max\{y_i\} \), \( z = re^{i\theta}, r > \max\{x_i\} \). In Figure (b) the contours of integration for the kernel \( \tilde{K}(\ell_1, \ell_2) \) in (3.2.30) after the change of variable \( v = z^{-1} \).

Once we have the kernel for the system \( \tilde{X}_N \), the kernel for \( \{\lambda_i - i + 1, 1 \leq i \leq N\} \) is just obtained by a shift of \( N - 1 \) of \( \ell_1, \ell_2 \), i.e. it has kernel

\[
\frac{1}{(2\pi i)^2} \int_{\Gamma_{y_1,\ldots,y_N}} dw \int_{\Gamma_{w,0}} dv \frac{w^{\ell_2-1}}{v^{\ell_1}} \frac{1}{v-w} \frac{H(\rho_1; v)H(\rho_2; w^{-1})}{H(\rho_1; w)H(\rho_2; v^{-1})}.
\]

(3.2.33)

This result can be applied to the case of multilayer PNG with parameter \( q \in (0,1) \).

Theorem 3.2.10. Let \( a_1 = a_2 = \cdots = \sqrt{q}, b_1 = b_2 = \cdots = \sqrt{q} \) for some \( q \in (0,1) \). Then the PNG measure at time \( n = 2m + 1 \) at position \( x = 0 \) is given by \( \text{const} \cdot s_3(\sqrt{q},\ldots,\sqrt{q})^m \).

Thus we have that the positions of the lines at \( x = 0 \) forms a determinantal point process with kernel

\[
K^{\text{PNG}}(\ell_1, \ell_2) = \frac{1}{(2\pi i)^2} \int_{\Gamma_{\sqrt{q}}} dw \int_{\Gamma_{w,0}} dv \frac{w^{\ell_2-1}}{v^{\ell_1}} \frac{1}{v-w} \left[ (1 - \sqrt{q} w)(1 - \sqrt{q} v^{-1}) \right]^m.
\]

(3.2.34)

Figure 3.2: Contours of integration for the kernel \( K^{\text{PNG}}(\ell_1, \ell_2) \) in (3.2.34) for the multilayer PNG.

3.3 The continuous time PNG model

Now we study the PNG in the continuous time limit, which is equivalent to consider the \( q \to 0 \) limit and take \( m = t/\sqrt{q} \). There is also a connection between this model and a combinatorial problem called the longest increasing subsequence of random permutations, as we shall point out later.
3.3. THE CONTINUOUS TIME PNG MODEL

In the $q \to 0$ limit, most of the $\omega_{i,j}$ of the discrete model will take value 0, once in while value 1 and a negligible amount of them value larger than 2. Scale all space variables by $\sqrt{q}$. Then, if we plot only the $(i,j)$ such that $\omega_{i,j} = 1$, one gets a Poisson point process with intensity 2. For the interested reader, let us describe how the model is defined and the construction of the multilayer.

3.3.1 Dynamics and multilayer

One way to view continuous time PNG is via a graphical construction involving Poisson points. Let $\omega$ be a realization of Poisson points in $\mathbb{R} \times [0,T]$. The dynamics of the continuous time PNG model is then the following. Let $x \mapsto h(x,t) \in \mathbb{Z}$ with discontinuities only of height 1.

- deterministic part: each up-step (resp. down-step) of the height function moves to the left (resp. right) with unit speed; an up-step and a down-step which collide merges,
- random part: at each event of the Poisson process, an pair of up- and down-step is created (and immediately starts spreading according to the deterministic dynamics).

We can determine the height function also graphically as follows. Let us start with initial condition $h(x,0) = 0$ for all $x \in \mathbb{R}$. Each nucleation event generates two lines, with slope +1 and −1 along its forward light cone. A line ends upon crossing another line (merging). In Figure 3.3 the dots are the nucleation events and the lines follow the forward light cones. The height $h(x,t)$ is then the number of lines crossed along the path from $(x,0)$ to $(x,t)$. The geometric setting of (2.1.2) corresponds in the limit to have Poisson points restricted to $\{|x| \leq t, t \geq 0\}$.

![Figure 3.3: Graphical construction generating the surface height from the Poisson points.](image)

Analogue to the discrete model, we can also associate to this model a multi-layer version: $h_\ell(x,t)(\omega)$, $(x,t) \in \mathbb{R} \times [0,T]$, $\ell \leq 0$. At $t = 0$ we set $h_\ell(x,0) = \ell$ with $\ell = 0, -1, \ldots$, $\ell$ denoting the level’s height. The first height is defined by $h_0(x,t)(\omega) \equiv h(x,t)(\omega)$. The meeting points of the forward light cones generated by the points of $\omega$ are called the annihilation events of level 0. $h_{-1}(x,t)(\omega)$ is constructed as $h_0(x,t)(\omega)$ but the nucleation events for level $-1$ are the annihilation events of level 0 and $h_{-1}(x,t)(\omega)-1$ equals the number of lines for level $-1$ crossed from $(0,0)$ to $(x,t)$. In Figure 3.3 the nucleation events of level $-1$ are the empty dots, whose forward light cones are the dotted lines. Setting the annihilation events of level $j$ as the nucleation events for level $j - 1$, the set of height functions $h_\ell(x,t)(\omega)$ is defined for all $\ell \leq 0$. The line ensemble for $t = T$, i.e., $\{h_\ell(x,T), \ell \leq 0\}$ is represented below.
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Figure 3.4: RSK construction up to time $t = T$. The nucleation events of level -1 are the empty dots and the light-like lines of level 2 are the dotted lines. The corresponding line ensemble is represented above.

3.3.2 Measure and correlation kernel

Proposition 3.3.1. Consider the Schur measure with $\rho_1 = \rho_2 = \rho = (\alpha; 0; 0)$, and $\alpha = (\sqrt{q}, \ldots, \sqrt{q}, 0, \ldots)$ with $m = t/\sqrt{q}$. Then

$$\lim_{q \to 0} \frac{s_\lambda(\rho) s_\lambda(\rho)}{H(\rho, \rho)} = \frac{s_\lambda(0; 0; t) s_\lambda(0; 0; t)}{e^{it^2}}. \quad (3.3.1)$$

Proof. It is a trivial limit of the Schur functions from $\alpha$ to $\gamma$ specialization.

In terms of line ensemble we have non-intersecting simple random walks with rate 1 in continuous time (see Figure 3.5).

Figure 3.5: PNG line ensemble from $x = -t$ to $x = t$.

Finally, the kernel can be written in the following forms.
3.3. THE CONTINUOUS TIME PNG MODEL

Proposition 3.3.2. The correlation kernel of the continuous time PNG at time $t$, for the point process at position $x = 0$, is given by

$$K_t^{PNG}(\ell_1, \ell_2) = \frac{1}{(2\pi i)^2} \int_{\Gamma_0} dw \int_{\Gamma_0, w} dv \frac{w^{\ell_2-1}}{v^{\ell_1}} \frac{e^{t(v-w)}}{v-w e^{t(w-\frac{1}{2})}}.$$ (3.3.2)

but also as

$$K_t^{PNG}(\ell_1, \ell_2) = \sum_{k \geq 0} J_{\ell_2+k}(2t) J_{\ell_1+k}(2t),$$ (3.3.3)

where $J_n(t)$ is the Bessel function. The Bessel function $J_n(t)$ is a bounded function, solution of the equation $t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} + (t^2 - n^2)x = 0$. It has the integral representation

$$J_n(t) = \frac{1}{2\pi i} \int_{\Gamma_0} dv \frac{e^{t(v-w)/2}}{v^{n+1}}.$$ (3.3.4)

Proof. The kernel of the continuous time PNG is defined as the limit for $q \to 0$ of the kernel of the discrete case:

$$K_t^{PNG}(\ell_1, \ell_2) = \lim_{q \to 0} K_t^{PNG}(\ell_1, \ell_2) = \frac{1}{(2\pi i)^2} \int_{\Gamma_0} dw \int_{\Gamma_0, w} dv \frac{w^{\ell_2-1}}{v^{\ell_1}} \frac{e^{t(v-w)}}{v-w e^{t(w-\frac{1}{2})}}.$$ (3.3.5)

We can choose the contours satisfying $|w| < |v|$. Then using

$$\frac{1}{v-w} = \frac{1}{v} \frac{1}{1-\frac{w}{v}} = \frac{1}{v} \sum_{k \geq 0} w^k$$ (3.3.6)

we can rewrite the kernel as

$$= \sum_{k \geq 0} \frac{1}{2\pi i} \int_{\Gamma_0} dw w^{\ell_2-1+k} e^{-t(w-\frac{1}{2})} \frac{1}{2\pi i} \int_{\Gamma_0} dv \frac{e^{t(v-w)}}{v^{\ell_1+1+k}}$$

$$= \sum_{k \geq 0} \frac{1}{2\pi i} \int_{\Gamma_0} dz e^{t(z-\frac{1}{2})} \frac{1}{2\pi i} \int_{\Gamma_0} dv \frac{e^{t(v-w)}}{v^{\ell_1+1+k}}$$

$$= \sum_{k \geq 0} J_{\ell_2+k}(2t) J_{\ell_1+k}(2t).$$ (3.3.7)

\[\square\]

3.3.3 Link to the longest increasing subsequence problem

For completeness, let us present a nice connection with a combinatorial model. Denote by $\mathcal{S}_N$ the permutation group of $\{1, \ldots, N\}$.

Definition 3.3.3. Let $\sigma \in \mathcal{S}_N$. We say that the sequence $(\sigma(1), \ldots, \sigma(N))$ has an increasing subsequence of length $k$, $(n_1, \ldots, n_k) \subset (\sigma(1), \ldots, \sigma(k))$, if $1 \leq n_1 < n_2 < \cdots < n_k \leq N$.

Example 3.3.4. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 1 & 7 & 5 & 2 & 6 \end{pmatrix}.$$ (3.3.8)

Then $(3, 4, 5, 6)$ is an increasing subsequence of length $4$.

Definition 3.3.5. Let us denote by $L_N(\sigma)$ the length of the longest increasing subsequence for the permutation $\sigma$. 
The Ulam’s problem studies the asymptotic law of $L_N$ for uniform distribution on $S_N$. In 1961 it was conjectured that $E[L_N] \sim c\sqrt{N}$, but only in 1968 the right value $c = 2$ was pointed out via numeric simulations. The final proof that $c = 2$ was completed in 1977 [20].

Let us give a geometric interpretation of $L_N$. Consider first an example with $N = 6$ in Figure 3.6. The permutation is $\sigma = (1\ 2\ 3\ 4\ 5\ 6)\ (6\ 2\ 4\ 3\ 1\ 5)$, which has two longest increasing subsequence, $(2, 4, 5)$ and $(2, 3, 5)$. We look at the paths from $(0, 0)$ to $(N + 1, N + 1)$ such that both $k$ and $\sigma(k)$ are increasing and define $L_N(\sigma)$ as the maximal number of points visited by such paths.

![Figure 3.6: Geometric representation of the Ulam’s problem for $N = 6$.](image)

There exists a correspondence between the Ulam’s problem and the Young tableaux, introduced in Proposition 2.3.39, known as the RSK correspondence. The RSK correspondence is a bijection between permutations $\sigma \in S_N$ and pairs of Young tableaux $(\mathcal{P}(\sigma), \mathcal{Q}(\sigma))$. The algorithm is the following:

- **$\mathcal{P}$-tableaux:** for $i = 1, \ldots, N$,
  - place $\sigma(i)$ in the top row as follows: if $\sigma(i)$ is higher than all numbers in the first row of $\mathcal{P}$, then append it to the right of them; otherwise, put it at the place of the smallest higher element of the first row;
  - if an element was replaced in row $j$, take it and apply the same procedure in row $j + 1$.
- **$\mathcal{Q}$-tableaux:** for $i = 1, \ldots, N$,
  - place $i$ in the position where a number appeared the first time at step $i$ in the $\mathcal{P}$-tableaux.

Note that by construction $\mathcal{P}$ and $\mathcal{Q}$ have the same shape. Notice that $L_N(\sigma) = \lambda_1(\sigma)$ by construction.

**Definition 3.3.6.** Let $\dim(\lambda)$ be the number of Young tableaux of shape $\lambda$. The measure on $\lambda$ induced by the uniform measure on $S_N$ is the Plancherel measure

$$P_{\lambda}N(\lambda) := \frac{(\dim(\lambda))^2}{\sum_{\mu \in \mathcal{Y}_N} (\dim(\mu))^2}, \quad \lambda \in \mathcal{Y}_N,$$

where $\mathcal{Y}_N = \{\lambda \in \mathcal{Y}, |\lambda| = N\}$.
3.3. THE CONTINUOUS TIME PNG MODEL

Due to the fact that the RSK correspondence is a bijection, one has

$$\sum_{\mu \in \mathbb{Y}_N} (\dim(\mu))^2 = N!$$

(3.3.10)

Poissonized version

Consider the Poissonized version of the Plancherel measure given as follows. Let $$N \sim \text{Poisson}(\theta)$$. First notice that if we take a Poisson point process on $$\mathbb{R}^2_+$$ with intensity 1 and consider the square $$Q_\theta = [0, \sqrt{\theta}] \times [0, \sqrt{\theta}]$$, then the number of points in $$Q_\theta$$, $$\xi(Q_\theta)$$, is Poisson($$\theta$$)-distributed.

For any $$(x, y) \in \mathbb{R}^2_+$$ define $$\tilde{L}(x, y)$$ as the maximal number of points visited by paths with slopes in $$[0, \infty]$$ from $$(0, 0)$$ to $$(x, y)$$. We can divide $$\mathbb{R}^2_+$$ into regions with $$\tilde{L}(x, y) = 0, 1, 2, \ldots$$, as illustrated in Figure 3.7.

![Figure 3.7](image)

Figure 3.7: The square $$Q_\theta = [0, \sqrt{\theta}] \times [0, \sqrt{\theta}]$$ divided into regions for $$\tilde{L}(x, y) = 0, 1, \ldots, 4$$ for $$\lambda \sim \text{Poisson}(\theta)$$. Rotating by 45 degrees one sees the similarities with Figure 3.3.

On $$Q_\theta$$ a Poisson configuration of points can be obtained as follows: first, sample $$N \sim \text{Poisson}(\theta)$$; then, sample $$x_1, \ldots, x_N, y_1, \ldots, y_N$$ independent and uniformly distributed on $$[0, \sqrt{\theta}]$$. We have $$N$$ independent points, $$(x_1, y_1), \ldots, (x_N, y_N)$$, uniformly distributed on $$Q_\theta$$. We can relabel the $$x_i$$’s such that $$x_1 < x_2 < \cdots < x_N$$; then , there exists a unique permutation such that $$y_{\sigma^{-1}(1)} < y_{\sigma^{-1}(2)} < \cdots < y_{\sigma^{-1}(N)}$$. A key observation is that $$\tilde{L}(\sqrt{\theta}, \sqrt{\theta})$$ depends only on the relative positions of the Poisson
points in $Q_\theta$, so $L_{N,\text{Poisson} (\theta)} \overset{d}{=} \tilde{L}(\sqrt{\theta}, \sqrt{\theta})$.

Now we construct a set of line ensemble for the Poissonized problem of increasing subsequence in the following way:

- to each Poisson point draw a vertical and a horizontal line, until they meet other lines from other Poisson points;
- the intersection are considered as points of ”second generation”;
- draw lines in the same way for the second generation;
- their intersection are points of the ”third generation” of points, and so on...
- then, we associate line ensembles as in Figure 3.8.

![Figure 3.8: Line ensemble for $\sigma = (3, 7, 1, 5, 2, 6, 4)$. Black dots represent points of the first generation, green dots points of the second and blue dots points of the third.](image)

**Remark 3.3.7.** The two line ensembles obtained are the ones of the $\gamma$-specialization with $\gamma = \sqrt{\theta}$. Furthermore the graphical construction gives back the tableaux obtained by the RSK construction. Indeed, the tableaux contains the relative positions of the jumps in the lines. Compare with Table 3.1 and see Figure 3.9.

To recover the line ensemble of the continuous time PNG one just has to consider a triangular region instead of the squared region.

Let us denote by $\lambda(\theta)$ the shape of the Young tableaux $P(\sigma), Q(\sigma)$, where $\sigma \sim \text{Unif}(S_N)$ and $N \sim \text{Poisson}(\theta)$.

**Theorem 3.3.8.** The distribution of $\lambda(\theta)$ is given by the Poissonized Plancherel measure,

$$
P(\lambda(\theta) = \mu) = e^{-\theta} \mu^2 \left( \frac{\dim(\mu)}{\mu!} \right)^2
= \frac{s_{\mu}(0; 0; \sqrt{\theta}) s_{\mu}(0; 0; \sqrt{\theta})}{e^{\theta}}, \mu \in \mathbb{Y}.
$$ (3.3.11)

**Proof.** This follows from the fact that $P(\lambda(\theta) = \mu|N) = P(\mu|N) = \frac{\dim(\mu)^2}{N!}$, $P(N) = e^{-n} n^N$.
3.4 Large time asymptotics of the PNG model

As we have shown in (3.3.7), the correlation kernel of the poissonized Plancherel measure is given by

\[ K^{PNG}_t(x, y) = \sum_{k \geq 0} J_{k+x}(2t) J_{k+y}(2t). \]  

(3.4.1)

The goal is to analyze the large time limit of this kernel and from this the large time limit of the top line of the continuous time PNG model.

First let us give some properties of Bessel functions, which can be proven by analyzing the integral representation, or by studying the ODE defining them. We are not going to derive these properties here.

**Lemma 3.4.1.**

(a) For any time \( t > 0 \),

\[ \sup_{n \in \mathbb{N}} t^{1/3} |J_n(2t)| \leq C \]  

for some \( C < \infty \).

(b) For \( u \) in a compact set,

\[ \lim_{t \to \infty} t^{1/3} J_{\lfloor 2t + au^{1/3} \rfloor}(2t) = \text{Ai}(u). \]  

(3.4.3)

(c) For \( u \) bounded from below, for all \( t \) large enough,

\[ |t^{1/3} J_{\lfloor 2t + au^{1/3} \rfloor}(2t)| \leq C e^{-cu}, \]  

(3.4.4)

for some \( C, c > 0 \) positive constants.

The function \( \text{Ai}(u) \) appearing in Lemma 3.4.1 (b) is known as the Airy function. It is defined as the unique solution of the ordinary differential equation

\[ \frac{d^2}{du^2} y(u) = uy(u), \]  

(3.4.5)

satisfying \( y(u) \downarrow 0 \) as \( u \to \infty \). In particular, it has the asymptotics

\[ y(u) \sim \frac{e^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi u^{1/4}}} \text{ as } u \to \infty. \]  

(3.4.6)
The Airy function can be written in terms of integral in the complex plane as follows:

\[
\text{Ai}(x) = \frac{1}{2\pi i} \int_{e^{-r_{1/3}\infty}}^{e^{r_{1/3}\infty}} dz e^{\frac{3}{z} - \frac{3}{2} x}.
\]  

(3.4.7)

As a consequence of the known asymptotics, we have

**Theorem 3.4.2.** For \( u, v \) in a bounded set,

\[
\lim_{t \to \infty} t^{1/3} K^{\text{PNG}}_t(2t + ut^{1/3}, 2t + vt^{1/3}) = \int_0^\infty d\lambda \text{Ai}(u+\lambda)\text{Ai}(v+\lambda) =: K_{\text{Ai}}(u,v).
\]  

(3.4.8)

Here \( K_{\text{Ai}}(u,v) \) is called the *Airy kernel*.

**Proof.** We have

\[
t^{1/3} K^{\text{PNG}}_t(2t + ut^{1/3}, 2t + vt^{1/3}) = t^{1/3} \sum_{\ell \geq 0} J_{[2t + ut^{1/3} + \ell]}(2t) J_{[2t + vt^{1/3} + \ell]}(2t)
\]

\[
= t^{1/3} \sum_{\lambda \in \mathbb{Z} t^{1/3} \mathbb{N}_0} J_{[2t + (u+\lambda) t^{1/3}]}(2t) J_{[2t + (v+\lambda) t^{1/3}]}(2t)
\]

\[
\simeq \int_0^\infty d\lambda t^{1/3} J_{[2t + (u+\lambda) t^{1/3}]}(2t) t^{1/3} J_{[2t + (v+\lambda) t^{1/3}]}(2t).
\]  

(3.4.9)

Since \( t^{1/3} J_{[2t + (u+\lambda) t^{1/3}]}(2t) \) converges pointwise to \( \text{Ai}(u+\lambda) \) and

\[
t^{1/3} J_{[2t + (u+\lambda) t^{1/3}]}(2t) t^{1/3} J_{[2t + (v+\lambda) t^{1/3}]}(2t) \leq C^2 e^{-c(u+v+\lambda)}
\]  

(3.4.10)

by Lemma 3.4.1 (c). Thus by dominated convergence we can take the limit \( t \to \infty \) inside the integral, which is the result.

**Lemma 3.4.3.** For \( u, v \) bounded from below,

\[
|t^{1/3} K^{\text{PNG}}_t(2t + ut^{1/3}, 2t + vt^{1/3})| \leq C^2 e^{-c(u+v)}
\]  

(3.4.11)

for all \( t \geq t_0 \).

**Proof.** The proof follows directly from Lemma 3.4.1 (b) – (c).
3.4. LARGE TIME ASYMPTOTICS OF THE PNG MODEL

Remark 3.4.4. Why do we need the factor \( t^{1/3} \) in front of the kernel? Consider a determinantal point process on \( \mathbb{R} \) with \( n \)-points correlation functions \( \rho^{(n)}(x_1, \ldots, x_n) \) with respect to \( dx_1 \cdots dx_n \). If we perform a change of variable \( x_i = \alpha + \beta y_i \), then
\[
\rho^{(n)}(\alpha + \beta y_1, \ldots, \alpha + \beta y_n) \beta^n dy_1 \cdots dy_n = \rho^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n. \tag{3.4.12}
\]
Call \( \tilde{\rho}^{(n)}(y_1, \ldots, y_n) = \rho^{(n)}(\alpha + \beta x_1, \ldots, \alpha + \beta x_n) \beta^n \) the \( n \)-points correlation function with respect to \( dy_1 \cdots dy_n \). Since the process is determinantal, there exists a kernel \( K(x, y) \) such that \( \rho^{(n)}(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{1 \leq i, j \leq n} \); in terms of the \( y_i \)'s, we have \( \tilde{\rho}^{(n)}(y_1, \ldots, y_n) = \det[\tilde{K}(y_i, y_j)]_{1 \leq i, j \leq n} \), with
\[
\tilde{K}(y_i, y_j) = \beta K(\alpha + \beta y_i, \alpha + \beta y_j). \tag{3.4.13}
\]

Definition 3.4.5. The GUE Tracy-Widom distribution \( F_{\text{GUE}} \) is defined by
\[
F_{\text{GUE}}(s) = \det(\mathbb{I} - K_{\text{Ai}})_{L^2(\mathbb{R})} = \det(\mathbb{I} - \chi(s) K_{\text{Ai}} \chi(s))_{L^2(\mathbb{R})} = \sum_{n \geq 0} (-1)^n \int_{s}^{\infty} dx_1 \cdots \int_{s}^{\infty} dx_n \det[K_{\text{Ai}}(x_i, x_j)]_{1 \leq i, j \leq n}, \tag{3.4.14}
\]
with \( \chi(s) = \mathbb{I}_{(s, \infty)} \).

Theorem 3.4.6. Let \( \lambda \) be distributed as \( e^{-\langle 0,0:t \rangle_{x} \mathcal{P}(0,0)} \). Then
\[
\lim_{t \to \infty} P(\lambda_1 \leq 2t + st^{1/3}) = F_{\text{GUE}}(s). \tag{3.4.15}
\]

Proof. We know that \( \xi = \sum_{i \geq 1} \delta_{\lambda_i - i + 1} \) is a determinantal point process with correlation kernel \( K_{\text{PNG}}^{(s, s)}(x, y) \). Thus
\[
P(\lambda_1 \leq 2t + st^{1/3}) = P(\xi(\mathbb{I}_{[2t+st^{1/3}, \infty)}) = 0)
= \sum_{n \geq 0} \left( \frac{-1}{n!} \right)^n \sum_{j_1, \ldots, j_n \in \mathbb{Z}} t^{-n/3} \det\left[ t^{1/3} K_{\text{PNG}}^{(s, s)}(2t + x_k t^{1/3}, 2t + x_\ell t^{1/3}) \right]_{1 \leq k, \ell \leq n}.
\]
with \( I_s = \mathbb{Z}_{\geq 0} - (s, \infty) \). We know that \( t^{1/3} K_{\text{PNG}}(2t + x_k t^{1/3}, 2t + x_\ell t^{1/3}) \) converges to \( K_{\text{Ai}}(x_k, x_\ell) \) and by Lemma 3.4.3 that
\[
\left| \frac{t^{1/3} K_{\text{PNG}}^{(s, s)}(2t + x_k t^{1/3}, 2t + x_\ell t^{1/3})}{C^2 e^{-c(x_k + x_\ell)}} \right| \leq 1, \quad \forall x_k, x_\ell \geq s. \tag{3.4.17}
\]
Then we
\[
\sum_{n \geq 0} \left( \frac{-1}{n!} \right)^n \int_{s}^{\infty} dx_1 \cdots \int_{s}^{\infty} dx_n \sum_{j_1, \ldots, j_n \in \mathbb{Z}} t^{-n/3} \det\left[ t^{1/3} K_{\text{PNG}}^{(s, s)}(2t + x_k t^{1/3}, 2t + x_\ell t^{1/3}) \right]_{1 \leq k, \ell \leq n}
\]
and denote \( A_{k, \ell} := \frac{t^{1/3} K_{\text{PNG}}^{(2s t^{1/3}, 2s t^{1/3})}}{C^2 e^{-c(s + \ell s^{1/3})}} \). Using Hadamard's bound (see below)
\[
\left| \sum_{n \geq 0} \left( \frac{1}{n!} \right)^n \frac{n^{n/2} C^2 e^{-2c s n}}{n!} \right| < \infty. \tag{3.4.19}
\]
since \( n! \sim e^{n \ln n - n} \). We can use Dominated Convergence and exchange the sum with the limit, with the result

\[
\lim_{t \to \infty} P(\lambda_1 \leq 2t + st^{1/3}) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_s^\infty dx_1 \cdots \int_s^\infty dx_n \det [K_{Ai}(x_k, x_\ell)]_{1 \leq k, \ell \leq n}.
\]

(3.4.20)

**Remark 3.4.7.** The Hadamard bound states that given a \( n \times n \) matrix \( A \) with \( |A_{ij}| \leq 1 \), for all \( i, j \), then \( |\det(A)| \leq n^n/2 \).

Now we give some properties of the GUE Tracy-Widom distribution and the Airy kernel.

**Lemma 3.4.8.**

(a) 

\[
K_{Ai}(x, y) = \int_0^\infty d\lambda Ai(x + \lambda)Ai(y + \lambda)
\]

\[
= \begin{cases} 
\frac{\Lambda(x)\Lambda'(y) - \Lambda'(x)\Lambda(y)}{x - y}, & x \neq y, \\
\Lambda'(x)\Lambda''(y) - \Lambda'(y)\Lambda'(x), & x = y.
\end{cases}
\]

(3.4.21)

(b) 

\[
K_{Ai}^2 = K_{Ai}.
\]

(3.4.22)

(c) Let \( H = -\frac{d^2}{dx^2} + x \) be the Airy operator. The generalized eigenfunctions \( \psi_\lambda(x) = Ai(x - \lambda) \) satisfies \( H\psi_\lambda = \lambda\psi_\lambda \). So, \( K_{Ai} \) is the spectral projection onto \( \{H \leq 0\} \),

\[
K_{Ai} = \int_{\mathbb{R}_-} d\lambda \psi_\lambda \otimes \psi_\lambda.
\]

(3.4.23)

(d) \( K_{Ai} \) is locally trace-class and for \( s > -\infty \) \( \|P_s K_{Ai} P_s\|_1 < \infty \).

**Proof.** For (a):

\[
(x - y) \int_0^\infty d\lambda Ai(x + \lambda)Ai(y + \lambda) = \int_0^\infty d\lambda (x + \lambda)Ai(x + \lambda)Ai(y + \lambda)
\]

\[
- \int_0^\infty d\lambda (y + \lambda)Ai(x + \lambda)Ai(y + \lambda)
\]

\[
= \Lambda''(x)Ai(x) \int_0^\infty d\lambda Ai''(x + \lambda)Ai(y + \lambda) - \int_0^\infty d\lambda Ai(x + \lambda)Ai''(y + \lambda)
\]

\[
= \Lambda(x + \lambda)Ai''(x + \lambda) \big|_0^\infty - \int_0^\infty d\lambda Ai'(x + \lambda)Ai'(y + \lambda)
\]

\[
- \Lambda'(x + \lambda)Ai'(y + \lambda) \big|_0^\infty + \int_0^\infty d\lambda Ai'(x + \lambda)Ai'(y + \lambda)
\]

\[
= -\Lambda'(x)Ai(y) + Ai(x)\Lambda'(y).
\]

(3.4.24)

For (b):

\[
\int_0^\infty d\mu \int_0^\infty d\lambda Ai(x + \lambda) \int_{\mathbb{R}} dz Ai(z + \lambda)Ai(z + \mu)Ai(\mu + y)
\]

\[
= \int_0^\infty d\lambda Ai(x + \lambda)Ai(y + \lambda).
\]

(3.4.25)
(c) follows by definition. Finally, rename $B(x, y) := \text{Ai}(x+y)$, and recall the inequality $\|AB\|_1 \leq \|A\|_{\text{HS}}\|B\|_{\text{HS}}$ where $\| \cdot \|_{\text{HS}}$ is the Hilbert-Schmidt norm. Then
\[
\|P_sK_A\|_1 \leq (\|P_sBP_0\|_{\text{HS}})^2 < \infty
\] (3.4.26)
proves (d).

The GUE Tracy-Widom distribution can be defined by either their Fredholm determinant representations or their representations in terms of a Painlevé II function. Let $q$ denote the solution to the Painlevé II equation
\[
q'' = xq + 2q^3
\] (3.4.27)
satisfying
\[
q(x) \sim \text{Ai}(x) \text{ as } x \to \infty.
\] (3.4.28)
That such a solution exists and is unique was proved by Hastings and McLeod [11]. Then we have
\[
\mathcal{F}_{\text{GUE}}(s) = \exp \left( -\int_{s}^{\infty} dx (x-s)q^2(x) \right).
\] (3.4.29)
The asymptotics of these distributions as $x \to \infty$ is straightforward given the large $x$ asymptotics of the Airy function; however, the complete asymptotic expansion as $x \to \infty$ has only recently been given [4].

### 3.5 Tracy-Widom distribution in random matrices

We will outline how the GUE Tracy-Widom distribution arises in the random matrices setting. We recall the definition of Gaussian Unitary Ensemble given in Section 1.3.1.

**Definition 3.5.1.** The Gaussian Unitary Ensemble (GUE) is a probability measure on Hermitian matrices $H$ of size $N \times N$ with density
\[
P_{\text{GUE}}(H \in dH) = \frac{1}{Z_N} \exp \left( -\frac{1}{2N} \text{Tr}(H^2) \right) dH = p_{\text{GUE}}(H)dH,
\] (3.5.1)
where $dH = \Pi_{1 \leq i < j \leq N} d\text{Re}(H_{i,j})d\text{Im}(H_{i,j})$ is the reference measure and $Z_N$ the normalization constant.

It is easy to see that $p_{\text{GUE}}(H)$ is invariant under unitary transformations and the entries are independent Gaussian random variables (with appropriate variances). One can further compute explicitly the distribution of the $N$ eigenvalues $\mu = (\mu_1, \ldots, \mu_N)$, namely
\[
\mathbb{P}(\mu \in d\mu) = \text{const} \times \prod_{1 \leq i < j \leq N} (\mu_j - \mu_i)^2 \prod_{k=1}^{N} e^{-\frac{\mu_k^2}{2N}} d\mu_k,
\] (3.5.2)
which is a Vandermonde determinant squared times a product measure.

Let $\omega(x) = e^{-x^2/2N}$, then for any family of polynomials
\[
\{q_k(x) = c_kx^k + \text{poly}_k(x)\}_{k \geq 0},
\] (3.5.3)
where $c_k \neq 0$ and $\text{poly}_k(x)$ is a polynomial of degree $k-1$, we have that
\[
\mathbb{P}(\mu \in d\mu) = \tilde{c} \left( \det [q_{k-1}(\mu_j)]_{1 \leq j,k \leq N} \right)^2 \prod_{k=1}^{N} \omega(\mu_k) d\mu_k
\]
\[
= \tilde{c} \left( \det \left[ q_{k-1}(\mu_j)\omega(\mu_j)^{1/2} \right]_{1 \leq j,k \leq N} \right)^2 \prod_{k=1}^{N} d\mu_k.
\] (3.5.4)
CHAPTER 3. POINT PROCESSES

This defines a biorthogonal ensemble with \( \Phi_i(x) = \Psi_i(x) = q_{i-1}(x) \sqrt{\omega(x)} \).

Furthermore, if we choose the polynomials \( q_i \)’s such that
\[
\int_\mathbb{R} dx q_k(x) q_l(x) \omega(x) = \delta_{k,l}, \tag{3.5.5}
\]
then we have
\[
\int_\mathbb{R} dx \Phi_i(x) \Psi_j(x) = \delta_{i,j} \tag{3.5.6}
\]
and consequently the \( n \)-point correlation of \( \xi = \sum_{i=1}^{N} \delta_{\mu_i} \) (with respect to the \( \text{Leb}^n \) reference measure) has correlation kernel
\[
K_N(x,y) = \sum_{k=1}^{N} \Psi_k(x) \Phi_k(y). \tag{3.5.7}
\]

For the GUE a suitable choice of orthogonal polynomials are the \textit{Hermite polynomials}\(^1\) \( \{h_k(x)\}_{k \geq 0} \).

**Proposition 3.5.2.** Define the conjugated kernel \( K_N^{\text{Hermite}}(x,y) = e^{\frac{x^2}{4} - \frac{y^2}{4}} K_N(x,y) \).

Then,
\[
K_N^{\text{Hermite}}(x,y) := \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\varepsilon + i \mathbb{R}} dz \frac{e^{\frac{x^2}{4} - \frac{z^2}{4}}}{e^{\frac{w^2}{4} - \frac{z^2}{4}} - W} \frac{Z^N}{W^N} \frac{1}{Z - W} \tag{3.5.8}
\]
for \( \varepsilon > 0 \), where the contour \( \Gamma_0 \) satisfies \(|w| < \varepsilon\).

**Definition 3.5.3.** The \( k \)th Hermite polynomial is defined as
\[
h_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}. \tag{3.5.9}
\]

The Hermite polynomials have the following two integral representations.

**Proposition 3.5.4.** We have
\[
h_k(x) = \frac{k!}{2\pi i} \oint_{\Gamma_x} dw e^{-w^2 + 2wx} \frac{e^w}{w^{k+1}}. \tag{3.5.10}
\]

The second representation is
\[
h_k(x) = \frac{2^k}{i \sqrt{\pi}} e^{x^2} \int_{\varepsilon + i \mathbb{R}} dz e^{z^2 - 2xz} z^k, \tag{3.5.11}
\]
for any \( \varepsilon > 0 \).

**Proof.** By residue’s theorem we can write
\[
h_k(x) = (-1)^n e^{x^2} \frac{k!}{2\pi i} \oint_{\Gamma_x} dz e^{-\frac{z^2}{4} - \frac{2xz}{x}} \frac{1}{(z - x)^{k+1}}, \tag{3.5.12}
\]
where \( \Gamma_x \) is a closed anti-clockwise path around \( x \). The change of variable \( z = -w + x \) gives the first representation.

For the second representation, first notice that for \( \frac{1}{\sqrt{\pi}} \int_{\varepsilon + i \mathbb{R}} dz e^{z^2 - 2xz} = e^{-x^2} \).

Taking the \( k \)th derivative gives the \((-2z)^k\) factor in the integral. \( \square \)

\(^1\)Hermite polynomials are used in physics, where they give rise to the eigenstates of the quantum harmonic oscillator. The normalized polynomials \( q_n \) defined in (3.5.16) are indeed called \textit{normalized oscillator wave-function}.
Here are some properties of Hermite polynomials.

**Proposition 3.5.5.** It holds

\[ h_k(x) = 2^k x^k + \ldots, \]  

(3.5.13)

and

\[ \int_R dx h_k(x) h_\ell(x) e^{-x^2} = \delta_{k,\ell} \sqrt{\pi} 2^k k! . \]  

(3.5.14)

For \( x = \sqrt{2n+1} - 2^{-1/2} n^{-1/6} t, \ t \) bounded, the following asymptotics holds

\[ e^{-x^2/2} h_n(x) = (2\pi)^{1/4} 2^{n/2} \sqrt{n!} n^{-1/2} \left[ Ai(t) + \mathcal{O}(n^{-2/3}) \right] . \]  

(3.5.15)

**Proof of Proposition 3.5.2.** In our case the weight is \( w(x) = e^{-x^2/2N} \) instead of \( e^{-x^2} \), so the change of variable \( x \mapsto x/\sqrt{2N} \) leads to

\[ q_k(x) = \frac{1}{(2\pi N)^{1/4}} \frac{1}{\sqrt{2N}} h_k \left( \frac{x}{\sqrt{2N}} \right) \]  

(3.5.16)

and the correlation kernel becomes

\[ K_N(x, y) = e^{-x^2/2N} \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\pi N}} \frac{1}{2^k k!} h_k \left( \frac{x}{\sqrt{2N}} \right) h_k \left( \frac{y}{\sqrt{2N}} \right) . \]  

(3.5.17)

Using the two integral representations (3.5.10) and (3.5.11) we have

\[ K_N(x, y) = e^{-x^2/2N} \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\pi N}} \frac{1}{2^k k!} \int_{\Gamma_0} dw \frac{e^{-w^2 + 2w \frac{z}{\sqrt{2N}}}}{w^{k+1}} \]  

\[ \times \frac{2^k}{\pi k!} \int_{\epsilon+i\mathbb{R}} dz e^{z^2 - 2z \frac{z}{\sqrt{2N}} z^k} . \]  

(3.5.18)

Call \( \ell = N - 1 - k \) and using the fact that the integral in \( w \) is zero for \( \ell \geq N \), we obtain

\[ (3.5.19) \]

\[ e^{-x^2/2N} \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\pi N}} \frac{1}{2^k k!} \int_{\Gamma_0} dw \frac{e^{-w^2 + 2w \frac{z}{\sqrt{2N}}}}{w^{k+1}} \int_{\epsilon+i\mathbb{R}} dz e^{z^2 - 2z \frac{z}{\sqrt{2N}} z^k} z^{N-1-\ell} . \]  

Choosing \( |z| > |w| \) we can exchange the sum and the integrals, and using \( \frac{w^t}{z^t} = \frac{1}{z-w} \) we obtain

\[ (3.5.19) \]

\[ e^{-x^2/2N} \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\pi N}} \frac{1}{2^k k!} \int_{\Gamma_0} dw \int_{\epsilon+i\mathbb{R}} dz e^{z^2 - 2z \frac{z}{\sqrt{2N}} z^k} \frac{1}{w^{k+1}} . \]  

(3.5.20)

The change of variable \( w = W \sqrt{N}/2, \ z = Z \sqrt{N}/2 \) leads to the claimed result. \( \square \)

Similarly, as for the Bessel kernel (the one of the PNG model), one obtains

\[ \lim_{N \to \infty} N^{1/3} K_N^{\text{Hermite}}(2N + s_1 N^{1/3}, 2N + s_2 N^{1/3}) = K_{Ai}(s_1, s_2) . \]  

(3.5.21)

This, together with the exponential bound on the decay of the Hermite kernel, leads to the following result.

**Theorem 3.5.6.** Let \( \lambda_{N, \max} \) be the largest eigenvalue of a GUE \( N \times N \) matrix. Then,

\[ \lim_{N \to \infty} \mathbb{P}(\lambda_{N, \max} \leq 2N + s N^{1/3}) = F_{\text{GUE}}(s) . \]  

(3.5.22)
Chapter 4

Schur processes

The Schur processes are measures on sequences of partitions whose weights are defined as products of suitable skew Schur functions. They were introduced in [3] for the purpose of asymptotic analysis of large random plane partitions. The Schur processes are very useful in asymptotic analysis of a variety of combinatorial and probabilistic models, such as finding the length of the longest increasing subsequence of random permutations, random growth processes in (1 + 1) and (2 + 1) dimensions, random tilings.

**Definition 4.0.1.** A Schur process (on rank N) is a probability measure on sequences of Young diagrams \(\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \ldots, \mu^{(N-1)}, \lambda^{(N)}\), parametrized by \(2N\) Schur positive specializations \(\rho^+_0, \rho^+_1, \ldots, \rho^+_{N-1}, \rho^-_1, \rho^-_2, \ldots, \rho^-_N\) and it is given by

\[
P(\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \ldots, \mu^{(N-1)}, \lambda^{(N)}) = \frac{1}{Z} s_{\lambda^{(1)}}(\rho^+_0) s_{\lambda^{(1)}}(\rho^-_1) s_{\lambda^{(2)}}(\rho^+_1) \ldots s_{\lambda^{(N-1)}}(\rho^+_1) s_{\lambda^{(N)}}(\rho^-_N),
\]

where \(Z\) is a positive normalization constant.

A representation with diagrams is

\[
\begin{array}{c}
\rho^0 \downarrow \lambda^{(1)} \\
\mu^{(1)} \downarrow \rho^+_1 \\
\rho^-_1 \downarrow \lambda^{(2)} \\
\ldots \mu^{(N-1)} \downarrow \rho^+_N \\
\rho^-_N \downarrow \emptyset
\end{array}
\]

(4.0.2)

**Remark 4.0.2.** For \(N = 1\), \(P(\lambda^{(1)}) = \frac{1}{Z} s_{\lambda^{(1)}}(\rho^+_0) s_{\lambda^{(1)}}(\rho^-_1)\), which is simply a Schur measure.

We are not obliged to add and remove pieces of the partition at alternating time. We can also have models where one adds only pieces for a while and then remove for a while. This can be achieved by choosing some of the specializations to be trivial. Indeed, if we choose \(\rho^-_1 = (0; 0; 0)\), then \(\mu^{(1)} \equiv \lambda^{(1)}\) and thus the diagram representation becomes

\[
\begin{array}{c}
\rho^0 \downarrow \lambda^{(1)} \\
\mu^{(1)} \downarrow \rho^+_1 \\
\rho^-_1 \downarrow \lambda^{(2)} \\
\rho^+_1 \downarrow \lambda^{(1)} \\
\rho^-_1 \downarrow \emptyset
\end{array}
\]

(4.0.3)
4.1 Joint measure of Schur process

Now we give two results:

**Proposition 4.1.1.** Let \( H(p_1; p_2) = \exp \left( \sum_{k \geq 1} \frac{p_k(p_2)}{k} \right) \) and assume that \( \forall i < j, H(p_i^+; p_j^-) < \infty \). Then, the Schur process is well-defined and
\[
Z = \prod_{0 \leq i < j \leq N} H(p_i^+; p_j^-). \tag{4.1.1}
\]

**Proposition 4.1.2.** The projection of the Schur process on \( \lambda^{(k)} \) is the Schur measure \( \mathbb{P}_{\rho_1, \rho_2} \) with \( \rho_1 = (\rho_0^+, \rho_1^+, \ldots, \rho_{k-1}^+), \rho_2 = (\rho_k^-, \rho_{k+1}^-, \ldots, \rho_N^+) \).

The proofs of both the proposition are consequences of the Skew-Cauchy identity (4.1.6) and the consistency relation (4.1.11). In the context of the PNG model we already computed them, see the proofs of Proposition 2.3.25 and Proposition 2.3.26.

The algebraic steps are the same, but for sake of completeness let us write them again. The goal is to find an expression for \( \mathbb{P}(\lambda^{(k_1)}, \ldots, \lambda^{(k_m)}) \) for \( 0 < k_1 < k_2 < \cdots < k_m \leq N \).

Let us represent the Skew-Cauchy identity and consistency relation with diagrams. The Skew-Cauchy identity states that for any \( \lambda, \nu \in \mathbb{Y} \),
\[
\sum_{\mu \in \mathbb{Y}} s_{\mu/\lambda}(\rho)s_{\mu/\nu}(\tilde{\rho}) = H(p; \tilde{\rho}) \sum_{\kappa \in \mathbb{Y}} s_{\lambda/\kappa}(\tilde{\rho})s_{\nu/\kappa}(\rho), \tag{4.1.2}
\]
or equivalently
\[
\sum_{\mu \in \mathbb{Y}} \left( \begin{array}{c} \rho \rightarrow \\ \lambda \end{array} \right) \left( \begin{array}{c} \mu \rightarrow \\ \nu \end{array} \right) = H(p; \tilde{\rho}) \sum_{\kappa \in \mathbb{Y}} \left( \begin{array}{c} \lambda \rightarrow \\ \kappa \end{array} \right) \left( \begin{array}{c} \rho \rightarrow \\ \nu \end{array} \right). \tag{4.1.3}
\]

The consistency relation is
\[
s_{\nu/\lambda}(\rho_1, \rho_2) = \sum_{\mu \in \mathbb{Y}} s_{\nu/\mu}(\rho_1)s_{\nu/\mu}(\rho_2), \tag{4.1.4}
\]
for any \( \lambda, \nu \in \mathbb{Y} \), or equivalently
\[
\sum_{\rho_1, \rho_2} \left( \begin{array}{c} \rho_1 \rightarrow \\ \lambda \end{array} \right) \left( \begin{array}{c} \rho_2 \rightarrow \\ \nu \end{array} \right) = \sum_{\mu \in \mathbb{Y}} \left( \begin{array}{c} \mu \rightarrow \\ \rho_1 \end{array} \right) \left( \begin{array}{c} \rho_2 \rightarrow \\ \nu \end{array} \right). \tag{4.1.5}
\]

**Lemma 4.1.3.** We have
\[
\sum_{\lambda^{(1)}, \ldots, \lambda^{(k-1)}} s_{\lambda^{(1)}}(\rho_0^+)s_{\lambda^{(1)}}(\rho_1^-)s_{\lambda^{(2)}}(\rho_1^-)s_{\lambda^{(1)}}(\rho_2^+) \cdots s_{\lambda^{(k-1)}}(\rho_{k-1}^+) \rho_{k-1}^+ = c \cdot s_{\lambda^{(k)}}(\rho_0^+, \ldots, \rho_{k-1}^+) \tag{4.1.6}
\]
for some constant \( c > 0 \).

**Proof.** With the diagram representation
Summing over $\lambda^{(k-1)}$, by the Skew-Cauchy identity, we obtain the sum over $\kappa$ of

\[
\sum_{\lambda^{(k-1)} / \mu^{(k-1)}} s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) \cdots s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+})
\]

Then, we sum over $\mu^{k-2}, \mu^{k-1}$ and the consistency relation gives the sum over $\kappa$ of

\[
\sum_{\lambda^{(k-1)} / \mu^{(k-1)}} s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) \cdots s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+})
\]

Iterating, we get

\[
\sum_{\lambda^{(k-1)} / \mu^{(k-1)}} s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) \cdots s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+})
\]

but $s_{\emptyset / \kappa}$ is non-zero only if $\kappa = \emptyset$. 

\section{4.1. Joint Measure of Schur Process}

\[\text{Lemma 4.1.4. We have}
\]

\[
\sum_{\lambda^{(k-1)} / \mu^{(k-1)}} s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+}) \cdots s_{\lambda^{(k-1)} / \mu^{(k-1)}} (\rho_{\tilde{k}^+})
\]

\[\begin{equation}
= c \cdot \sum_{\kappa \in \mathcal{Y}} s_{\lambda^{(k-1)} / \kappa} (\rho_{\tilde{k}^+}, \ldots, \rho_{\tilde{k}^+ - 1}) s_{\lambda^{(k-1)} / \kappa} (\rho_{\tilde{k}^+}, \ldots, \rho_{\tilde{k}^+ - 1})
\end{equation}
\]

\[\text{for some constant } c > 0.
\]

\[\text{Proof. Analogous to the proof of Lemma 4.1.3.}\]

We recall some notions that will be useful for the next results. For a Schur-positive specialization $\rho = (\alpha; \beta; \gamma)$, the generating function is

\[
H(\rho)(z) = \sum_{k \geq 0} h_k(\rho) z^k = c^\gamma z \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z},
\]

where $h_k$ is the $k$th complete omogeneous function \[2.3.18\]. By Cauchy residue theorem it can be expressed through a complex integral, as

\[
h_k(\rho) = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{d\rho}{\rho^{n+1}} H(\rho)(z),
\]

or equivalently, as

\[
h_k(\rho) = \frac{1}{2\pi i} \oint_{\Gamma_{0,\alpha}} dw w^{n-1} H(\rho)(w^{-1}),
\]

where $\Gamma_{0,\alpha}$ is a closed anticlockwise path around 0 and the $\alpha_i$s. We can obtain \[4.1.9\] from \[4.1.8\] with a change of variable $z = w^{-1}$. The closed anticlockwise path $\Gamma_0$ including 0 but not $\alpha_i^{-1}$’s is mapped into a closed clockwise path $\Gamma_{0,\alpha}$ which includes 0 and $\alpha_i$’s; we reorient $\Gamma_{0,\alpha}$ in the standard direction by a change of sign,
that compensate the sign – of the differential \( dz = -w^{-2}dw \).

Let us first consider the case where for some \( M \) arbitrary large, \( \ell(\lambda^{(k)}) \leq M \), \( 1 \leq k \leq N \). As a consequence of the integral representation and the fact that (Proposition 2.3.20)

\[
\zeta_{\lambda/\mu}(\rho) = \det \left[ h_{k,-i-(\mu_j-j)}(\rho) \right]_{1 \leq i,j \leq \max\{\ell(\lambda),\ell(\mu)\}},
\]

we have a straightforward result:

**Corollary 4.1.5.**

\[
s_{\lambda(k)}(\rho^+_0, \ldots, \rho^+_{k-1}) = \det \left[ \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dz}{z^\xi_i + j} \right]_{1 \leq i,j \leq M},
\]

where \( \xi_i^{(k)} = \lambda_i^{(k)} - i + 1 \) and

\[
H(\rho^+_0, \ldots, \rho^+_{k-1})(z) = \prod_{\ell=0}^{k-1} H(\rho^+_\ell)(z).
\]

Another consequence is

**Corollary 4.1.6.**

\[
\sum_{\kappa \in \mathcal{Y}} s_{\lambda^{(k_1)}/\kappa}(\rho^-) s_{\lambda^{(k_2)}/\kappa}(\rho^+),
\]

where \( \xi_i^{(k)} = \lambda_i^{(k)} - i + 1 \).

**Proof.** Let \( \zeta_i := k_i - i + 1, i \geq 1 \). Then

\[
= \sum_{\zeta_1 > \zeta_2 > \cdots > \zeta_M} \det \left[ h_{\xi_i^{(k_1)} - \zeta}(\rho^-) \right]_{1 \leq i,j \leq M} \cdot \det \left[ h_{\xi_i^{(k_2)} - \zeta}(\rho^+) \right]_{1 \leq i,j \leq M}.
\]

Since the determinants are antisymmetric in \( (\zeta_1, \ldots, \zeta_M) \) (it is zero if there exist \( i \neq j \) such that \( \zeta_i = \zeta_j \)), Eq. (4.1.15) becomes

\[
\frac{1}{M!} \sum_{\zeta_1, \zeta_2, \ldots, \zeta_M} \det \left[ h_{\xi_i^{(k_1)} - \zeta}(\rho^-) \right]_{1 \leq i,j \leq M} \cdot \det \left[ h_{\xi_i^{(k_2)} - \zeta}(\rho^+) \right]_{1 \leq i,j \leq M},
\]

where we used Cauchy-Binet identity.

By (4.1.14), (4.1.15)

\[
\sum_{\zeta \in \mathcal{Z}} h_{\xi_i^{(k_1)} - \zeta}(\rho^-) h_{\xi_j^{(k_2)} - \zeta}(\rho^+)
\]

\[
= \sum_{\zeta} \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dz}{z^\xi_i^{(k_1)} + 1} \cdot \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{w^\xi_j^{(k_2)} + 1} H(\rho^-)(z^{-1}) \left( \frac{z}{w} \right)^\zeta.
\]
As $\Gamma_{0,o+}$ is a closed anticlockwise path including 0 and the $\alpha_i$'s and excluding $\alpha_i^{-1}$'s, for the path in $z$ we have two possible choices: a contour around 0 not including $w$ ($|z| < |w|$) and a contour that includes $w$ ($|z| > |w|$).

In order to exchange the sum in $\zeta$ with the integral we need $\sum_{\zeta} \left( \frac{z}{w} \right) \zeta < \infty$. So we split the sum of (4.1.17) in two terms, a sum over $\zeta \geq 0$ and a sum over $\zeta < 0$: for the first sum we choose the first contour $|z| < |w|$ and for the second one we choose the contour $|z| > |w|$. Since

$$
\sum_{\zeta \geq 0} \left( \frac{z}{w} \right) \zeta = \frac{w}{w - z} \text{ for } |z| < |w|
$$

$$
\sum_{\zeta < 0} \left( \frac{z}{w} \right) \zeta = -\frac{w}{w - z} \text{ for } |z| > |w|
$$

(4.1.17) becomes

$$
\frac{1}{(2\pi i)^2} \oint_{\Gamma_0} \oint_{\Gamma_{0,o+}} \frac{dz}{z} \frac{dw}{w} \left( \frac{H(\rho^-)(z)}{\xi_{i+1}^{(k_1)}w} \right) H(\rho^+)(w^{-1}) \frac{1}{w - z}
$$

$$
= \frac{1}{(2\pi i)^2} \oint_{\Gamma_{0,o+}} \oint_{\Gamma_{0,w}} \frac{dz}{z} \frac{dw}{w} \left( \frac{H(\rho^-)(z)}{\xi_{i+1}^{(k_1)}w} \right) H(\rho^+)(w^{-1}) \frac{1}{w - z}
$$

$$
= \frac{1}{2\pi i} \oint_{\Gamma_{0,o+}} \frac{dz}{z} \left( \frac{H(\rho^-)(z)H(\rho^+)(z^{-1})}{\xi_{i+1}^{(k_1)}\zeta_j^{(k_2)}} \right). \tag{4.1.19}
$$

The last equation is a consequence of Cauchy Residue Theorem. \hfill \square

Applying Corollaries 4.1.5 and 4.1.6 we get

**Theorem 4.1.7.** Let $0 < k_1 < k_2 < \cdots < k_m \leq N$. Denote by $\xi^{(\ell)}_i = \lambda_i^{(k_\ell)} - i + 1$, $\ell = 1, \ldots, m$. Then,

$$
\mathbb{P}(\lambda^{(k_1)}, \ldots, \lambda^{(k_m)}) = \frac{1}{Z_N} \det \left[ \Phi_i(\xi^{(1)}_j) \right]_{1 \leq i,j \leq M} \cdot \prod_{\ell=1}^{m-1} \det \left[ T_{\ell,\ell+1}(\xi^{(\ell)}_i, \xi^{(\ell+1)}_j) \right]_{1 \leq i,j \leq M} \det \left[ \Psi_i(\xi^{(m)}_j) \right]_{1 \leq i,j \leq M}, \tag{4.1.20}
$$

where

$$
\Phi_i(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} \oint_{\Gamma_{0,o-}} \frac{dz}{z} \frac{H(\rho_0^+, \ldots, \rho_{k_1-1})(z)H(\rho_1^+, \ldots, \rho_{k_1-1})(z^{-1})}{z^{x+i-1}}, \tag{4.1.21}
$$

$$
T_{\ell,\ell+1}(x,y) = \frac{1}{2\pi i} \oint_{\Gamma_{0,o+}} \oint_{\Gamma_{0,w}} \frac{dz}{z} \frac{H(\rho_{k_{\ell}}^-, \ldots, \rho_{k_{\ell+1}-1})(z)H(\rho_{k_{\ell}}^+, \ldots, \rho_{k_{\ell+1}-1})(z^{-1})}{z^{x+y}}
$$

$$
= \frac{1}{2\pi i} \oint_{\Gamma_{0,o-}} \oint_{\Gamma_{0,w}} \frac{dw}{w} \frac{H(\rho_{k_{\ell}}^-, \ldots, \rho_{k_{\ell+1}-1})(w^{-1})H(\rho_{k_{\ell}}^+, \ldots, \rho_{k_{\ell+1}-1})(w)}{w^{y-x}}, \tag{4.1.22}
$$

and

$$
\Psi_j(y) = \frac{1}{2\pi i} \oint_{\Gamma_{0,o-}} \oint_{\Gamma_{0,w}} \frac{dz}{z} \frac{H(\rho_{k_{m}}^-, \ldots, \rho_N)(z^{-1})H(\rho_{k_{m}}^+, \ldots, \rho_{N-1})(z)z^{y+j-1}}{z^{y+j-1}}. \tag{4.1.23}
$$
Remark 4.1.8. By applying Cauchy-Binet iteratively, we obtain the normalization constant
\[
Z_N = \det \left[ \frac{1}{2\pi i} \oint_{\Gamma_{0,-}} \frac{dz}{z^i} H(\rho_0^+, \ldots, \rho_{N-1}^+)(z) H(\rho_N^+)(z^{-1}) \right]_{1 \leq i,j \leq M}.
\] (4.1.24)

It is not necessary to put the extra \( H(\rho_{k_1}^-, \ldots, \rho_{k_m}^-)(z^{-1}) \) and \( H(\rho_{k_m}^+, \ldots, \rho_N^+)(z) \) in the \( \Phi_i \)'s and \( \Psi_j \)'s. But it will be convenient when we do the analogue of a biorthogonalization later to determine the correlation kernel later.

Remark 4.1.9. The distribution of \( \mathcal{P}(\lambda^{(k_1)}, \ldots, \lambda^{(k_m)}) \) is the same if we take \( \Phi_i \) and \( \Psi_j \) as
\[
\Phi_i(x) = \frac{1}{2\pi i} \oint_{\Gamma_{0,+}} \frac{dz}{z^i} H(\rho_0^+, \ldots, \rho_{k_1}^+)(z),
\] (4.1.25)
\[
\Psi_j(y) = \frac{1}{2\pi i} \oint_{\Gamma_{0,+}} \frac{dz}{z^j} H(\rho_{k_m}^+, \ldots, \rho_N^+)(z) z^{y+j-1}.
\] (4.1.26)

This change, in fact, affects only the normalization constant, which becomes
\[
Z_N = \det \left[ \frac{1}{2\pi i} \oint_{\Gamma_{0,+}} \frac{dz}{z^i} H(\rho_0^+, \ldots, \rho_{k_m}^+)(z) H(\rho_{k_m}^+, \ldots, \rho_N^+)(z^{-1}) \right]_{1 \leq i,j \leq M}.
\] (4.1.27)

The next goal is to see that the measure of Theorem 4.1.7 is determinantal and compute the correlation kernel.

### 4.2 Conditional L-ensembles

The biorthogonal ensemble is just a specific type of determinantal point process, then we want to discuss another general framework of processes. For simplicity, consider finite state space first. The countable state space case will be fine provided all the sums are integrable, while the continuous state space case can be obtained as appropriate limit of discrete cases.

Let \( \mathcal{X} \) be a finite set. For any subset \( X \subset \mathcal{X} \) and a matrix \( L : \mathcal{X} \times \mathcal{X} \to \mathbb{C} \) define
\[
L_X := [L(x_i, x_j)]_{x_i, x_j \in X}
\] (4.2.1)
as the submatrix of size \( |X| \times |X| \) with entries of \( L \) restricted to \( X \).

**Definition 4.2.1.** Assume that \( L \) is positive definite, that is, \( \forall X \subseteq \mathcal{X}, \det [L_X] \geq 0 \). Then we define a simple point process \( \eta \) on \( \mathcal{X} \) by
\[
\mathbb{P}(\eta \equiv 1 \text{ on } X, \eta \equiv 0 \text{ on } X^c) = \frac{\det [L_X]}{\det [I + L]}, \quad \forall X \subset \mathcal{X}.
\] (4.2.2)

The point process \( \eta \) is called L-ensemble.

**Proposition 4.2.2.** The L-ensemble is determinantal with correlation kernel \( K \) given by \( K = L(I + L)^{-1} \).

We will prove this result later for a more general case.

**Remark 4.2.3.** The L-ensembles form a restricted class of determinantal point processes, which does not include the biorthogonal ensembles.
Consider a non-empty subset Λ of $\mathcal{X}$. Given a $L$-ensemble $\eta$ on $\mathcal{X}$, we define now a new simple point process $\xi$ on $\Lambda$ by considering the intersection of $\eta$ with $\Lambda$, provided $\Lambda \cap \Lambda^c = \emptyset$ is contained in $\eta$.

**Definition 4.2.4.** Let $\Lambda \subset \mathcal{X}$, $\Lambda \neq \emptyset$. Define $\xi$ by

$$P(\xi \equiv 1 \text{ on } Y, \xi \equiv 0 \text{ on } Y^c = \Lambda \setminus Y) = \frac{\det[\mathbb{I}_Y \cup \Lambda]}{\det[\mathbb{I}_\Lambda + L]}, \forall Y \subset \Lambda. \quad (4.2.3)$$

The point process $\xi$ is called conditional $L$-ensemble.

The block representation of $\mathbb{I}_\Lambda$ on $\mathcal{X} = \Lambda \cup \Lambda^c$ is

$$\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}. \quad (4.2.6)$$

**Theorem 4.2.5** (Borodin, Rains). The conditional $L$-ensemble is a determinantal point process with correlation kernel given by

$$K = \mathbb{I}_\Lambda - (\mathbb{I}_\Lambda + L)^{-1}|_{\Lambda \times \Lambda}. \quad (4.2.4)$$

Note that if $\Lambda^c = \emptyset$, then $\mathbb{I}_\Lambda = \mathbb{I}$ and the kernel is

$$K = \mathbb{I} - (\mathbb{I} + L)^{-1} = (\mathbb{I} + L)(\mathbb{I} + L)^{-1} - (\mathbb{I} + L)^{-1} = L(\mathbb{I} + L)^{-1}, \quad (4.2.5)$$

which is the kernel of the $L$-ensemble in Proposition 4.2.2.

To prove this result we need the following lemma.

**Lemma 4.2.6.** Let $M = \text{diag}(\mu_1, \ldots, \mu_n)$ and $L$ a $n \times n$ matrix. Then,

$$\det[M + L] = \sum_{\sigma, \tau \in S_n} \prod_{i=1}^n \mu_i \delta_{\tau(i), \sigma(i)}.$$
When \( \mu_1 = \cdots = \mu_n = 1 \), the formula reduces to \( \det [I + L] = \sum_{X \subseteq \{1, \ldots, n\}} \det [LX] \). Moreover if \( L = -K \), then
\[
\det [I - K] = \sum_{X \subseteq \{1, \ldots, n\}} (-1)^{|X|} \det [K_X]
\]
\[
= \sum_{m=0}^{n} (-1)^m \frac{1}{m!} \sum_{x_1 < \cdots < x_m} \det [K_{x_i, x_j}]_{1 \leq i, j \leq m}
\]
which is the Fredholm determinant expansion for finite matrices.

**Remark 4.2.7.** One can show that every determinantal point process is actually a conditional L-ensemble.

Before proving Theorem 4.2.5, we still need an identity from linear algebra:

**Lemma 4.2.8.** We have
\[
\det \left[ (M^{-1})_X \right] = \frac{\det [M_X^{-1}]}{\det [M]}.
\]

**Proof.** Let \( M \) be a block matrix, \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), where \( A = M_X \) and \( D = M_{X^c} \). The inverse is
\[
M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BQCA^{-1} & * \\ * & Q \end{bmatrix}
\]
with \( Q^{-1} = D - CA^{-1}B \), which implies \( CA^{-1}B = D - Q^{-1} \). Then
\[
\det \left[ (M^{-1})_X \right] = \det \left[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right] = \det \left[ \begin{bmatrix} \mathbb{1} + A^{-1}BQC & A^{-1}B(\mathbb{1} + QCA^{-1}B) \\ C & D \end{bmatrix} \right]
\]
\[
= \det \left[ \begin{bmatrix} \mathbb{1} + A^{-1}BQC & A^{-1}B + A^{-1}BQD - A^{-1}BQ^{-1} \\ C & D \end{bmatrix} \right]
\]
\[
= \det \left[ \begin{bmatrix} \mathbb{1} & A^{-1}BQ \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ C & D \end{bmatrix} \right] = \det [D].
\]

**Proof of Theorem 4.2.5** By Lemma 4.2.6 we get
\[
\det [K_Y] = \sum_{X \subseteq Y} (-1)^{|X|} \det \left[ (I_\Lambda + L)^{-1} \left|_{X \setminus \Lambda} \right. \right]
\]
Applying Lemma 4.2.8 we get
\[
\det [K_Y] = \sum_{Z: X \subseteq Y^c} (-1)^{|X|} \frac{\det [I_\Lambda + L]_{X^c}}{\det [I_\Lambda + L]}
\]

**Claim:** \( \frac{\det [(I_\Lambda + L)_{X^c}]}{\det [I_\Lambda + L]} = \mathbb{P} (\xi \equiv 0 \text{ on } X) \).
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It follows that
\[
\det [K_Y] = \sum_{X \subseteq Y} (-1)^{|X|} \mathbb{P} (\xi \equiv 0 \text{ on } X) = \mathbb{P} (\xi \equiv 1 \text{ on } Y),
\]
where the last equality follows from the inclusion-exclusion formula.

To conclude the proof, we show that the claim is correct.

\[
\det [(\mathbb{I}_\Lambda + L)|_{X^c}] = \sum_{M \subseteq X^c} \prod_{Y \not\subseteq M} \delta_{Y \Lambda} \det [L_M].
\]

If \( Y \subseteq \Lambda \), then \( \delta_{Y \Lambda} = 1 \), otherwise is zero. Thus \( M \) must include all \( \Lambda^c \) to have a non-zero contribution. Denoting \( M = \Lambda^c \cup Y \) (disjoint union) we get

\[
\det [(\mathbb{I}_\Lambda + L)|_{X^c}] = \sum_{M \subseteq \Lambda^c} \det [L_M] = \sum_{Y \subseteq X \cap \Lambda} \det [L_{Y \cup \Lambda^c}].
\]

Since

\[
\frac{\det [L_{Y \cup \Lambda^c}]}{\det [\mathbb{I}_\Lambda + L]} = \mathbb{P} (\xi \equiv 1 \text{ on } Y, \xi \equiv 0 \text{ on } Y^c),
\]
we have that

\[
\sum_{Y \subseteq X \cap \Lambda} \frac{\det [L_{Y \cup \Lambda^c}]}{\det [\mathbb{I}_\Lambda + L]} = \mathbb{P} (\xi \equiv 0 \text{ on } X).
\]

\[\blacksquare\]

4.2.1 Biorthogonal ensembles as conditional L-ensemble

Let us see how biorthogonal ensembles fits into the framework of conditional L-ensembles. Consider \( \mathcal{X} = \{1, \ldots, N\} \cup \Lambda = \Lambda^c \cup \Lambda \). Let

\[
L = \begin{bmatrix} 0 & \Phi^T \\ \Psi & 0 \end{bmatrix},
\]

with \( \Phi = (\Phi_1, \ldots, \Phi_N) \) and \( \Psi = (\Psi_1, \ldots, \Psi_N) \).

We ask when \( \det [L_{(1, \ldots, M)\cup Y}] = 0 \). If \( Y = (Y_1, \ldots, Y_M) \),

\[
L_{(1, \ldots, N)\cup Y} = \begin{bmatrix} 0 & \ldots & 0 & \Phi_1(Y_1) & \ldots & \Phi_1(Y_M) \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \Phi_N(Y_1) & \ldots & \Phi_N(Y_M) \\ \Psi_1(Y_1) & \ldots & \Psi_N(Y_1) & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \Psi_1(Y_M) & \ldots & \Psi_N(Y_M) & 0 & \ldots & 0 \end{bmatrix}.
\]

The determinant of this matrix is zero if \( M \neq N \); indeed, for any \( \sigma \in S_{M+N} \), \( \prod_{i=1}^{M+N} (L_{(1, \ldots, N)\cup Y})_{i,\sigma(i)} \) contains always a 0. On the other hand, if \( |Y| = N \), then

\[
\det [L_{(1, \ldots, N)\cup Y}] = c \cdot \det [\Phi_i(Y_j)]_{1 \leq i, j \leq N} \cdot \det [\Psi_i(Y_j)]_{1 \leq i, j \leq N},
\]
which is the biorthogonal ensemble.

Remark 4.2.9. It is easy to verify (using the block representation) that \((\mathbb{I}_\Lambda - (\mathbb{I}_\Lambda + L)^{-1})_{\Lambda \times \Lambda}\) is the kernel \((4.2.5)\) of the biorthogonal ensemble.
4.3 Correlation kernel for Schur processes

Now we are ready to see that the Schur processes have determinantal correlations and compute the correlation kernel. Consider the conditional $L$-ensemble $X = \{1, \ldots, M\} \cup \Lambda$, with $\Lambda = X^{(1)} \times \cdots \times X^{(m)}$ (in this case $X^{(i)} = \mathbb{Z}$) and

\[
L = \begin{pmatrix}
0 & \Phi^T & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & -T_{1,2} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -T_{2,3} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -T_{m-1,m}
\end{pmatrix},
\]

(4.3.1)

where $\Psi = (\Psi_1, \ldots, \Psi_M)$ and $\Phi = (\Phi_1, \ldots, \Phi_M)$.

**Lemma 4.3.1.** Let $Y = (Y^{(1)}, \ldots, Y^{(m)})$ with $Y^{(k)} = (Y_{1}^{(k)}, \ldots, Y_{N(k)}^{(k)})$. Then,

(a) if $\exists k$ such that $N(k) \neq M$, then $\det (L_{Y \setminus \Lambda}) = 0$;

(b) if $N(1) = \cdots = N(m) = M$, then $\det (L_{Y \setminus \Lambda}) = \text{const} \cdot \mathbb{P}(\lambda^{(k_1)}, \ldots, \lambda^{(k_m)})$.

**Proof.** It follows easily from the fact that a determinant of a block-matrix of type $L$ is zero if and only if all blocks are square and of the same size. $\square$

To compute the correlation kernel, we need to determine $(I - L - I)_{\Lambda \times \Lambda}$, since $K = I - (I + L)^{-1}|_{\Lambda \times \Lambda}$ (see Theorem 4.2.5).

Before stating the main theorem, we antepone a technical lemma.

**Lemma 4.3.2.** Let $A, D$ be square matrices. Then

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} -M^{-1} & M^{-1}BD^{-1} \\ D^{-1}CM^{-1} & D^{-1}CM^{-1}BD^{-1} \end{pmatrix},
\]

(4.3.2)

where $M = BD^{-1}C - A$, provided that all the matrices are invertible.

The proof is an easy computation of linear algebra.

**Theorem 4.3.3 (Eynard-Mehta Theorem).** The measure \[4.20\] is determinantal with correlation kernel given by

\[
K(i, x; j, y) = -T_{ij}(x, y) + \sum_{k, \ell=1}^{M} (T_{i,m} \ast \Psi_k)(x) [M^{-1}]_{k,\ell} (\Phi_\ell \ast T_{1,j})(y),
\]

(4.3.3)

where $i, j \in \{1, \ldots, M\}$, $M_{ij} = \Phi_1 \ast T_{1,2} \ast \cdots \ast T_{m-1,m} \ast \Psi_j$ and

\[
T_{i,j}(x, y) = \begin{cases} (T_{i,j+1} \ast \cdots \ast T_{j-1,j})(x, y) & \text{if } i < j, \\
0 & \text{if } i \geq j.
\end{cases}
\]

(4.3.4)

**Remark 4.3.4.** Given operators $A, B$ and vectors $V, W$, we define

\[
(A \ast B)(x, y) = \sum_z A(x, z)B(z, y), \quad (A \ast V)(x) = \sum_z A(x, z)V(z),
\]

\[
(W \ast B)(y) = \sum_z W(z)B(z, y), \quad W \ast V = \sum_z W(z)V(z).
\]

(4.3.5)
4.4. SPECIAL CASE OF THE CONTINUOUS TIME PNG

Proof.

\[
\mathbb{1}_A + L = \begin{pmatrix}
0 & \Phi & 0 & 0 & \cdots & 0 \\
0 & \mathbb{1} & -T_{1,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \mathbb{1} & -T_{m-1,m} \\
\Psi & 0 & \cdots & 0 & 0 & \mathbb{1}
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

(4.3.6)

A simple computation gives

\[
D^{-1} = \begin{pmatrix}
\mathbb{1} & T_{1,2} & T_{1,3} & \cdots & T_{1,m} \\
0 & \mathbb{1} & T_{2,3} & \cdots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \mathbb{1} & -T_{m-1,m} \\
0 & 0 & \cdots & 0 & 0 & \mathbb{1}
\end{pmatrix}.
\]

(4.3.7)

Applying Lemma 4.3.2 with \(A = 0, B = (\Phi^T, 0, \ldots, 0), C = (0, \ldots, 0, \Psi)^T\),

\[
D = \begin{pmatrix}
\mathbb{1} & -T_{1,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \mathbb{1} & -T_{m-1,m} \\
0 & 0 & \cdots & 0 & 0 & \mathbb{1}
\end{pmatrix}
\]

(4.3.8)

and \(M = BD^{-1}C = \Phi^T * T_{1,m} * \Psi\), we have

\[
K = (\mathbb{1} - (D^{-1} - D^{-1}CM^{-1}BD^{-1}))_{i,j}
\]

\[
= -T_{i,j} \mathbb{1}_{j>i} + \sum_{k,\ell} (D^{-1}C)_{ik} (M^{-1})_{k,\ell} (BD^{-1})_{\ell j}
\]

\[
= -T_{i,j} \mathbb{1}_{j>i} + \sum_{k,\ell} T_{i,m} * \Psi_k (M^{-1})_{k,\ell} \Phi_\ell * T_{1,j}.
\]

(4.3.9)

The inverse matrix \(M^{-1}\) can be computed as we made for the Schur measures with the following result (for details, see Theorem 2.2 of [5]).

**Theorem 4.3.5.** For the Schur process,

\[
K(i, x; j, y) = \frac{1}{(2\pi i)^2} \int dw \int dv \frac{H(\rho_0^i, \ldots, \rho_{k_i-1}^i)(v) H(\rho_{k_i}, \ldots, \rho_N^i)(w^{-1})}{H(\rho_0^j, \ldots, \rho_{k_j-1}^j)(w) H(\rho_{k_j}, \ldots, \rho_N^j)(v^{-1})} \frac{1}{v - w^{-1}}.
\]

(4.3.10)

for \(i \geq j\) with the contour in Figure 4.2(a) and for \(i < j\) with the contour in Figure 4.2(b).

4.4 Special case of the continuous time PNG

The following result is an application of Theorem 4.3.5 to the continuous time PNG and recall that \(H(0; 0; \gamma)(z) = e^{z\gamma}\).
Corollary 4.4.1. Let \(-t < t_1 < t_2 < \cdots < t_m < t\). Then the continuous PNG has correlation kernel at times \(t_1, \ldots, t_m\) given by

\[
K^\text{PNG}_t(t_i, x; t_j, y) = \frac{1}{(2\pi i)^2} \oint dw \oint dv \frac{w^{x-1} e^{(t_j + t)w} e^{(t-t_i)w^{-1}}}{v^y e^{(t_i + t)v} e^{(t-t_j)v^{-1}}} \frac{1}{v - w}
\]

(4.4.1)

with the contour in Figure 4.3(a) for \(t_i \geq t_j\) and Figure 4.3(b) for \(t_i < t_j\).

The kernel (4.4.1) can be reformulated in terms of Bessel functions.

Proposition 4.4.2. For \(t_i \geq t_j\) we have

\[
K^\text{PNG}_t(t_i, x; t_j, y) = \sum_{\ell \geq 0} J_{\ell+y}(2\sqrt{t^2 - t_i^2}) \left(\frac{t - t_i}{t + t_i}\right)^{\ell+\frac{y}{2}} J_{\ell+x}(2\sqrt{t^2 - t_j^2}) \left(\frac{t + t_j}{t - t_j}\right)^{\ell+\frac{x}{2}}.
\]

(4.4.2)

and for \(t_i < t_j\) we have

\[
K^\text{PNG}_t(t_i, x; t_j, y) = -\sum_{\ell < 0} J_{\ell+y}(2\sqrt{t^2 - t_i^2}) \left(\frac{t - t_i}{t + t_i}\right)^{\ell+\frac{y}{2}} J_{\ell+x}(2\sqrt{t^2 - t_j^2}) \left(\frac{t + t_j}{t - t_j}\right)^{\ell+\frac{x}{2}}.
\]

(4.4.3)
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Proof. Recall the identity

\[ J_n(2t) = \frac{1}{2\pi i} \oint_{\Gamma_0} dz e^{t(z^{-1} - z)} z^{n-1} = \frac{1}{2\pi i} \oint_{\Gamma_0} dw e^{t(w-w^{-1})} \frac{w^{n+1}}{w^{n+1}}. \] (4.4.4)

With the change of variable \( z \mapsto z \sqrt{\frac{t+x}{t-x}} \),

\[ J_n(2\sqrt{t^2 - x^2}) \left( \frac{t+x}{t-x} \right)^{n/2} = \frac{1}{2\pi i} \oint_{\Gamma_0} dz e^{t(z^{-1} - z)} e^{x(z^{-1} + z)} z^{n-1} \]

\[ = \frac{1}{2\pi i} \oint_{\Gamma_0} dw e^{t(w-w^{-1})} e^{x(w+w^{-1})} \frac{w^{n+1}}{w^{n+1}}. \] (4.5.5)

For \( t_i \geq t_j \), we can choose the path \(|v| > |w|\), which gives \( \frac{1}{v-w} = \sum_{\ell \geq 0} \frac{w^\ell}{(v-w)^\ell} \) and

\[ K_t^{PNG}(t_i; x, t_j, y) = \sum_{\ell \geq 0} \oint_{\Gamma_0} dw e^{t(w^{-1}-w)} e^{-t_i(w^{-1}+w)} e^{x(w+w^{-1})} \frac{w^{\ell+x-1}}{w^{\ell+1}} \oint_{\Gamma_0} dv e^{t(v-v^{-1})} e^{-t_j(v+v^{-1})} \]

\[ = \sum_{\ell \geq 0} J_{t+x}(2\sqrt{t^2 - t_i^2}) \left( \frac{t-x}{t+x} \right) \frac{\ell+x+1}{\ell+x+1} J_{t+x}(2\sqrt{t^2 - t_j^2}) \left( \frac{t+j}{t-j} \right) \frac{\ell+j+1}{\ell+j+1}. \] (4.6.6)

Similarly, for \( t_i \geq t_j \), we can choose the path \(|v| < |w|\), which gives \( \frac{1}{v-w} = -\sum_{\ell < 0} \frac{w^\ell}{(v-w)^\ell} \) and the claimed formula.

4.5 Scaling limit for the continuous time PNG

We want to study the behavior for large times of the top line of the line ensemble of the multilayer PNG in continuous time. The law of large number for the height function is

\[ \lim_{t \to \infty} \frac{h(\tau t, t)}{t} = 2\sqrt{1 - \tau^2}, \quad \tau \in [-1, 1]. \] (4.5.1)

For simplicity, we consider the case \( \tau = 0 \). We need to study the \( t \to \infty \) limit of the rescaled height function

\[ h_t^{\text{esc}}(u) = \frac{h(u t^{2/3}, t) - 2t \sqrt{1 - u^2 t^{-2/3}}}{t^{1/3}}. \] (4.5.2)

and

\[ x = 2t \sqrt{1 - u^2 t^{-2/3}} + \xi t^{1/3} = 2t - u^2 t^{1/3} + \xi t^{1/3} + O(1) \]

\[ t = ut^{2/3} \] (4.5.3)

The rescaled kernel is given by

\[ K_t^{\text{esc}}(u, \xi; \tilde{\xi}) := t^{1/3} K_t^{PNG}(ut^{2/3}, 2t \sqrt{1 - u^2 t^{-2/3} + \xi t^{1/3}}; \tilde{u} t^{2/3}, 2t \sqrt{1 - \tilde{u}^2 t^{-2/3} + \tilde{\xi} t^{1/3}}). \] (4.5.4)
For $u > \bar{u}$, calling $\ell = \lambda^{1/3}$ we have

$$K_t^{\text{resc}}(u, \xi; \bar{u}, \bar{\xi}) = t^{-1/3} \sum_{\lambda \in t^{-1/3} \mathbb{N}_0} t^{1/3} J_{\lambda t^{1/3} + 2t - u^2 t^{1/3} + \xi t^{1/3}}(2t \sqrt{1 - u^2}) \left(1 - \frac{ut - 1/3}{1 + ut - 1/3}\right)^{\lambda t^{1/2}}$$

$$\times t^{1/3} J_{\lambda t^{1/3} + 2t - \bar{u}^2 t^{1/3} + \bar{\xi} t^{1/3}}(2t \sqrt{1 - \bar{u}^2}) \left(1 + \frac{\bar{u}t - 1/3}{1 - \bar{u}t - 1/3}\right)^{\lambda t^{1/2}}$$

$$\times \left(1 - ut - 1/3\right)^{2t - (\xi - \bar{u})^2 t^{1/3}} \left(1 + \bar{u}t - 1/3\right)^{2t - (\bar{\xi} - \bar{u})^2 t^{1/3}}. \quad (4.5.5)$$

The term in the last line is just a conjugation. Using the asymptotics

$$\lim_{t \to \infty} t^{1/3} J_{\lambda t^{1/3} + 2t - (\xi - u^2) t^{1/3}}(2t \sqrt{1 - u^2}) = \Lambda_i(\xi), \quad (4.5.6)$$

and

$$\lim_{t \to \infty} \left(1 - ut - 1/3\right)^{\lambda t^{1/2}} \left(1 + ut - 1/3\right)^{\lambda t^{1/2}} = e^{-\lambda u}, \quad (4.5.7)$$

together with the exponential bounds (Lemma 4.4.1(e)) we get by dominated convergence

$$\lim_{t \to \infty} K_t^{\text{resc}}(u, \xi; \bar{u}, \bar{\xi}) \cong \int_0^\infty d\lambda \Lambda_i(\xi + \lambda) \Lambda_i(\bar{\xi} + \lambda) e^{-\lambda (u - \bar{u})}, \quad (4.5.8)$$

For $u < \bar{u}$, in a similar way we obtain

$$\lim_{t \to \infty} K_t^{\text{resc}}(u, \xi; \bar{u}, \bar{\xi}) \cong -\int_{-\infty}^0 d\lambda \Lambda_i(\xi + \lambda) \Lambda_i(\bar{\xi} + \lambda) e^{-\lambda (u - \bar{u})}. \quad (4.5.9)$$

**Definition 4.5.1.** The extended Airy kernel is defined by

$$K_{\bar{\Lambda}}(u, \xi; v, \bar{\xi}) = \left\{\begin{array}{ll} \int_0^\infty d\lambda \Lambda_i(\xi + \lambda) \Lambda_i(\bar{\xi} + \lambda) e^{-\lambda (u - v)} & \text{for } u \geq v, \\ -\int_0^\infty d\lambda \Lambda_i(\xi + \lambda) \Lambda_i(\bar{\xi} + \lambda) e^{-\lambda (u - v)} & \text{for } u < v. \end{array}\right. \quad (4.5.10)$$

The marginal of the point process with extended Airy kernel is called the **Airy$_2$ process.**

**Definition 4.5.2.** The Airy$_2$ process, $A_2$, is the process with $m$-points joint distributions at $u_1 < u_2 < \cdots < u_m$ given by

$$\mathbb{P} \left( \bigcap_{k=1}^m \{A_2(u_k) \leq s_k\} \right) = \det (\mathbb{1} - \chi_s K_{\bar{\Lambda}}(\chi_s))_{L^2}((u_1, \ldots, u_m) \times \mathbb{R}), \quad (4.5.11)$$

where $\chi_s(u_k, x) = \mathbb{1}_{x > u_k}$ and $K_{\bar{\Lambda}}$ is the extended Airy kernel.

**Remark 4.5.3.** In terms of Fredholm series expansion,

$$\det (\mathbb{1} - \chi_s K_{\bar{\Lambda}}(\chi_s))_{L^2}((u_1, \ldots, u_m) \times \mathbb{R})$$

$$= \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{i_1, \ldots, i_n \in \{u_1, \ldots, u_m\}} \int_{s_{i_1}}^\infty dx_1 \cdots \int_{s_{i_n}}^\infty dx_n \det \left[ K_{\bar{\Lambda}}(u_{i_k}, x_k; u_{i_j}, x_j) \right]_{1 \leq k, j \leq n}. \quad (4.5.12)$$
Using the pointwise convergence of the kernel together with exponential decay of the kernel as $\xi_k \to \infty$ one proves that the Fredholm determinants also converge. This gives the following result.

**Theorem 4.5.4.** We have

$$\lim_{t \to \infty} h_{t}^{\text{resc}} = A_2$$

in the sense of finite-dimensional distributions. Explicitly, for any $m \in \mathbb{N}$, $u_1 < \ldots < u_m$ and $s_1, \ldots, s_m$ fixed,

$$\lim_{t \to \infty} \mathbb{P}\left( \bigcap_{\ell=1}^{m}\{h_{t}^{\text{resc}}(u_\ell) \leq s_\ell\} \right) = \mathbb{P}\left( \bigcap_{\ell=1}^{m}\{A_2(u_\ell) \leq s_\ell\} \right).$$

We give some properties of the Airy$_2$ process.

**Lemma 4.5.5.**

(a) Let $H_{\text{Ai}} = -\frac{d^2}{dx^2} + x$ and $K_2(x, y) = \int_0^\infty d\lambda \text{Ai}(x + \lambda)\text{Ai}(y + \lambda)$. Then,

$$K_{\text{Ai}}(u, x; \tilde{u}, \tilde{x}) = -\left(e^{-(u-\tilde{u})H_{\text{Ai}}} \right)(x, \tilde{x}) 1_{(u < \tilde{u})} + \left(e^{uH_{\text{Ai}}}K_2 e^{-(\tilde{u}H_{\text{Ai}})} \right)(x, \tilde{x}).$$

(b) The process $u \mapsto A_2(u)$ is stationary with $\mathbb{P}(A_2(u) \leq s) = F_{\text{GUE}}(s)$.

(c) Let $g(u) = \text{Var} \left( A_2(u) - A_2(0) \right)$. Then

$$g(u) = \begin{cases} 2u + O(u^2) & \text{for small } u, \\ 2 \text{ Var} \left( A_2(0) \right) - \frac{2}{\pi} + O(u^{-4}) & \text{for } |u| \text{ large}. \end{cases}$$

(d) There is a continuous version of $A_2$.

(e) The Airy$_2$ process is not a Markov process.

The behaviour (4.5.16) means that the Airy$_2$ process shows very decorrelation at large $|u|$.

Property (a) can be seen using the fact that the Airy function is an eigenfunction of the operator $H_{\text{Ai}}$. Property (b) is obvious from the definition. Property (c) is much less trivial and it was proven is a paper of Widom.

### 4.6 The Airy$_2$ process in random matrices

Consider the matrix-valued stationary Ornstein-Uhlenbeck process on $N \times N$ Hermitian matrices with one-time distribution

$$\mathbb{P}(H \in dH) = \frac{1}{Z_N} e^{-\frac{1}{2} \text{Tr} H^2} dH.$$  

(4.6.1)

We have already seen that the largest eigenvalue of $H$, $\lambda_{\text{max}}^N$, converges in distribution to $F_{\text{GUE}}$ for $N \to \infty$. The transition probability on matrices is given by

$$\mathbb{P}(H(t) \in dH|H(0) = H_0) = \text{const} \cdot e^{-\frac{\text{Tr}(H-H_0)^2}{2N(1-q^2)}} dH,$$

(4.6.2)

where $q = q(t) = e^{-t/2N}$.

**Definition 4.6.1.** The stationary matrix-valued process on matrices with transition probability (4.6.2) is called Dyson’s Brownian Motion (DBM).
Proposition 4.6.2.

Define the point process with measure supported on \((t_k; \lambda_i(t_k)), i = 1, \ldots, N, k = 1, \ldots, m\), i.e. the eigenvalues process at times \(t_1 < t_2 < \cdots < t_m\),

\[
\eta = \sum_{k=1}^{m} \sum_{i=1}^{N} \delta_{(t_k; \lambda_i(t_k))}.
\]

(4.6.3)

Let \(H_k\) be the matrix at time \(t_k\). Let \(0 = t_0 < t_1 < \ldots < t_m\) and \(q_j = e^{-(t_{j+1} - t_j)/2N}\). Then the multi-time measure reads

\[
\frac{1}{Z_{N,m}} \exp \left( - \frac{\text{Tr}(H_k^2)}{2N} \right) \prod_{j=0}^{m-1} \exp \left( - \frac{\text{Tr}(H_{j+1} - q_j H_j)^2}{2N(1 - q_j^2)} \right) dH_0 \cdots dH_m.
\]

(4.6.4)

The joint distribution of the eigenvalues is obtained using the Harish-Chandra/Itzykson-Zuber formula: let \(A = \text{diag}(a_1, \ldots, a_N)\) and \(B = \text{diag}(b_1, \ldots, b_N)\) two diagonal \(N \times N\) matrices. Let \(d\mu\) denote the Haar measure on the unitary group \(U(N)\). Then,

\[
\int_{U(N)} d\mu(U) \exp(\text{Tr}(A U B U^*)) = \frac{\det(e^{a_i b_j})_{1 \leq i,j \leq N}}{\Delta(a) \Delta(b)} \prod_{p=1}^{N-1} p!,
\]

(4.6.5)

where \(\Delta(a)\) is the Vandermonde determinant of the vector \(a = (a_1, \ldots, a_N)\).

Denote by \(U(t_k)\) the unitary matrix which diagonalize \(H(t_k)\), namely

\[
H(t_k) = U(t_k) \Lambda(t_k) U(t_k)^*, \quad k = 1, \ldots, m,
\]

(4.6.6)

with \(\Lambda(t_k) = \text{diag}(\lambda_1(t_k), \ldots, \lambda_N(t_k))\). The Jacobian of the transformation \(H(t_k) \rightarrow (\Lambda(t_k), U(t_k))\) gives

\[
dH(t_k) = \Delta_N(\Lambda(t_k))^2 d\Lambda(t_k) d\mu_N(U(t_k)),
\]

(4.6.7)

where \(d\mu_N\) denotes the Haar measure on the unitary ground \(U_N\) (which is the analogue of the Lebesgue measure). Thus the measure at two times, \(t_0\) and \(t_1\) is given by

\[
e^{-a_1 \text{Tr}(H_1^2)} e^{b_1 \text{Tr}(H_1)} e^{-a_2 \text{Tr}(H_2^2)} dH_0 dH_1
e^{-a_1 \text{Tr}(\Lambda(t_0)^2)} e^{b_1 \text{Tr}(U(t_1) \Lambda(t_1) U^*(t_1) U(t_0) \Lambda(t_0) U^*(t_0))}
\times \Delta_N(\Lambda(t_0))^2 \Delta_N(\Lambda(t_1))^2 d\Lambda(t_0) d\Lambda(t_1) d\mu_N(U_0) d\mu_N(U_1).
\]

(4.6.8)

for some constants \(a_1, a_2, b\).

Denoting \(V = U(t_1) U(t_0)^*\), \(d\mu_N(U_0) d\mu_N(U_1) = d\mu_N(V) d\mu_N(U_0)\). Then using the HCIZ formula (4.6.5) for the integral over \(d\mu_N(V)\) and (4.6.7) one obtains the following result.

Proposition 4.6.2. The joint distribution of the eigenvalues is given by

\[
\frac{1}{Z_{N,m}} \Delta_N(\lambda_0) \prod_{i=1}^{N} e^{-a_i \lambda_i^2} \int \prod_{k=0}^{m-1} \det \left[ e^{\beta_k \lambda_i, \lambda_{k+1,j}} \right]_{i,j=1,\ldots,N} \Delta_N(\lambda_m) \prod_{i=1}^{N} e^{-a_i \lambda_i^2} \prod_{k=0}^{m} d\lambda_k
\]

(4.6.9)

where \(\lambda_{k,j} = \lambda_j(t_k), j = 1, \ldots, N\), are the eigenvalues of the matrix \(H_k, k = 0, \ldots, m\),

\[
\alpha_k = \frac{1}{(1 - q_k^2)^{2N}}, \quad \beta_k = \frac{q_k}{1 - q_k^2}, \quad q_k = e^{-(t_{k+1} - t_k)/2N},
\]

(4.6.10)
4.7. PLANE PARTITIONS

We see that this measure has the same form as Theorem 4.1.7 and thus it is determinantal. Some computations give that the extended kernel is given by

$$K_N(t, x; t, x) = \left\{ \begin{array}{ll}
\sum_{k=-N}^{t-1} e^{(t-k) \frac{x^2}{2N}} p_N(x) p_N(x) e^{-\frac{x^2}{2N}}, & t \geq 1, \\
- \sum_{k=0}^{t-1} e^{(t-k) \frac{x^2}{2N}} p_N(x) p_N(x) e^{-\frac{x^2}{2N}}, & t < 1,
\end{array} \right. $$

(4.6.11)

where $p_k(x) = h_k \left( \frac{x}{\sqrt{2N}} \right) \sqrt{1 - 2 \pi N x^2 / \sqrt{4}}$, with $h_k(x)$ the standard Hermite polynomials.

Consider the scaling $t = 2u N^{2/3}$ and $x = 2N + s N^{1/3}$. Then some computations give

$$\lim_{N \to \infty} N^{1/3} K_N(2u N^{2/3}, 2N + s N^{1/3}; 2u' N^{2/3}, 2N + s' N^{1/3}) = K_{A_2}(u, s; u', s').$$

(4.6.12)

This, together with some appropriate bounds for the large values of $s, s'$ to have convergence of the Fredholm determinant one proves the following result.

**Theorem 4.6.3.** Let $\lambda_N^{\max}$ denote the largest eigenvalue of the stationary GUE Dyson’s Brownian Motion. Then,

$$u \mapsto \frac{\lambda_N^{\max}(2u N^{2/3}) - 2N}{N^{1/3}}$$

(4.6.13)

converges to $u \mapsto A_2(u)$ in the sense of finite-dimensional distributions.

4.7 Plane partitions

4.7.1 Modelling of a crystal corner: the 3D-Ising corner

As a very common phenomenon, crystals are faceted at sufficiently low temperatures with facets joined through rounded pieces. Of course, on the atomic scale the crystal surface must be stepped. These steps meander through thermal fluctuations. On a facet the steps are regularly arranged except for small errors, whereas on a rounded piece the steps have more freedom to fluctuate. The following model aims to understand the precise step statistics, where the step bordering the crystal facet is of particular interest. To gain some insight we will study a simplified statistical mechanics model of a cubic crystal.

Let us first explain our model for the corner of a crystal. The crystal is assumed to be simple cubic with lattice $\mathbb{Z}^3$. We use lattice gas language and associate to each site $x \in \mathbb{Z}^3$, the occupation variable $n_x = 0, 1$ with 1 standing for site $x$ occupied by an atom and 0 for site $x$ empty. Up to a chemical potential the binding energy of the configuration $n$ is

$$H(n) = J \sum_{|x-y|=1} (n_x - n_y)^2, \quad J > 0.$$  

(4.7.1)

We consider the low temperature limit, meaning that all allowed configurations have the same energy, i.e., the same number of broken bonds. To define properly, we introduce the reference configuration $n^{\text{ref}}$ in which only the octant $\mathbb{Z}^3_+$ is occupied,

$$n^{\text{ref}}_x = \begin{cases} 1 & \text{for } x \in \mathbb{Z}^3_+, \\ 0 & \text{for } x \in \mathbb{Z}^3 \setminus \mathbb{Z}^3_+. \end{cases}$$

(4.7.2)

$n$ is an allowed configuration if for a sufficiently large box $\Lambda$ one has

$$n_x = n^{\text{ref}}_x \text{ for all } x \in \mathbb{Z}^3_+ \setminus \Lambda \text{ and } H(n) - H(n^{\text{ref}}) = 0.$$  

(4.7.3)

The set of allowed configurations is denoted by $\Omega$. By construction $\Omega$ is countable. To favor a crystal corner, we introduce the fugacity $q$, $0 < q < 1$, and assign to each $n \in \Omega$ the weight

$$q^{V(n)},$$

(4.7.4)
where \( V(n) \) is the number of atoms removed from \( n^{\text{ref}} \), i.e.

\[
V(n) = \sum_{x \in \mathbb{Z}^3_+} (1 - n_x).
\]  

(4.7.5)

A configuration \( n \in \Omega \) can uniquely be represented by a plane partition, i.e., a height function \( h \) over \( \mathbb{Z}^3_+ \) having the following properties. For the column at \((i,j) \in \mathbb{Z}^3_+\), all sites below \( h(i,j) \), excluding \( h(i,j) \), are empty and all sites above \( h(i,j) \) are filled. Thus \( n \in \Omega \) if and only if

\[
h(i+1,j) \leq h(i,j), \quad h(i,j+1) \leq h(i,j), \quad h(i,j) \to 0 \text{ for } (i,j) \to \infty.
\]  

(4.7.6)

By abuse of notation, the set of height functions satisfying (4.7.6) is also denoted by \( \Omega \). For \( h \in \Omega \) let \( V(h) = \sum_{(i,j) \in \mathbb{Z}^3_+} h(i,j) \) be the volume in \( \mathbb{Z}^3_+ \) below \( h \). Then the weight for the height \( h \) is \( q^{V(h)} \).

### 4.7.2 Connection with last passage percolation

We are interested in the line \( i \mapsto h(0,i), i \in \mathbb{Z}_+ \). We now give the connection between the 3D-Ising corner and a last passage percolation model on \( \mathbb{Z}^3_+ \). Consider independent random variables \( \omega(i,j), (i,j) \in \mathbb{Z}^3_+ \), geometrically distributed with mean value \( q^{i+j+1} \), \( q \in (0,1) \) as above:

\[
P(\omega(i,j) = k) = (1 - q^{i+j+1})q^{i+j+1+k}, \quad k \in \mathbb{Z}._+. 
\]  

(4.7.7)

Denote by \( L(i,j) \) the LPP from \((i,j)\) to \((\infty, \infty)\). This quantity is well defined because \( q < 1 \). In fact, consider the random variable \( \alpha(m) = \sum_{i+j \geq m} \omega(i,j) \). Then

\[
P(\alpha(m) \geq 1) \leq \mathbb{E}(\alpha(m)) \leq \frac{mq^m}{(1 - q)} \to 0, \quad m \to \infty.
\]  

(4.7.8)

Therefore when a maximizer of the LPP goes to infinity, with probability one it only passes a finite number of sites \((i,j)\) with \( \omega(i,j) > 0 \). By symmetry \( L(i,j) \) is also the maximal length of directed polymers from infinity to \((i,j)\), i.e., with down-left steps. We already know the relation between LPP and PNG growth. The connection between LPP and 3D-Ising corner is

\[
h(i,0) = L(i,0), \quad h(0,i) = L(0,i),
\]  

(4.7.9)

for \( i \in \mathbb{Z}_+ \) in law. On the other hand, there is not a simple connection between \( L(i,j) \), \( i,j > 0 \), and the heights \( h(i,j) \).

### 4.7.3 3D-Young diagram and line ensembles

A plane partition can be identified with a 3D Young diagram, as in Figure 4.54. Consider the description of the interface in terms of gradient lines, as in Figures 4.55 and 4.56. Introduce the coordinates \( t = j - i \) and define

\[
h_{\ell}(t) = h_{t,j} - \ell + 1, \quad \text{where } \ell = \ell(i,j) = \frac{i + j - |i - j|}{2}.
\]  

(4.7.10)

Then we have a bijection from \( h \) to a non-intersecting line ensemble (see Figure 4.59).

The volume of \( h \) is given by the sum of the areas under the \( h_k(t) \), \( k \geq 1 \) with respect to their lower possible positions, namely \(-k\). This can be achieved as follows: if there is a jump of size \( M \) at \( s \in \mathbb{Z} + \frac{1}{2} \), then we associate a weight \( q^{M/2} \).
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Figure 4.4: On the left a plane partition of volume 5. On the right the corresponding 3D Young diagram.

Figure 4.5: Gradient lines for a Young diagram.

Figure 4.6: Line ensembles for the 3D Young diagram in Figure 4.5. Up/down jumps at time $t$ have weights $q(t)$.

4.7.4 Schur process for 3D Young diagrams

From the non-intersecting line ensembles description we can immediately read off the associated Schur process (from $-\infty$ to $\infty$). Define

$$\alpha_t = \begin{cases} q^{-\frac{1}{2} - t}, & t \in \{-1, -2, \ldots\} \\ q^{\frac{1}{2} + t}, & t \in \{0, 1, 2, \ldots\} \end{cases}$$

then the Schur process is the measure on doubly-infinity partition given by

$$\prod_{k \leq -1} s_{\lambda(k+1)/\lambda(k)}((\alpha_k); 0; 0) \prod_{k \geq 0} s_{\lambda(k)/\lambda(k+1)}((\alpha_k); 0; 0).$$
Graphically, this is represented by

\[
\begin{align*}
&\lambda(0) \\
&\lambda(1) \\
&\lambda(2) \\
&\lambda(3)
\end{align*}
\]

4.7.5 Scaling limit

Figure 4.7: 3D Ising corned viewed from the (111)-direction for \( q = 0.98 \) [2].

The limit of large volume can be obtained by setting \( q = 1 - \frac{1}{T} \) and taking \( T \to \infty \).

Let \( h_T \) denote the random height function distributed according to

\[
\frac{1}{Z_T} \exp \left[ \ln \left( 1 - \frac{1}{T} \right) V(h) \right],
\]

with \( Z_T \) the normalizing partition function. Also, define the interface of the disordered region by

\[
b_T(t) = h_T(0, t).
\]

Then, precise asymptotic analysis on the correlation kernel gives

\[
\lim_{T \to \infty} \frac{b_T(\tau T)}{T} = b_{\infty}(\tau) = -2 \ln(1 - e^{-\tau/2}),
\]

and furthermore

\[
A_T(u) := \frac{b_T(\tau T + uT^{2/3}) - T b_{\infty}(\tau + uT^{-1/3})}{T^{1/3}} \xrightarrow{T \to \infty} \kappa \mathcal{A}_2(u\kappa/2),
\]

with \( \kappa = (2b_{\infty}'(\tau))^{1/3} \), in the sense of finite-dimensional distribution [2].
4.8 Aztec diamond

4.8.1 The random tiling model

Domino tilings of the Aztec diamond were introduced in [7]. For any $N \in \mathbb{N}$, we define the Aztec diamond $A_N$ of size $N$ as the union of all lattice squares $[m, m+1] \times [n, n+1]$, \(m, n \in \mathbb{Z}\), such that they are inside the region $\{(x, y) : |x| + |y| \leq N + 1\}$. A tiling of the Aztec diamond is a configuration of $N^2$ dominoes (i.e., $2 \times 1$ rectangles) such that the Aztec diamond $A_N$ is fully covered by dominoes. Alternatively, one can think of a tiling as a perfect matching on the bipartite graph obtained by considering the dual graph inside the Aztec diamond, see Figure 4.8. Denote by $T_N$ the set of all possible tilings of $A_N$. It is well-known [7] that $|T_N| = 2^{N(N+1)/2}$. A random tiling of the Aztec diamond $A_N$ is obtained by choosing a probability measure on $T_N$. Specifically we consider the probability measure $\mathbb{P}_N$ given by

$$\mathbb{P}_N(T) = \frac{a^{v(T)}}{\sum_{S \in T_N} a^{v(S)}}, \quad T \in T_N,$$

(4.8.1)

where $v(S)$ is the number of vertical dominoes in $S$, and $a > 0$ is a parameter. Below for simplicity we consider $a = 1$.

4.8.2 The line ensembles representation

To each tiling of the Aztec diamond one can associate a set of non-intersecting lines bijectively as shown by Johansson in [7]. We use a slightly different but equivalent representation. Add horizontal lines in the middle of the South-type dominoes, a 45-degrees oriented segment in the West-type dominoes and a step-down in the East-type dominoes as indicated in Figure 4.9. For a tiling of $A_N$ this results in $N$ non-intersecting lines with fixed initial and end points.

Further, one can think of the Aztec diamond of size $N$ as being embedded into tilings of $\mathbb{R}^2$, where outside the Aztec diamond we add only horizontal dominoes that do not overlap and fill the whole space, see the gray tiles in Figure 4.10 (left). By a simple geometric transformation, which can be easily recovered by comparing the left and the right illustrations in Figure 4.10, we obtain the final non-intersecting line ensemble representation of the random tiling of Figure 4.9.
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Figure 4.9: A random tiling of an Aztec diamond of size $N = 10$ and its associates set of lines. The North dominoes are red, the South are yellow, the East are green, and finally the West dominoes are blue.

Figure 4.10: A random tiling of an Aztec diamond of size $N = 10$. Left panel: The original tiling of Figure 4.9 with the lines extended deterministically outside $A_N$. Right pane: Final representation of the ensemble of non-intersecting lines. The dotted lines is the underlying LGV graph. The lines which are fixed and correspond to the gray dominoes of the left picture are dashed.
The above procedure gives a bijection between a tiling of an Aztec diamond and a configuration of non-intersecting lines on a Lindström-Gessel-Viennot (LGV) directed graph. The basic building block of the LGV graph is the one here on the right. Then, if we consider an Aztec diamond of size $N$, we have a one-to-one bijection between the set of non-intersecting lines on the LGV graph that consists of $N$ copies of the basic building bloc and where the lines start and end at all vertical positions $-1, -2, \ldots$, see Figure 4.10 for an example.

4.8.3 Schur process for the Aztec diamond

The line ensemble of the Aztec diamond of size $N$ can be also described in terms of Schur processes. Let $h_j(t - 1) = \lambda_j^{(t)} - j + 1, j \geq 1, t \in \{1, \ldots, N\}$ and $h_j(t) = \mu_j^{(t)} - j + 1, j \geq 1, t \in \{1, \ldots, N\}$. Then measure on partitions is given by the Schur process

$$\mathbb{P}(\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \ldots, \mu^{(N-1)}, \lambda^{(N)}) = \frac{1}{Z} s_{\lambda^{(1)}}(\rho^+) s_{\lambda^{(1)}}(\rho^-) s_{\lambda^{(2)}}(\rho^+) \ldots s_{\lambda^{(N-1)}}(\rho^+) s_{\lambda^{(N)}}(\rho^-), \quad (4.8.2)$$

with

$$\rho^+ = (\alpha, 0, 0), \quad \rho^- = (0, (\beta), 0), \text{ with } \alpha \beta = 1 (= a^2). \quad (4.8.3)$$

Thus, as in the previous models, in the $N \to \infty$ limit, the border of the disordered region is governed by the Airy$_2$ process.

**Theorem 4.8.1.** Let $h^N(t)$ be the north polar boundary of the Aztec diamond of size $N$. Then,

$$\lim_{N \to \infty} \frac{h^N(2^{-1/6}uN^{2/3}) - N/\sqrt{2}}{2^{-5/6}N^{1/3}} = A_2(u) - u^2, \quad (4.8.4)$$

in the sense of finite-dimensional distributions.

The shuffling algorithm

Finally, we would like to mention an algorithm that allows to generate a random tiling of the Aztec diamond of size $N$. For simplicity consider $a = 1$, that is, we want to generate a uniform random tiling of size $N$. One can proceed iteratively from the Aztec diamond of size $N - 1$ by means of the **shuffling algorithm** introduced in [7], that we recall briefly (see Figure 4.11 for an illustration). For that purpose, let us give some names to the different types of dominoes: by superimposing the Aztec diamond to a checkerboard table (which of the chosen two possible ways is irrelevant), there are actually four types of dominoes, as the black site might be either to the right/left (resp. top/bottom) site of a horizontal (resp. vertical) domino. We call them North, East, South, and West dominoes according to the following rule: the North (resp. South) domino is the horizontal one that fits into the top (resp. bottom) most part of the Aztec diamond, and similarly the East (resp. West) domino is the vertical one that fits into the right (resp. left) most part of the Aztec diamond.

The shuffling algorithm is the following. Start with a domino tiling of $A_N$ distributed according to $\mathbb{P}_N$:

**Step 1:** Move all the dominoes in the direction of their names by one unit. If in doing it a North and a South domino (resp. a East and a West domino) exchange their positions, remove them.
Step 2: At this point, the dominoes partially tile $A_{N+1}$. The empty region can be uniquely decomposed into $2 \times 2$ blocks. Independently of each other, each block is tiled with two horizontal dominoes with probability $1/2$ (in general $1/(1 + a^2)$) or two vertical dominoes with probability $1/2$ (in general $a^2/(1 + a^2)$).

This procedure generates a random tiling of the Aztec diamond $A_{N+1}$ with distribution $P_{N+1}$ as proven in [7].

This algorithm is a Markov chain on Aztec diamond configurations. As they are in bijection with a Schur process, the shuffling algorithm generates a Markov chain on a Schur process. In the next chapter we will study such kind of Markov chains.
Chapter 5

Markov Chains on Schur processes

5.1 Construction of Markov Chains

5.1.1 Stochastic matrices

Motivated by the Shuffling algorithm, we now construct Markov chains on Schur processes. For that purpose, we first introduce two stochastic matrices indexed by partitions, that will be the basis of this construction.

For any two given Schur-positive specializations $\rho, \rho'$ s.t. $H(\rho, \rho') < \infty$, we define the transition matrices indexed by Young diagrams $\lambda, \mu$ by

(a) $p_{\lambda \rightarrow \mu}(\rho; \rho') := \frac{1}{H(\rho, \rho')} \frac{s_\mu(\rho)}{s_\lambda(\rho)} s_{\mu/\lambda}(\rho')$;

(b) $p_{\lambda \rightarrow \mu}(\rho; \rho') := \frac{s_\mu(\rho)}{s_\lambda(\rho, \rho')} s_{\lambda/\mu}(\rho')$.

Let us verify that they are indeed stochastic matrices.

**Proposition 5.1.1.** $p_{\lambda \rightarrow \mu}$ and $p_{\lambda \rightarrow \mu}$ are stochastic, i.e. their entries are non-negative and $\forall \lambda \in \mathcal{Y}$,

\[ \sum_{\mu \in \mathcal{Y}} p_{\lambda \rightarrow \mu}(\rho; \rho') = 1, \]  

and

\[ \sum_{\mu \in \mathcal{Y}} p_{\lambda \rightarrow \mu}(\rho; \rho') = 1. \]

**Proof.** The positivity follows from the fact that $\rho, \rho'$ are Schur-positive specializations. By definition of $p_{\lambda \rightarrow \mu}(\rho; \rho')$

\[ \sum_{\mu \in \mathcal{Y}} p_{\lambda \rightarrow \mu}(\rho; \rho') = \sum_{\mu \in \mathcal{Y}} \frac{1}{H(\rho; \rho')} \frac{s_\mu(\rho)}{s_\lambda(\rho)} \frac{s_{\mu/\lambda}(\rho')}{s_{\lambda/\mu}(\rho')} = \frac{H(\rho; \rho')s_\lambda(\rho)}{H(\rho; \rho')} = 1, \]

where we applied the Skew-Cauchy identity (2.3.60) with $\lambda = \emptyset$. This proves (5.1.3a). By definition of $p_{\lambda \rightarrow \mu}(\rho; \rho')$

\[ \sum_{\mu \in \mathcal{Y}} p_{\lambda \rightarrow \mu}(\rho; \rho') = \sum_{\mu \in \mathcal{Y}} \frac{s_\mu(\rho)}{s_\lambda(\rho, \rho')} \frac{s_{\lambda/\mu}(\rho')}{s_{\lambda/\mu}(\rho')} = \frac{s_{\lambda}(\rho, \rho')}{s_{\lambda}(\rho, \rho')} = 1, \]

where we applied the Skew-Cauchy identity (2.3.60) with $\lambda = \emptyset$. This proves (5.1.3b).
Proposition 5.1.4. Indeed, where it can be viewed as a trajectory of a Markov chain with transition matrices 5.1.2 Action on Schur measures bridges, Schur processes can be obtained dynamically through Doob transforms.

The Schur process with distribution

$$\frac{1}{Z} s_{\lambda(1)} (\rho_0^+ ; \rho_1^+) s_{\lambda(2)} (\rho_1^+ ; \rho_2^+) \cdots s_{\lambda(N) / \lambda(1)} (\rho_N^+ ; \rho_1^+ ) s_{\lambda(N)} (\rho^-)$$

(5.1.6)
can be rewritten as

$$S_{\rho_0^+ ; \rho_1^+ ; \rho_2^+ ; \rho_3^+ ; \rho_N^+ ; \rho_1^- ; \rho_2^- ; \rho_N^-} (\lambda(N)) p_{\lambda(N-1)}^+ (\rho_0^+ ; \cdots ; \rho_N^+ ; \rho_N^- ; \rho_{N-1}^+) \cdots p_{\lambda(2)}^+ (\rho_0^+ ; \rho_1^+ ) s_{\lambda(1)} (\rho_1^+ ; \rho_2^+ )$$

(5.1.7)

where $S_{\rho_0^+ ; \rho_1^+}$ is the Schur measure. This generalizes to any Schur process, which means it can be viewed as a trajectory of a Markov chain with transition matrices $p^+$ and $p^-$ and initial distribution given by a Schur measure. For example,

$$\frac{1}{Z} s_{\lambda(1)} (\rho_0^+ ; \rho_1^+) s_{\lambda(2)} (\rho_1^+ ; \rho_2^+) s_{\lambda(3)} (\rho_2^+ ; \rho_3^+ ) s_{\lambda(4)} (\rho_3^+ ; \rho_4^+ ) s_{\lambda(5)} (\rho_4^+ ; \rho_5^+ )$$

(5.1.8)
can be written as

$$S_{\rho_0^+ ; \rho_1^+ ; \rho_2^+ ; \rho_3^+ ; \rho_4^+ ; \rho_5^+ ; \rho_6^+ ; \rho_7^+ ; \rho_8^+ ; \rho_9^+ ; \rho_0^- ; \rho_1^- ; \rho_2^- ; \rho_3^- ; \rho_4^- ; \rho_5^- ; \rho_6^- ; \rho_7^- ; \rho_8^- ; \rho_9^-} (\lambda(5)) p_{\lambda(4)}^+ (\rho_0^+ ; \rho_1^+ ; \rho_2^+ ; \rho_3^+ ) p_{\lambda(3)}^+ (\rho_0^+ ; \rho_1^+ ; \rho_2^+ ) p_{\lambda(2)}^+ (\rho_0^+ ; \rho_1^+ ; \rho_2^+ )$$

(5.1.9)

Indeed,

$$S_{\rho_0^+ ; \rho_1^+ ; \rho_2^+ ; \rho_3^+ ; \rho_4^+ ; \rho_5^+ ; \rho_6^+ ; \rho_7^+ ; \rho_8^+ ; \rho_9^+ ; \rho_0^- ; \rho_1^- ; \rho_2^- ; \rho_3^- ; \rho_4^- ; \rho_5^- ; \rho_6^- ; \rho_7^- ; \rho_8^- ; \rho_9^-} (\lambda(5)) p_{\lambda(4)}^+ (\rho_0^+ ; \rho_1^+ ; \rho_2^+ ; \rho_3^+ ) p_{\lambda(3)}^+ (\rho_0^+ ; \rho_1^+ ; \rho_2^+ ) p_{\lambda(2)}^+ (\rho_0^+ ; \rho_1^+ ; \rho_2^+ )$$

(5.1.10)

Remark 5.1.3. The transition matrices $p^+_{\lambda \rightarrow \mu}$ and $p^+_{\lambda \rightarrow \mu}$ are kind of Doob transforms. Schur processes are kind of random walks conditioned to start and end at the empty partition. For that reason, in complete analogy of the case of simple random walks bridges, Schur processes can be obtained dynamically through Doob transforms.

5.1.2 Action on Schur measures

Let us see now how these stochastic matrices acts on Schur measures $S_{\rho_1 ; \rho_2}$.

Proposition 5.1.4. For all $\mu \in \mathbb{Y}$,

$$\sum_{\lambda \in \mathbb{Y}} S_{\rho_1 ; \rho_2 ; \rho_3 ; \rho_4} (\lambda) p^+_{\lambda \rightarrow \mu} (\rho_2 ; \rho_3 ; \rho_4 ; \rho_5 ; \rho_6 ; \rho_7 ; \rho_8 ; \rho_9) = S_{\rho_1 ; \rho_2 ; \rho_3} (\mu)$$

(5.1.11a)

and

$$\sum_{\lambda \in \mathbb{Y}} S_{\rho_1 ; \rho_2 ; \rho_3 ; \rho_4} (\lambda) p^+_{\lambda \rightarrow \mu} (\rho_2 ; \rho_3 ; \rho_4 ; \rho_5 ; \rho_6 ; \rho_7 ; \rho_8 ; \rho_9) = S_{\rho_1 ; \rho_2 ; \rho_3} (\mu)$$

(5.1.11b)

Proof. We have

$$\sum_{\lambda \in \mathbb{Y}} \frac{s_{\lambda} (\rho_2) s_{\lambda} (\rho_3) s_{\lambda} (\rho_4) s_{\lambda} (\rho_5) s_{\lambda} (\rho_6) s_{\lambda} (\rho_7) s_{\lambda} (\rho_8) s_{\lambda} (\rho_9)}{H (\rho_0 ; \rho_1 ; \rho_2 ; \rho_3 ; \rho_4 ; \rho_5 ; \rho_6 ; \rho_7 ; \rho_8 ; \rho_9)}$$

(5.1.12)
using (5.1.10). Similarly,

\[ \sum_{\lambda \in \mathcal{Y}} s_\lambda(p_1)s_\lambda(p_2, p_3) \frac{s_\mu(p_2)}{H(p_1; p_2, p_3)} \frac{s_\lambda(p_3)}{s_\mu(p_3)} = \frac{s_\mu(p_2)}{H(p_1, p_2; p_3)} \sum_{\lambda \in \mathcal{Y}} s_\lambda(p_1)s_\mu(p_3) \]

(5.1.13)

\[ = s_\mu(p_2) \frac{H(p_1; p_3)}{H(p_1; p_2)H(p_1; p_3)} \]

using (2.3.60) with \( \lambda = \emptyset \).

5.1.3 The commutation relation

We present now a key property.

**Proposition 5.1.5 (Commutation relation).** For all \( \lambda, \nu \in \mathcal{Y} \), it holds

\[ \sum_{\mu \in \mathcal{Y}} p_{\lambda \rightarrow \mu}^\dagger (p_1, p_2; p_3)p_{\mu \rightarrow \nu}^\dagger (p_1; p_2) = \sum_{\mu \in \mathcal{Y}} p_{\lambda \rightarrow \mu}^\dagger (p_1; p_2)p_{\mu \rightarrow \nu}^\dagger (p_1; p_3) \]

(5.1.14)

**Proof.** The left hand side of (5.1.14) is equal to

\[ \sum_{\mu \in \mathcal{Y}} \frac{1}{H(p_1, p_2; p_3)} \frac{s_\mu(p_1, p_2)}{s_\lambda(p_1, p_2)} s_{\mu/\lambda}(p_1) \cdot \frac{s_\nu(p_1)}{s_\mu(p_1, p_2)} s_{\mu/\nu}(p_2) \]

(5.1.15)

while the right hand side is

\[ \sum_{\mu \in \mathcal{Y}} \frac{s_\mu(p_1)}{s_\lambda(p_1, p_2)} s_{\lambda/\mu}(p_2) \cdot \frac{1}{H(p_1; p_3)} \frac{s_\nu(p_1)}{s_\mu(p_1)} s_{\nu/\mu}(p_3), \]

(5.1.16)

which is equal to the left hand side of the equation.

As acting on Schur measures, the commutation relation says that adding \( p_3 \) and then removing \( p_2 \) is the same as first removing \( p_2 \) and then adding \( p_3 \), that is,

\[ S_{p_4; p_1, p_2} p_{\lambda}^\dagger (p_1, p_2; p_3) = S_{p_3, p_4; p_1, p_2} \]

(5.1.17)

followed by

\[ S_{p_3, p_4; p_1, p_2} p_{\lambda}^\dagger (p_1; p_2) = S_{p_3, p_4; p_1} \]

(5.1.18)

is the same as

\[ S_{p_4; p_1, p_2} p_{\lambda}^\dagger (p_1; p_2) = S_{p_4; p_1} \]

(5.1.19)

followed by

\[ S_{p_4; p_1} p_{\lambda}^\dagger (p_1; p_3) = S_{p_3, p_4; p_1} \]

(5.1.20)

Graphically we have

[Diagram showing the commutation relation with Schur measures]
5.1.4 Markov chains on $N$-tuples of Young diagrams

So far we have two types of transition matrices, one adding and one removing blocks from one Young diagram. Now we want to construct Markov chains on pairs of Young diagrams. Later we will generalize to Markov chains on $N$-tuples of Young diagrams and then we will apply this to special cases.

Definition 5.1.6. Let $\mathcal{Y}^{(2)}$ be the state space of pairs of Young diagrams $(\lambda^{(2)}, \lambda^{(1)})$ and let $\rho_1, \rho_2$ be Schur-positive specializations s.t. $p_{\lambda^{(2)} \rightarrow \lambda^{(1)}}(\rho_1; \rho_2) > 0$. Then we define the transition probabilities

$$P_\lambda \left( \left( \frac{\lambda^{(2)}}{\lambda^{(1)}} \right) \rightarrow \left( \frac{\mu^{(2)}}{\mu^{(1)}} \right) \right) := p_{\lambda^{(1)} \rightarrow \mu^{(1)}}(\rho_1; \rho') \frac{p_{\lambda^{(2)} \rightarrow \mu^{(2)}}(\rho_1, \rho_2; \rho') p_{\mu^{(2)} \rightarrow \mu^{(1)}}(\rho_1; \rho_2)}{\sum_{\mu} p_{\lambda^{(2)} \rightarrow \mu}(\rho_1, \rho_2; \rho')} p_{\mu \rightarrow \mu^{(1)}}(\rho_1; \rho_2)$$

(5.1.21)

In terms of diagrams,

The action of the transition matrix $P_\lambda$ is the following:

- first, $\lambda^{(1)}$ evolves to $\mu^{(1)}$ according to $p_{\lambda^{(1)} \rightarrow \mu^{(1)}}(\rho; \rho')$
- then, given $\lambda^{(2)}, \mu^{(1)}$, the distribution of $\mu^{(2)}$ is the one of the middle point in the two steps Markov chain with transitions probabilities $p_{\lambda^{(2)} \rightarrow \mu}(\rho_1, \rho_2; \rho')$ and $p_{\mu \rightarrow \mu^{(1)}}(\rho_1; \rho_2)$.

The transition matrix $P_\lambda$ has the nice property that, due to the commutation relation [5.1.14], it preserves a measure of a specific form:

Proposition 5.1.7. Set

$$\mathcal{M}_{\rho_1, \rho_2; \rho} \cdot (\lambda^{(1)}, \lambda^{(2)}) := S_{\rho_1, \rho_2; \rho} \cdot (\lambda^{(2)}) p_{\lambda^{(2)} \rightarrow \lambda^{(1)}}(\rho_1; \rho_2).$$

(5.1.22)

Then

$$\left( \mathcal{M}_{\rho_1, \rho_2; \rho} \cdot P_\lambda \right)(\mu^{(1)}, \mu^{(2)}) = \mathcal{M}_{\rho_1, \rho_2; \rho} \cdot (\mu^{(1)}, \mu^{(2)}).$$

(5.1.23)

Proof.

$$\left( \mathcal{M}_{\rho_1, \rho_2; \rho} \cdot P_\lambda \right)(\mu^{(1)}, \mu^{(2)}) = \sum_{\lambda^{(2)} \subset \lambda^{(1)}} \sum_{\lambda^{(2)} \subset \lambda^{(1)}} S_{\rho_1, \rho_2; \rho} \cdot (\lambda^{(2)}) p_{\lambda^{(2)} \rightarrow \lambda^{(1)}}(\rho_1; \rho_2) \
\cdot p_{\lambda^{(1)} \rightarrow \mu^{(1)}}(\rho_1; \rho') \frac{p_{\lambda^{(2)} \rightarrow \mu^{(2)}}(\rho_1, \rho_2; \rho') p_{\mu^{(2)} \rightarrow \mu^{(1)}}(\rho_1; \rho_2)}{\sum_{\mu} p_{\lambda^{(2)} \rightarrow \mu}(\rho_1, \rho_2; \rho')} p_{\mu \rightarrow \mu^{(1)}}(\rho_1; \rho_2)$$

(5.1.24)

Summing over $\lambda^{(1)}$ using Proposition [5.1.5] and then twice Proposition [5.1.4] we have

$$\left( \mathcal{M}_{\rho_1, \rho_2; \rho} \cdot P_\lambda \right)(\mu^{(1)}, \mu^{(2)}) = \sum_{\lambda^{(2)} \subset \lambda^{(1)}} S_{\rho_1, \rho_2; \rho} \cdot (\lambda^{(2)}) p_{\lambda^{(2)} \rightarrow \mu^{(2)}}(\rho_1, \rho_2; \rho') p_{\mu^{(2)} \rightarrow \mu^{(1)}}(\rho_1; \rho_2)$$

$$= S_{\rho_1, \rho_2; \rho} \cdot (\mu^{(2)}) p_{\mu^{(2)} \rightarrow \mu^{(1)}}(\rho_1; \rho_2)$$

$$= \mathcal{M}_{\rho_1, \rho_2; \rho} \cdot (\mu^{(1)}, \mu^{(2)}).$$

(5.1.25)
Now we are ready to extend Definition 5.1.6.

**Definition 5.1.8.** Let $\mathcal{Y}^{(N)}$ be the state space of $N$-tuples of Young diagrams. Let $\rho_1, \ldots, \rho_N$ be Schur-positive specializations and assume that $p_{\lambda(k) \to \lambda(k-1)}^{\downarrow}(\rho_1, \ldots, \rho_k; 0, 0) > 0$, $k = 2, \ldots, N$. We define the transition probabilities

$$P_{\lambda} \left( (\lambda^{(1)}, \ldots, \lambda^{(N)}) \to \left( \mu^{(1)}, \ldots, \mu^{(N)} \right) \right) := p_{\lambda^{(1)} \to \mu^{(1)}}^{\downarrow}(\rho_1; \rho'),$$

$$\prod_{k=2}^{N} p_{\lambda(k) \to \mu(k)}^{\downarrow}(\rho_1, \ldots, \rho_k; \rho') p_{\mu(k) \to \mu(k-1)}^{\downarrow}(\rho_1, \ldots, \rho_{k-1}; \rho_k).$$

(5.1.26)

This defines a Markov chain on $\mathcal{Y}^{(N)}$ with sequential update: first $\mu^{(1)}$, then $\mu^{(2)}$, then $\mu^{(3)}$ and so on until $\mu^{(N)}$.

For this process the generalization of Proposition 5.1.7 is immediate.

**Proposition 5.1.9.** Set

$$\mathcal{M}_{\rho_1, \ldots, \rho_N; \rho^{-}}(\vec{\lambda}) := S_{\rho_1, \ldots, \rho_N; \rho^{-}}(\lambda^{(N)}) p_{\lambda^{(N)} \to \lambda^{(N-1)}}^{\downarrow}(\rho_1, \ldots, \rho_{N-1}; \rho_N) \cdots p_{\lambda^{(2)} \to \lambda^{(1)}}^{\downarrow}(\rho_1; \rho_2).$$

Then

$$\sum_{\vec{\lambda} \in \mathcal{Y}^{(N)}} \mathcal{M}_{\rho_1, \ldots, \rho_N; \rho^{-}}(\vec{\lambda}) P_{\lambda} (\vec{\lambda} \to \vec{\mu}) = \mathcal{M}_{\rho_1, \ldots, \rho_N; \rho^{-}, \rho}(\vec{\mu}).$$

(5.1.27)

The measure $\mathcal{M}_{\rho_1, \ldots, \rho_N; \rho}$ is the measure of a Schur process and thus we know that it has determinantal correlation functions and how to compute its correlation kernel.

### 5.2 An application

#### 5.2.1 Evolution of partitions

Now we present an application in the simple case where the specializations are $\rho_k = ((1, 0, \ldots); 0; 0)$ and $\rho = (0; (b, 0, \ldots); 0)$, $b \in (0, 1)$ and later we will see how it defines an interacting particle system.

Consider the homogeneous discrete time Markov chain $\vec{\lambda}(t) = (\lambda^{(1)}(t), \ldots, \lambda^{(N)}(t))$ starting from the Schur process $\mathcal{M}_{\rho_1, \ldots, \rho_N; \rho^{-}}(\vec{\lambda}(0))$ with $\rho^{-} = (0; 0; 0)$.

First we ask what is the measure at $t = 0$. By definition of Schur measure

$$S_{\lambda^{(1)}, \ldots, \lambda^{(N)}}^{(N)}(0; 0; 0) s_{\lambda^{(N)}}^{(N)}(1, \ldots, 1; 0; 0)$$

$$= \delta_{\lambda^{(N)} = \emptyset}.$$  

(5.2.1)

The condition $p_{\emptyset \to \lambda^{(N-1)}}^{\downarrow}(\rho_1, \ldots, \rho_{N-1}; \rho_N) > 0$ implies that $\lambda^{(N-1)} = \emptyset$. Iterating we obtain that at $t = 0$ the measure is concentrated on the configuration $\vec{\lambda} = (\emptyset, \ldots, \emptyset)$, i.e.,

$$\mathcal{M}_{\rho_1, \ldots, \rho_N; \emptyset}(\vec{\lambda}(0)) = \delta_{\lambda^{(1)} = \emptyset} \cdots \delta_{\lambda^{(N)} = \emptyset}.$$  

(5.2.2)

Now we run start the dynamics and look what happens at time $t$. By Proposition 5.1.9 the measure at time $t$ is

$$\mathcal{M}_{\rho_1, \ldots, \rho_N; \rho}(\vec{\lambda}(t)) = S_{\lambda^{(1)}, \ldots, \lambda^{(N)}}^{(N)}(0; 0; 0) \cdot \prod_{k=2}^{N} p_{\lambda(k) \to \lambda(k-1)}^{\downarrow}(1, \ldots, 1; 0; 0).$$  

(5.2.3)
A consequence of this is that \( \lambda^{(k)}(t) \) has at most \( k \) non-empty rows and their coordinates satisfy the interlacing condition

\[
\lambda_1^{(k)} \geq \lambda_1^{(k-1)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_{k-1}^{(k-1)} \geq \lambda_k^{(k)}.
\] (5.2.4)

Indeed, one can see by iteration, using (5.1.11b) that the measure on \( \lambda^{(k)} \) is

\[
S_{(((1, \ldots, 1); 0; 0); (0, \ldots, b, 0))} \lambda^{(k)}(t) \prod_{j=2}^{k} p_{\lambda_j^{(j-1)}(1; 0; 0); (0; 0)}^{\lambda_j^{(j-1)}} \lambda^{(j-1)}((1, \ldots, 1); 0; 0) \) (5.2.5)

and

\[
S_{(((1, \ldots, 1); 0; 0); (0, \ldots, b, 0))} \lambda^{(k)}(t) = \frac{1}{(1 - b)^{\lambda^{(k)}(t)}} s_{\lambda^{(k)}(t)}((1, \ldots, 1); 0; 0) s_{\lambda^{(k)}(t)}(0; b, \ldots, b; 0) = 0
\] (5.2.6)

whenever \( \lambda^{(k)}_{k+1}(t) \neq 0 \).

Now we want to compute the transition probabilities. \( \lambda^{(1)} \) has a single non-zero row with transition probabilities

\[
p_{\lambda^{(1)}}^{(1)}((1; 0; 0); (0; b; 0)) = \frac{1}{1 + b s_{\lambda^{(1)}}((1; 0; 0))} s_{\mu^{(1)}}^{(1)} \lambda^{(1)}((0; b; 0))
\] (5.2.7)

since

\[
s_{\mu^{(1)}}^{(1)}((0; b; 0)) = \begin{cases} b, & \text{if } \mu^{(1)} = \lambda^{(1)} + 1 \\ 1, & \text{if } \mu^{(1)} = \lambda^{(1)} \\ 0, & \text{otherwise,} \end{cases}
\] (5.2.8)

More generally, for \( k \geq 2 \),

\[
\mathbb{P} \left( \lambda^{(k)}(t + 1) = \nu | \lambda^{(k)}(t) = \lambda, \lambda^{(k-1)}(t + 1) = \mu \right) = \frac{s_{\nu/\lambda}((0; b; 0)) s_{\mu/\nu}((1; 0; 0))}{\sum_{\nu \in \mathbb{Y}} s_{\nu/\lambda}((0; b; 0)) s_{\mu/\nu}((1; 0; 0))} \sim b^{\nu - |\lambda|},
\] (5.2.9)

provided that \( 0 \leq \nu - \lambda \leq 1 \) and that \( \mu, \nu \) are interlacing.

This means that the length of each row of \( \lambda^{(k)} \) independently increases by 1 with probability \( \frac{b}{1 + b} \) unless this contradicts the interlacing relation. In the latter case, the length either stays the same or increases by 1 with probability 1.

### 5.2.2 Evolution of a particle system

We want to describe the process with measure (5.2.3) as an interacting particle system. Let us introduce the \( \frac{N(N+1)}{2} \) particles with positions

\[
x_n^{(k)}(t) := \lambda_{n+1-k}^{(n)}(t) - n + k, 1 \leq k \leq n \leq N.
\] (5.2.10)

In particular, if all \( \lambda_{n}^{(n)}(0) = 0 \), this corresponds to the initial condition \( x_n^{(n)}(0) = -n + k \).

The interlacing relation for the particle system becomes

\[
x_1^{(1)}(t) < x_1^{(n-1)}(t) \leq x_2^{(1)}(t) < x_2^{(n-1)}(t) \leq \cdots \leq x_n^{(1)}(t).
\] (5.2.11)
The dynamics evolves as follows: at every time \( t \), each particle would like to jump to its right with probability \( \frac{b}{1+b} \). This means
\[
x_{1}^{1}(t+1) = \begin{cases} x_{1}^{1}(t) + 1 & \text{with probability } \frac{b}{1+b}, \\ x_{1}^{1}(t) & \text{with probability } \frac{1}{1+b}. \end{cases} \tag{5.2.12}
\]

Recursively from bottom to top, if all \( x_{k}^{n-1}(t+1), 1 \leq k \leq n-1 \) are updated, then

(a) if \( k > 1 \) and \( x_{k}^{n}(t) = x_{k-1}^{n-1}(t+1) - 1 \), then \( x_{k}^{n}(t+1) = x_{k-1}^{n-1}(t+1) \) (particle \( (k, n) \) is pushed by particle \( (k-1, n-1) \)),

(b) if \( x_{k}^{n}(t) = x_{k}^{n-1}(t+1) - 1 \), then \( x_{k}^{n}(t+1) = x_{k}^{n}(t) \) (particle \( (k, n) \) is blocked by particle \( (k, n-1) \)),

(c) otherwise
\[
x_{k}^{n}(t+1) = \begin{cases} x_{k}^{n}(t) + 1 & \text{with probability } \frac{b}{1+b}, \\ x_{k}^{n}(t) & \text{with probability } \frac{1}{1+b}. \end{cases} \tag{5.2.13}
\]

The evolution of the particles with \( k = 1 \), \( x_{1}^{n}(t) \) is independent to the evolution of the particles with \( k \geq 2 \): the projection of the Markov chain \( \{x_{k}^{n}(t), 1 \leq k \leq n \leq N\} \) onto \( \{x_{1}^{n}(t), 1 \leq n \leq N\} \) is still a Markov process. This (discrete time) process is known as \textit{Totally Asymmetric Simple Exclusion Process} (TASEP) with sequential update.

### 5.2.3 Continuous time limit

One can obtain the continuous time process by replacing time \( t \) with \( t/b \) and taking limit \( b \to 0 \); the dynamics of the continuous time TASEP is simpler: each particle tries to jump to its right with rate 1 and

- if \( x_{k}^{n}(t) \) tries to jump and \( x_{k}^{n}(t) = x_{k}^{n-1}(t) - 1 \), then the jump is suppressed;

- if \( x_{k}^{n}(t) \) jumps and before the jump \( x_{k}^{n}(t) = x_{k+1}^{n+1}(t) = \cdots = x_{k+j}^{n+j}(t) \), for some \( j \), then all these particles are moved to right by 1.

![TASEP dynamics in continuous time.](image)
Remark 5.2.1. The projection onto \( \{ x_n^a(t), 1 \leq n \leq N \} \) is a Markov process called PushTASEP. Finally, the projection onto \( \{ x_k^a(t), 1 \leq k \leq N \} \) is a Markov process called Charlier process: it is a discrete version of Dyson’s Brownian motion. This gives an explanation why the GUE Tracy-Widom distribution discovered first in random matrix theory also describes the fluctuations of TASEP particles in the large time limit.

Continuous time TASEP is an interacting particle system on \( \mathbb{Z} \) with state space \( \Omega = \{ 0, 1 \}^\mathbb{Z} \). For a configuration \( \eta \in \Omega, \eta = (\eta_j, j \in \mathbb{Z}) \), \( \eta_j \) is the occupation variable at site \( j \), which is 1 if and only if \( j \) is occupied by a particle. Define \( \eta^{x,y} \) as the configuration \( \eta \) with the occupations at sites \( x \) and \( y \) interchanged,

\[
\eta^{x,y}(z) = \begin{cases} 
\eta(x), & z = y, \\
\eta(y), & z = x, \\
\eta(z) & \text{otherwise.}
\end{cases}
\] (5.2.14)

TASEP has generator \( \mathcal{L} \) given by

\[
\mathcal{L} f(\eta) = \sum_{x \in \mathbb{Z}} \eta_x (1 - \eta_{x+1}) (f(\eta^{x,x+1}) - f(\eta)),
\] (5.2.15)

where \( f \) are local functions (depending only on finitely many sites). Naming the initial position of the \( n \)-th particle \( x_n(0) \) and ordering them from right to left as \( x_{n+1}(0) < x_n(0), n \in \mathbb{Z} \), we notice that the ordering of particles is preserved for all times \( t \geq 0: x_{n+1}(t) < x_n(t), n \in \mathbb{Z} \).

TASEP can be also thought as a growth process by introducing the height function \( h(x,t) \) as

\[
h(x,t) = \begin{cases} 
2J(t) + \sum_{y=1}^{x}(1 - 2\eta_y(t)) & \text{for } x \geq 1, \\
2J(t) & \text{for } x = 0, \\
2J(t) - \sum_{y=-x+1}^{0}(1 - 2\eta_y(t)) & \text{for } x \leq -1,
\end{cases}
\] (5.2.16)

for \( x \in \mathbb{Z}, t \geq 0, \) where \( J(t) \) counts the number of jumps from site 0 to site 1 during the time-span \([0,t]\).

Consider the initial condition corresponding to \( \lambda^{(n)}(0) = \emptyset: x_k^a(0) = -n + k, 1 \leq k \leq n \leq N \). TASEP particles \((x_1(t), x_2(t), \ldots, x_N(t))\) have positions \( x_k(0) = -k + 1 \), with \( x_k(t) = x_k^1(t) \). We ask if it is possible to determine the joint distribution of these particles, i.e. for any \( 1 \leq n_1 < n_2 < \cdots < n_m \leq N \),

\[
P\left( \bigcap_{k=1}^{m} \{ x_{n_k}(t) \geq a_k \} \right). \tag{5.2.17}
\]

It can be expressed as a gap probability, since (5.2.17) can be written as

\[
P\left( \bigcap_{k=1}^{m} \{ \text{the set } \{(x, n_k)(t), x < a_k \} \text{ is empty} \} \right). \tag{5.2.18}
\]
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By (5.2.16), this can also be rephrased as

\[
\mathbb{P} \left( \bigcap_{k=1}^{m} \{ x_{nk}(t) \geq m_k - n_k \} \right) = \mathbb{P} \left( \bigcap_{k=1}^{m} \{ h(m_k - n_k, t) \geq m_k + n_k \} \right). \tag{5.2.19}
\]

Since the measure on the point process

\[
\xi = \sum_{n \geq 1} \sum_{k=1}^{n} \delta(x_{nk}(t), n)
\]

is determinantal, the probability \[(5.2.20)\] is a Fredholm determinant.

5.2.4 Correlation kernel

Now we want to compute the correlation kernel of the measure at time \(t\). In discrete time the measure is given by

\[
S_{((1, \ldots, 1);(0, b, \ldots, b);0)}(\lambda^{(N)}) \prod_{k=2}^{N} p_{k}^{\lambda^{(k-1)}} ((1, \ldots, 1); (1; 0; 0)) = \frac{1}{Z} s_{\lambda^{(1)}}(\rho_{0}^{+}) s_{\lambda^{(2)}}(\rho_{1}^{+}) \cdots s_{\lambda^{(N-1)}}(\rho_{N-1}^{+}) s_{\lambda^{(N)}}(\rho_{N}^{-}), \tag{5.2.21}
\]

with \(\rho_{0}^{+} = \cdots = \rho_{N-1}^{+} = (1; 0; 0)\), for which the generating function is \(H(\rho_{k}^{+})(z) = \frac{1}{1-z}\) and \(\rho_{N}^{-} = (0; b, \ldots, b; 0)\), for which \(H(\rho_{N}^{-})(z) = (1 + bz)^N\).

The continuous time limit is obtained as the \(b \to 0\) limit for \(t\) replaced by \(t/b\). The measure a time \(t\) is given by

\[
\mathcal{M}_{((1, \ldots, 1);(0, b, \ldots, b);0)}(\lambda^{(N)}, \lambda^{(1)}, \ldots, \lambda^{(N)}) = \frac{1}{Z} s_{\lambda^{(1)}}(1; 0; 0) \cdots s_{\lambda^{(N-1)}}(1; 0; 0) s_{\lambda^{(N)}}((0; 0; t)). \tag{5.2.22}
\]

The correlation kernel of this Schur process has been computed in Theorem 4.3.5, and we only have to specialize the formula with \(H(\rho_{0}^{+}, \ldots, \rho_{N-1}^{+})(v) = \left( \frac{1}{1-v} \right)^k\) and \(H(\rho_{N}^{-})(v) = e^{tv}\).

**Theorem 5.2.2.** The process \(\xi\) defined in \[(5.2.20)\] is a determinantal point process on \(\{1, \ldots, N\} \times \mathbb{Z}\) with correlation kernel

\[
K(n_1, x_1; n_2, x_2) = \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dv \frac{w^{n_1-1}}{v^{n_2}} \frac{1}{v-w} \frac{(1-w)^{n_1} e^{t/w}}{(1-v)^{n_2} e^{t/v}}, \tag{5.2.23}
\]

where the contours are as in Figure 4.3(a) for \(n_1 \geq n_2\) and as in Figure 4.3(b) for \(n_1 < n_2\).

For \(n_1 \geq n_2\) the poles of \(v\) are \(v = 1\) and \(v = w\), so we can redefine the contours in the following way: Instead, for \(n_1 < n_2\), we can exchange the contours and get the pole at \(w = v\) as well (see Figure 5.5).

This leads to

\[
K(n_1, x_1; n_2, x_2) = -\frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{1}{(1-v)^{n_2-n_1} v^{n_2-x_1+1} 1_{[n_1<n_2]}} + \frac{1}{2\pi i} \oint_{\Gamma_1} dw \frac{w^{n_1-1}}{v^{n_2}} \frac{1}{v-w} \frac{(1-w)^{n_1} e^{t/w}}{(1-v)^{n_2} e^{t/v}}, \tag{5.2.24}
\]

where \(\Gamma_0\) and \(\Gamma_1\) are non-intersecting contours around 0 and 1 respectively.
5.2.5 Large time asymptotics

We want to determine the large time asymptotics of $x_n(t)$ for $n = \mathcal{O}(t)$. Let us fix first some notation.

Let $j(x, t) = \mathbb{E}[\eta_x(t) (1 - \eta_{x+1}(t))]$ be the average current of particles from $x$ to $x + 1$ and $\rho(x, t) = \mathbb{E}[\eta_x(t)]$ the particle density at $x$.

Lemma 5.2.3. The conservation of the number of particles implies

$$\frac{d}{dt}\rho(x, t) + \nabla_x j(x, t) = 0,$$

where $\nabla_x j(x, t) := j(x, t) - j(x - 1, t)$.

Proof. Let $T(t)$ be the semigroup generated by the TASEP $\mathcal{L}$. The forward Kolmogorov equation is

$$\frac{d}{dt}T(t)f = T(t)\mathcal{L}f.$$

Taking $f(\eta) = \eta_x$ and integrating with respect to the initial condition $\mu$ we get (recall $\mu_t = \mu T(t)$)

$$\frac{d}{dt}\mu_t(f) = \frac{d}{dt}\rho(x, t).$$

On the other hand, for the same $f$,

$$(\mathcal{L}f)(\eta) = \sum_{y \in \mathbb{Z}} \eta_y (1 - \eta_{y+1}) \left[ f(\eta_{y,y+1}) - f(\eta) \right]
= -\eta_x (1 - \eta_{x+1}) + \eta_{x-1} (1 - \eta_x).$$
Thus,
\[ \mu_L f = \mathbb{E}[\eta_{x-1}(t)(1 - \eta_x(t))] - \mathbb{E}[\eta_x(t)(1 - \eta_{x+1}(t))] = j(x-1,t) - j(x,t). \] (5.2.29)

**Hydrodynamic scaling**

Now we look at what happens to the dynamics at large scale. For this we rescale the distance of the particles of a factor \( \epsilon \) and we set \( x = \lfloor \epsilon^{-1} X \rfloor \) (equivalently \( X = \epsilon x \)), with \( X \in \mathbb{R} \) and \( x \in \mathbb{Z} \), and \( \tilde{j}(X,t) = j(\lfloor \epsilon^{-1} X \rfloor, t) \). We have
\[ \nabla_x j(x,t) = \epsilon \frac{\partial}{\partial X} \tilde{j}(X - \epsilon, t) + O(\epsilon^2). \] (5.2.30)

Since we are looking at particles at distance \( O(\epsilon^{-1}) \), we expect to wait a time of order \( O(\epsilon) \) to see a non-trivial behavior for \( \epsilon \to 0 \), so we need to take \( t = \epsilon^{-1} T \). The change of variables \( (x,t) \mapsto (\epsilon^{-1} X, \epsilon^{-1} T) \) (5.2.31) is called the **hydrodynamical scaling**.

We define the macroscopic current and density as
\[ J(X,T) = \lim_{\epsilon \to 0} j(\lfloor \epsilon^{-1} X \rfloor, \epsilon^{-1} T) \] (5.2.32)
and
\[ \rho_{ma}(X,T) = \lim_{\epsilon \to 0} \rho(\lfloor \epsilon^{-1} X \rfloor, \epsilon^{-1} T). \] (5.2.33)

From Lemma 5.2.3 one can show that
\[ \partial_T \rho_{ma}(X,T) + \partial_X J(X,T) = 0. \] (5.2.34)
In our case
\[ \partial_T \rho_{ma}(X,T) + \partial_X \rho_{ma}(X,T)(1 - \rho_{ma}(X,T)) = 0, \] (5.2.35)
which is also known as **Burgers equation**.

We have
\[ \rho_{ma}(X,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \] (5.2.36)
It is possible to show, using e.g. the method of characteristics, that
\[ \rho_{ma}(X,T) = \begin{cases} 1, & X < -T, \\ \frac{T-X}{T}, & X \in [-T,T] \\ 0, & X > T. \end{cases} \] (5.2.37)

From this one can estimate how many particles are around the origin at time \( T \), which is approximately \( T/4 \), since the number of particles which jumped over the origin is
\[ \int_0^\infty dX \rho_{ma}(X,T) = T/4, \] as shown in Figure 5.6.

**Scaling limit at the edge**

Now we want to see what happens in the \( t \to \infty \) limit after a proper rescaling. Let us consider \( n_i = \frac{1}{t} + u_i \left( \frac{t}{2} \right)^{2/3} \). Define \( \xi_i \) as the point s.t.
\[ t \int_{\xi_i}^\infty dX \rho_{ma}(X,t) = n_i, \] (5.2.38)
i.e., choose \( \xi_i \) such according to the macroscopic approximation. In our case we get that \( x_{n_i}(t) \) is roughly around \( \xi_i = -2u_i \left( \frac{t}{2} \right)^{2/3} + u_i^2 \left( \frac{t}{2} \right)^{1/3} \). Due to the \( \left( \frac{2}{3}, \frac{1}{3} \right) \) KPZ scaling, we set \( a_i = \xi_i - s_i \left( \frac{t}{2} \right)^{1/3} \). Then, we can prove the following result
Figure 5.6: The macroscopic density $\rho_{m\alpha}(X,T)$. The light blue region has area $T/4$.

**Theorem 5.2.4.** For any fixed $u_1 < u_2 < \cdots < u_m$, $s_1, s_2, \ldots, s_m \in \mathbb{R}$,

$$
\lim_{t \to \infty} \mathbb{P}\left( \bigcap_{k=1}^{m} \{ x_{n_k}(t) \geq a_k \} \right) = \mathbb{P}\left( \bigcap_{k=1}^{m} \{ A_2(u_k) \leq s_i \} \right). \quad (5.2.39)
$$

We will not prove completely this theorem, but we will present the main steps of the asymptotic analysis to illustrate what has to be done.

**Step 1.** By Theorem 5.2.2, the joint probability of TASEP can be expressed as a gap probability,

$$
P\left( \bigcap_{k=1}^{m} \{ x_{n_k}(t) \geq a_k \} \right) = \det \left( \mathbb{1} - \chi_a K_t(x_{n_k}, x_{n_k}) \right), \quad (5.2.40)
$$

with $\chi_a(x_{n_k}, x_{n_k}) = \frac{1}{BD} \{ x < a \}$. Let

$$
n(u) = \frac{1}{4} + u(\frac{t}{2})^{2/3}, \quad x(u, s) = -2u(\frac{t}{2})^{2/3} + u^2(\frac{t}{2})^{1/3} - s(\frac{t}{2})^{1/3}. \quad (5.2.41)
$$

Then we need to consider the rescaled kernel

$$
K^{\text{resc}}_{1}(u_1, s_1; u_2, s_2) = (-1)^{u_1-u_2}(\frac{t}{2})^{1/3}K_{1\times1}(n(u_1), x(u_1, s_1); n(u_2), x(u_2, s_2)). \quad (5.2.42)
$$

Here $(-1)^{u_1-u_2}$ is conjugation term that we added for convenience. Using (5.2.23) and separating terms in $t$ with different exponents, we get

$$
K^{\text{resc}}_{1} = \left( \frac{t}{2} \right)^{1/3} \int dw \int dv \frac{e^{tf_0(w) + (t/2)^{2/3}f_1(u_1, w) + (t/2)^{1/3}f_2(u_1, s_1, w) - w^{-1}v - v}}{e^{tf_0(v) + (t/2)^{2/3}f_1(u_2, v) + (t/2)^{1/3}f_2(u_2, s_2, v)}} \quad (5.2.43)
$$

with

$$
f_0(w) = \frac{1}{w} + \frac{1}{4} \ln(w - 1), \quad f_1(u, w) = u \ln(w - 1) - 2u \ln w, \quad f_2(u, s, w) = (w^2 - s) \ln w.
$$

**Step 2.** After scaling the kernel, we need to understand from which region the integral gets the dominant contribution. Consider $u_1 \geq u_2$ to explain how to do. For that purpose, we first look at the leading term, namely $f_0$. It has the following critical points:

$$
\frac{df_0}{dw} = -\frac{1}{w^2} + \frac{1}{4(w - 1)} = \frac{-4 + 4w - w^2}{4w^2(1-w)} = \frac{(2-w)^2}{4w^2(1-w)}. \quad (5.2.44)
$$
from which we have that \( \frac{df}{dw} = 0 \) for \( w = w_c = 2 \). Close to \( w_c = 2 \), we can expand \( f_0 \) as
\[
f_0(w) = \frac{1}{2} + \frac{1}{48}(w-2)^3 + O((w-2)^4). \tag{5.2.45}
\]
We need to find steep descent path, i.e., we need to take the path for \( w \) in the region where \( \Re f_0(w) < \Re f_0(w_c) \) (except at \( w = w_c = 2 \)) and for \( v \) in the region where \( -\Re f_0(v) < -\Re f_0(w_c) \) (except at \( v = w_c = 2 \)). Then the contribution to the double integral with \( |w-w_c| > \delta \) and/or \( |v-w_c| > \delta \) is exponentially small in \( t \), i.e., it is of order \( O(e^{-c(\delta)t}) \), for some \( c(\delta) > 0 \) (for small \( \delta \), we have \( c(\delta) \sim \delta^3 \).

**Step 3.** Next one has to estimate the contribution around the critical points. Using Taylor expansion, we define the new variables
\[
w := 2 + 2W \left( \frac{t}{2} \right)^{-1/3}, \quad v = 2 + 2V \left( \frac{t}{2} \right)^{-1/3}.
\]
Then,
\[
K_t^{\text{resc}}(u_1, s_1; u_2, s_2) = O(e^{-c(\delta)t}) + 4 \left( \frac{1}{2} \right)^{-2/3} \frac{e^{(t/2)^{2/3}} f_1(u_1, 2) + (t/2)^{1/3} f_2(u_1, s_1, 2)}{e^{(t/2)^{2/3}} f_1(u_2, 2) + (t/2)^{1/3} f_2(u_2, s_2, 2)}
\]
\[
\times \frac{1}{(2\pi i)^2} \left( \frac{1}{2} \right)^{1/3} \int dW \int dV \frac{1}{4(W-V)(\frac{1}{2})^{1/3}}
\]
\[
\times \frac{e^{\frac{u_1^2}{2} + O(W^{4/1/3})} e^{-u_1 W^2 + O(W^{3/1/3})} e^{(u_1^2 - s_1) W + O(W^{2/1/3})}}{e^{\frac{u_2^2}{2} + O(V^{4/1/3})} e^{-u_2 V^2 + O(V^{3/1/3})} e^{(u_2^2 - s_2) V + O(V^{2/1/3})}}. \tag{5.2.46}
\]
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Figure 5.8: Integration lines $\Gamma_W$ and $\Gamma_V$ of (5.2.47).

The term $e^{(t/2)^2/3}f_1(u_1, s) + e^{(t/2)^1/3}f_2(u_1, s)$ is just a conjugation term and thus we can conjugate it out. Controlling the error term is possible if we choose $\delta$ small enough. The correction obtained by removing the remainder in Taylor development is $O(t^{-1/3})$ smaller than the leading term. Finally we get that

$$K_{t}^{\text{resc}}(u_1, s_1; u_2, s_2) = O(t^{-1/3}) + \frac{1}{(2\pi i)^2} \int_{\Gamma_W} \int_{\Gamma_V} dW dV \frac{e^{W^3 - u_1 W^2 - (s_1 - u_1^2) W} - 1}{W - V} \cdot$$

(5.2.47)

where $\Gamma_W$ and $\Gamma_V$ are shown in Figure 5.8. Since $\Re W > \Re V$, we can write $\frac{1}{W - V} = \int_0^\infty d\lambda e^{-\lambda(W - V)}$, so that the double integral becomes

$$\int_0^\infty d\lambda \frac{1}{2\pi i} \int_{\Gamma_W} dW e^{W^3 - u_1 W^2 - (s_1 - u_1^2) W} \frac{1}{W - V} \int_{\Gamma_V} dV e^{V^3 - u_2 V^2 - (s_2 - u_2^2) V}.$$  

(5.2.48)

**Step 4.** Now we show that $K_{t}^{\text{resc}}$ converges to the Airy kernel (4.5.10). With the change of variables

$$W = w + u_1, \quad V = v + u_2,$$

we obtain

$$\lim_{t \to \infty} K_{t}^{\text{resc}}(u_1, s_1; u_2, s_2) = \int_0^\lambda \frac{1}{2\pi i} \int_{\Gamma_W} dwe^{w^3 - w - \lambda w} e^{w^3 - w - \lambda w} e^{-\lambda u_1} \frac{1}{2\pi i} \int_{\Gamma_V} dve^{v^3 - v - \lambda v} e^{-\lambda u_2} \cdot$$

$$\lim_{t \to \infty} K_{t}^{\text{resc}}(u_1, s_1; u_2, s_2)^{\text{conj}} = K_{\text{Ai}}(u_1, s_1; u_2, s_2)$$

(5.2.49)

which is the extended Airy kernel for $u_1 \geq u_2$.

In this way one shows that

$$\lim_{t \to \infty} K_{t}^{\text{resc}}(u_1, s_1; u_2, s_2) = K_{\text{Ai}}(u_1, s_1; u_2, s_2)$$

(5.2.50)

and some slightly more efforts are needed to prove

$$|K_{t}^{\text{resc}}(u_1, s_1; u_2, s_2)| \leq C e^{-(s_1 + s_2)} + \bar{C} \mathbb{1}_{[u_1 < u_2]} e^{-|s_1 - s_2|},$$

(5.2.51)

for some $C, \bar{C}$ independent of $t$. 
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Figure 5.9: From particle configuration (left) to 3d visualization via lozenge tilings (right). The corner with the white circle has coordinates \((x, n, h) = (-1/2, 0, 0)\).

5.2.6 Scaling limit in the bulk

Using the correlation kernel and doing asymptotic analysis other results can be obtained. Here we give a short description of the most important ones obtained in [1].

We define a height function

\[
h(x, n; t) = \text{number of particles at level } n \text{ on the right of the position } x \text{ at time } t.
\]

(5.2.52)

Take Figure 5.9 as reference. We want to describe the limit shape at the edge. Let \(n_i = \eta_i L\), \(x_i = -\eta_i + \nu_i L\) and \(t_i = \tau_i L\): we study the behavior for \(L \to \infty\). The random region where there is only one type of color/lozenge is

\[
D = \{(\nu, \eta, \tau) \in \mathbb{R}_+^3 | |\sqrt{\eta} - \sqrt{\tau}| < \sqrt{\nu} < \sqrt{\eta} + \sqrt{\tau}\}.
\]

(5.2.53)

This is obtained by computing the density of the particles, \(\rho(n, x) = K_t(n, x; n, x)\), and see when this is equal to 0 or 1. In the disordered phase \(D\) the fluctuations of the height function are not the ones of the one-dimensional KPZ class, but are asymptotically Gaussian in the \(\sqrt{\ln L}\) scale.

**Theorem 5.2.5** (Theorem 1.2 of [1]). For any \((\nu, \eta, \tau) \in D\) we have the moment convergence of random variables

\[
\lim_{L \to \infty} \frac{h([((\nu - \eta)L] + \frac{1}{4}, [\eta L], \tau L) - \mathbb{E}[h([((\nu - \eta)L] + \frac{1}{4}, [\eta L], \tau L)]]}{\sqrt{\kappa \ln L}} = \xi \sim \mathcal{N}(0, 1),
\]

(5.2.54)

with \(\kappa = (2\pi^2)^{-1}\).

We can observe that \(D\) is exactly the set of triples \((\nu, \eta, \tau) \in \mathbb{R}_+^3\) for which there exists a nondegenerate triangle with side lengths \((\sqrt{\nu}, \sqrt{\eta}, \sqrt{\tau})\). Denote by \((\pi_{\nu}, \pi_{\eta}, \pi_\tau)\) the angles of this triangle that are opposite to the corresponding sides (see Figure 5.11).

The condition in the definition of \(D\) (5.2.53) is also equivalent to saying that the circle centered at 0 of radius \(\sqrt{\eta/\tau}\) has two disjoint intersections with the circle centered at 1 of radius \(\sqrt{\nu/\tau}\). In that case, the two intersections are complex conjugate. Denote by \(\Omega(\nu, \eta, \tau)\) the intersection in

\[
\mathbb{H} = \{z \in \mathbb{C} | \text{Im}(z) > 0\},
\]

(5.2.55)

as in Figure 5.11. We can associate an angle (or equivalently a frequency) to the different facet’s types of Figure 5.9.
Figure 5.10: A configuration of the model analyzed with $N = 100$ particles at time $t = 25$, using the same representation as in Figure 5.9.

Figure 5.11: The triangle of (5.2.53) on the left and its scaled version defined by intersection of circles on the right.
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We can give an explicit formula for the limit shape:

**Proposition 5.2.6** (Corollary 3.7 of [1]).

\[
\lim_{L \to \infty} \mathbb{E}[h((\nu - \eta)L + \frac{1}{2}, [\eta L], \tau L)] = h_{\text{ma}}(\nu, \eta, \tau) = \frac{1}{\pi} \left( -\nu\pi + \eta(\pi - \pi) + \frac{\tau}{\sin \pi} \sin \pi \right), \quad (5.2.56)
\]

As an immediate consequence we obtain values for the slopes of the macroscopic height

\[
\frac{\partial}{\partial \nu} h_{\text{ma}}(\nu, \eta, 1) = -\frac{\pi}{\pi}
\]

\[
\frac{\partial}{\partial \eta} h_{\text{ma}}(\nu, \eta, 1) = 1 - \frac{\pi}{\pi}
\]

and the speed of growth

\[
\frac{\partial}{\partial \tau} h_{\text{ma}}(\nu, \eta, \tau) = \frac{\sin \pi \sin \pi}{\pi} = \frac{\text{Im}(\Omega)}{\pi}
\]

Now we look at the correlation in the “bulk”. We introduce the notation

\[G(z, w) = -\frac{1}{2\pi} \ln \left| \frac{z - w}{z - \bar{w}} \right|, \quad (5.2.57)\]

This is the Green function of the Laplace operator on \(\mathbb{H}\) with Dirichlet boundary condition on \(\partial \mathbb{H}\).

**Theorem 5.2.7** (Theorem 1.3 of [1]). Let \(\mathcal{H}_i = (\nu_i, \eta_i, \tau = 1)\) be \(N\) distinct triples in \(\mathcal{D}\). Let \(H_L(\nu, \eta, 1) = \sqrt{\pi} [h((\nu - \eta)L + \frac{1}{2}, [\eta L], \tau L) - \mathbb{E}[h((\nu - \eta)L + \frac{1}{2}, [\eta L], \tau L)]]\) and \(\Omega_j = \Omega(\nu_j, \eta_j, 1)\). Then,

\[
\lim_{L \to \infty} \mathbb{E}[H_L(\mathcal{H}_1) \ldots H_L(\mathcal{H}_N)] = \begin{cases} \sum_{\sigma \in \mathcal{F}_N} \prod_{k=1}^{N/2} G(\Omega_{\sigma(2k-1)}, \Omega_{\sigma(2k)}), & \text{if } N \text{ even}, \\
0, & \text{if } N \text{ odd}, \end{cases} \quad (5.2.58)
\]

where the summation is taken over all fixed point free involutions \(\sigma\) on \(\{1, \ldots, N\}\) (also known as pairings).

The result of the theorem means that as \(L \to \infty\), \(H_L(\Omega^{-1}(z))\) is a Gaussian process with covariance given by \(G\), i.e., it has correlation of the Gaussian Free Field on \(\mathbb{H}\).

**Example 5.2.8.** For \(N = 2\), \(\mathbb{E}[H_L(\mathcal{H}_1)H_L(\mathcal{H}_2)]\) converges to \(G(\Omega_1, \Omega_2)\).
For $N = 4$ we have
\[ G(\Omega_1, \Omega_2)G(\Omega_3, \Omega_4) + G(\Omega_1, \Omega_3)G(\Omega_2, \Omega_4) + G(\Omega_1, \Omega_4)G(\Omega_2, \Omega_3). \]

Note that Theorem 5.2.5 and 5.2.7 use different normalization. The reason is that the limit of $H_L$ is not a smooth random field. In particular, $H_L$ converges to a distribution, not to a function. The correlations of Theorem 5.2.7 are the ones of the so-called Gaussian Free Field (GFF) on $\mathbb{H}$.

**Gaussian Free Field**

The Gaussian Free Field on $\mathbb{H}$, see e.g. [19], is a generalized Gaussian process (i.e. it is a probability measure on a suitable class of generalized functions on $\mathbb{H}$) that can be characterized as follows. If we denote by GFF the random generalized function and take any sequence $\{\phi_k\}$ of (compactly supported) test functions, the pairings $\{\text{GFF}(\phi_k)\}$ form a sequence of mean 0 normal variables with covariance matrix
\[
\mathbb{E}[\text{GFF}(\phi_k)\text{GFF}(\phi_l)] = \int_{\mathbb{H}} |dz|^2 (\nabla \phi_k(z), \nabla \phi_l(z))
\]
\[= \int_{\mathbb{H}} |dz_1|^2 |dz_2|^2 \phi_k(z_1)\phi_l(z_2)G(z_1, z_2). \quad (5.2.59)\]
The value of GFF at a point cannot be defined. However, one can think of expectations of products of values of GFF at different points as being finite and equal to
\[
\mathbb{E} [\text{GFF}(z_1) \ldots \text{GFF}(z_m)] = \begin{cases} 
\sum \text{pairings } & G(z_{\sigma(1)}, z_{\sigma(2)}) \ldots G(z_{\sigma(m-1)}, z_{\sigma(m)}), \\
0, & \text{if } m \text{ odd.}
\end{cases} \quad (5.2.60)
\]
The justification for the notation is the fact that for any finite number of test functions,
\[
\mathbb{E} [\text{GFF}(\phi_1) \ldots \text{GFF}(\phi_m)] = \int_{\mathbb{H}^m} \prod_{k=1}^m |dz_k|^2 \phi_k(z_k)\mathbb{E} [\text{GFF}(z_1) \ldots \text{GFF}(z_m)]. \quad (5.2.61)
\]
The moments uniquely determine the Gaussian Free Field.

To state the convergence results, we consider any (smooth) space-like surface $U \subset \mathbb{R}^3$ in the rounded part of the surface. Namely $U \subset \mathcal{D}$, and for any two triples $(\nu_i, \eta_i, \tau_i) \in U$, $i = 1, 2$, $\eta_1 \leq \eta_2$ implies $\tau_1 \geq \tau_2$. The mapping $\Omega$, defined in the previous section, restricted to $U$ is a bijection. Consider any smooth parametrization $u = (u_1, u_2)$ of $U$. Denote by $\Omega_u$ the map from $u$ to $\mathbb{H}$, which is the composition of the map from $u$ to $(\nu, \eta, \tau)$ and $\Omega$. Then, for any smooth compactly supported test function $f$ on $U$, we define
\[
\langle f, H_L \rangle = \int_U du f(u)H_L(u), \quad (5.2.62)
\]
where $H_L$ is as in Theorem 5.2.7. Then,
\[
\langle f, H_L \rangle = \int_{\mathbb{H}} |dz|^2 J(z)f(\Omega^{-1}_u(z))H_L(\Omega^{-1}_u(z)), \quad (5.2.63)
\]
where $J(z)$ is the Jacobian of the change of variables $z \mapsto u$ by $\Omega_u^{-1}$.
Theorem 5.2.9. For any $m \in \mathbb{N}$, and any smooth compactly supported functions $f_1, \ldots, f_m$ on $\mathcal{U}$,

$$
\lim_{L \to \infty} \mathbb{E} \left[ \prod_{k=1}^{m} \langle f_k, H_L \rangle \right] = \int_{\mathbb{R}^m} \prod_{k=1}^{m} |d z_k|^2 f_k^{\mathbb{R}}(z_k) \mathbb{E} [\text{GFF}(z_1) \ldots \text{GFF}(z_m)], \quad (5.2.64)
$$

where $f_k^{\mathbb{R}}(z) = J(z) f(\Omega^{-1} U(z))$.

Since moment convergence to a (multidimensional) Gaussian implies convergence in distribution, Theorem 5.2.9 implies that the random vector $(\langle f_k, H_L \rangle)_{k=1}^{m}$ converges in distribution (and with all moments) to the Gaussian vector with mean zero and covariance matrix $\| \int_{\mathbb{R}} |d z|^2 (\nabla f_k^{\mathbb{R}}(z), \nabla f_l^{\mathbb{R}}(z)) \|$, $k, l = 1, \ldots, m$. 
Bibliography


