Markov chains on Schur processes

- First we introduce two stochastic matrices, which will be the basis of the construction of Markov chains.

- For any two given Schur-positive specializations ρ, ρ' s.t. $H(\rho;\rho') < \infty$, we define the transition matrices indexed by Young diagrams λ and μ by:

(a) $$P^{\uparrow}_{\lambda\to\mu}(\rho;\rho') := \frac{1}{H(\rho;\rho')} \frac{s_\mu(\rho)}{s_\lambda(\rho)} s_{\mu/\lambda}(\rho')$$

(b) $$P^{\downarrow}_{\lambda\to\mu}(\rho;\rho') := \frac{s_\mu(\rho)}{s_\lambda(\rho,\rho')} s_{\lambda/\mu}(\rho')$$

Prop 1: $P^{\uparrow}_{\lambda\to\mu}$ and $P^{\downarrow}_{\lambda\to\mu}$ are stochastic, i.e., the entries are ≥ 0 and $\forall \lambda \in \mathbb{Y}$,

$$\sum_{\mu\in\mathbb{Y}} P^{\uparrow}_{\lambda\to\mu}(\rho;\rho') = 1$$

$$\sum_{\mu\in\mathbb{Y}} P^{\downarrow}_{\lambda\to\mu}(\rho;\rho') = 1$$

For the proof of this and other propositions we will use identities which we recall:

(Skew-Cauchy) (A) $$\sum_{\mu\in\mathbb{Y}} s_{\mu/\lambda}(\rho) s_{\mu/\nu}(\rho') = H(\rho;\rho') \sum_{\kappa\in\mathbb{Y}} s_{\lambda/\kappa}(\rho') s_{\nu/\kappa}(\rho)$$

(B) $$\sum_{\lambda\in\mathbb{Y}} s_\lambda(\rho_1) s_{\mu/\lambda}(\rho_2) = s_\mu(\rho_1,\rho_2)$$

(c): (a) with $\lambda=\phi$: $\sum_{\mu\in\mathbb{Y}} s_\mu(\rho)\, s_{\mu/\nu}(\rho') = H(\rho;\rho')\, s_\nu(\rho)$

· the Schur measure $\mathbb{S}_{\rho_1;\rho_2}(\lambda) := \dfrac{s_\lambda(\rho_1)\, s_\lambda(\rho_2)}{H(\rho_1;\rho_2)}$.

Proof of Prop 1: · Positivity is by definition of Schur-positive specialisations ρ,ρ'.

$$\text{(a)}\quad \sum_\mu P^\uparrow_{\lambda\to\mu}(\rho;\rho') = \sum_\mu \frac{1}{H(\rho;\rho')}\frac{s_\mu(\rho)\, s_{\mu/\lambda}(\rho')}{s_\lambda(\rho)}$$

$$\overset{(c)}{=} \frac{1}{H(\rho;\rho')}\frac{1}{s_\lambda(\rho)}\cdot H(\rho;\rho')\, s_\lambda(\rho) = 1.$$

$$\text{(b)}\quad \sum_\mu P^\downarrow_{\lambda\to\mu}(\rho;\rho') = \sum_\mu \frac{s_\mu(\rho)\, s_{\lambda/\mu}(\rho')}{s_\lambda(\rho,\rho')}$$

$$\overset{(B)}{=} \frac{s_\lambda(\rho;\rho')}{s_\lambda(\rho;\rho')} = 1. \qquad \#$$

· Rem: · Since $s_{\mu/\lambda}\equiv 0$ if $\lambda\not\subset\mu$, we have that $P^\uparrow_{\lambda\to\mu}=0$ unless $\lambda\subset\mu$ (the Young diagram increases) and $P^\downarrow_{\lambda\to\mu}=0$ unless $\mu\subset\lambda$ (the Young diagram decreases).

· Let us see how these stochastic matrices acts on Schur measures:

Prop 2: $\forall \mu\in\mathbb{Y}$:

$$\text{(a)}\quad \sum_{\lambda\in\mathbb{Y}} \mathbb{S}_{\rho_1;\rho_2}(\lambda)\, P^\uparrow_{\lambda\to\mu}(\rho_2;\rho_3) = \mathbb{S}_{\rho_1,\rho_3;\rho_2}(\mu)$$

$$\text{(b)}\quad \sum_{\lambda\in\mathbb{Y}} \mathbb{S}_{\rho_1;\rho_2,\rho_3}(\lambda)\, P^\downarrow_{\lambda\to\mu}(\rho_2;\rho_3) = \mathbb{S}_{\rho_1;\rho_2}(\mu).$$

Proof: (a) $$\sum_{\lambda\in\mathbb{Y}} \frac{s_\lambda(\rho_1)\cancel{s_\lambda(\rho_2)}}{H(\rho_1;\rho_2)} \cdot \frac{1}{H(\rho_2;\rho_3)} \cdot \frac{s_\mu(\rho_2)}{\cancel{s_\lambda(\rho_2)}} s_{\mu/\lambda}(\rho_3)$$

$$= \frac{1}{H(\rho_1,\rho_3;\rho_2)} \cdot s_\mu(\rho_2) \cdot \underbrace{\sum_{\lambda\in\mathbb{Y}} s_\lambda(\rho_1)\, s_{\mu/\lambda}(\rho_3)}_{\overset{(B)}{=}\; s_\mu(\rho_1,\rho_3)} \quad \checkmark.$$

(b) $$\sum_{\lambda\in\mathbb{Y}} \frac{s_\lambda(\rho_1)\cancel{s_\lambda(\rho_2,\rho_3)}}{H(\rho_1;\rho_2,\rho_3)} \cdot \frac{s_\mu(\rho_2)\, P_{\lambda,\mu}(\rho_3)}{\cancel{s_\lambda(\rho_2,\rho_3)}}$$

$$\overset{(c)}{=} s_\mu(\rho_2) \frac{\cancel{H(\rho_1,\rho_3)}\, s_\mu(\rho_1)}{H(\rho_1;\rho_2)\cancel{H(\rho_1,\rho_3)}} \quad \checkmark.$$

#

Remark: The Schur process, with distribution,
$$\frac{1}{Z} s_{\lambda^{(1)}}(\rho_0^+)\, s_{\lambda^{(2)}/\lambda^{(1)}}(\rho_1^+) \cdots s_{\lambda^{(N)}/\lambda^{(N-1)}}(\rho_{N-1}^+) \cdot s_{\lambda^{(N)}}(\rho_-)$$

can be rewritten as:

$$\mathbb{S}_{\rho_0^+,\dots,\rho_{N-1}^+;\rho_-}(\lambda^{(N)})\, P^{\downarrow}_{\lambda^{(N)}\to\lambda^{(N-1)}}(\rho_0^+,\dots,\rho_{N-2}^+;\rho_{N-1}^+) \cdots$$
$$\cdots P^{\downarrow}_{\lambda^{(2)}\to\lambda^{(1)}}(\rho_0^+;\rho_1^+).$$

- This generalizes to any Schur process, i.e., it can be viewed as a trajectory of a Markov chain with transition matrices $P^\uparrow$ and $P^\downarrow$ and an initial distribution given by a Schur measure.

Example: $\frac{1}{Z} S_{\lambda^{(1)}}(\rho_0^+) S_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) S_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) S_{\lambda^{(2)}}(\rho_2^-)$

$$= \mathbb{S}_{\rho_0^+,\rho_1^+;\rho_2^-}(\lambda^{(2)}) \cdot P^{\downarrow}_{\lambda^{(2)}\to\mu^{(1)}}(\rho_0^+;\rho_1^+) \cdot P^{\uparrow}_{\mu^{(1)}\to\lambda^{(1)}}(\rho_0^+;\rho_1^-)$$

Indeed:
$$= \frac{\cancel{S_{\lambda^{(2)}}(\rho_0^+,\rho_1^+)} S_{\lambda^{(2)}}(\rho_2^-)}{\text{const}} \cdot \frac{S_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) \cancel{S_{\mu^{(1)}}(\rho_0^+)}}{\cancel{S_{\lambda^{(2)}}(\rho_0^+,\rho_1^+)}}$$
$$\cdot \frac{S_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) S_{\lambda^{(1)}}(\rho_0^+)}{\text{const} \cdot \cancel{S_{\mu^{(1)}}(\rho_0^+)}}$$
$$= \text{const} \cdot S_{\lambda^{(2)}}(\rho_2^-) S_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) S_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) S_{\lambda^{(1)}}(\rho_0^+)$$

- The key property is the following commutation relation:

Prop. 3: $\forall \lambda, \nu \in \mathbb{Y}$, it holds:
$$\sum_{\mu\in\mathbb{Y}} P^{\uparrow}_{\lambda\to\mu}(\rho_1,\rho_2;\rho_3)\, P^{\downarrow}_{\mu\to\nu}(\rho_1;\rho_2) = \sum_{\mu\in\mathbb{Y}} P^{\downarrow}_{\lambda\to\mu}(\rho_1;\rho_2)\, P^{\uparrow}_{\mu\to\nu}(\rho_1;\rho_3)$$

Proof: $$\text{l.h.s.} = \sum_{\mu} \frac{1}{H(\rho_1,\rho_2;\rho_3)} \frac{\cancel{S_\mu(\rho_1,\rho_2)}}{S_\lambda(\rho_1,\rho_2)} S_{\mu/\lambda}(\rho_3) \cdot \frac{S_\nu(\rho_1) S_{\mu/\nu}(\rho_2)}{\cancel{S_\mu(\rho_1,\rho_2)}}$$
$$\overset{(A)}{=} \underbrace{\frac{H(\rho_2;\rho_3)}{H(\rho_1,\rho_2;\rho_3)}}_{=\frac{1}{H(\rho_1;\rho_3)}} \frac{S_\nu(\rho_1)}{S_\lambda(\rho_1,\rho_2)} \sum_{\varkappa} S_{\lambda/\varkappa}(\rho_2)\, S_{\nu/\varkappa}(\rho_3)$$

$$\text{r.h.s.} = \sum_{\mu} \frac{\cancel{S_\mu(\rho_1)}}{S_\lambda(\rho_1,\rho_2)} S_{\lambda/\mu}(\rho_2) \cdot \frac{1}{H(\rho_1;\rho_3)} \frac{S_\nu(\rho_1) S_{\nu/\mu}(\rho_3)}{\cancel{S_\mu(\rho_1)}}$$
$$= \text{l.h.s.} \qquad \#$$

Remark: As acting on Schur measures, the commutation relation says that adding ρ_3 and then removing ρ_2 is the same as first removing ρ_2 and then adding ρ_3:

$$\mathbb{S}_{\rho_4;\rho_1,\rho_2} P^{\uparrow}(\rho_1,\rho_2;\rho_3) = \mathbb{S}_{\rho_3,\rho_4;\rho_1,\rho_2}$$

followed by $\mathbb{S}_{\rho_3,\rho_4;\rho_1,\rho_2} P^{\downarrow}(\rho_1;\rho_2) = \mathbb{S}_{\rho_3,\rho_4;\rho_1}$

is the same as:

$$\mathbb{S}_{\rho_4;\rho_1,\rho_2} P^{\downarrow}(\rho_1;\rho_2) = \mathbb{S}_{\rho_4;\rho_1}$$

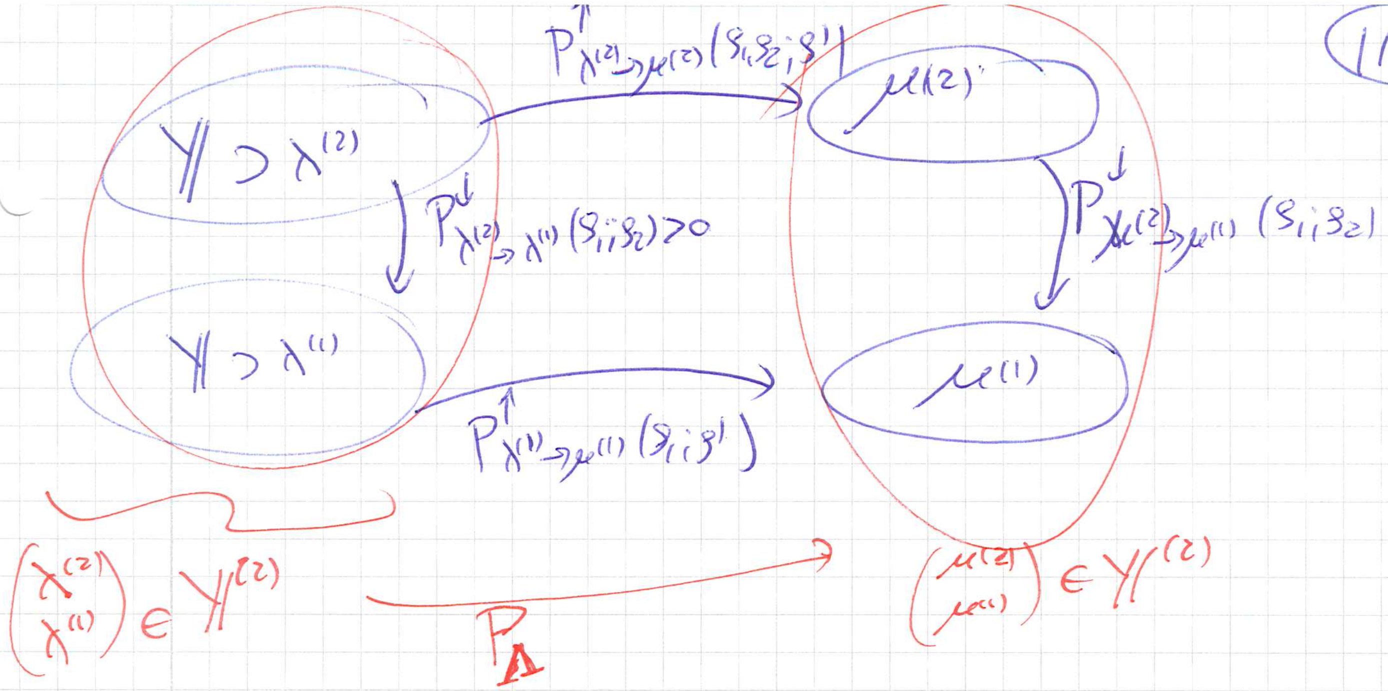
followed by $\mathbb{S}_{\rho_4;\rho_1} P^{\uparrow}(\rho_1;\rho_3) = \mathbb{S}_{\rho_3,\rho_4;\rho_1}$.

- So-far we have two types of transition matrices, one adding and one removing blocs from one Young diagram.

- Now we want to construct (and then apply to special cases) Markov chains on ~~couple~~ pairs of Young diagrams.

Def. 4 Let $\mathbb{Y}^{(2)}$ be the state space of pairs of Young diagrams $\binom{\lambda^{(2)}}{\lambda^{(1)}}$ and ρ_1, ρ_2 Schur-positive spec. s.t. $P^{\downarrow}_{\lambda^{(2)}\to\lambda^{(1)}}(\rho_1;\rho_2) > 0$. Then we define the (sequential) transition probabilities:

$$\mathbb{P}_{\Lambda}\left(\begin{pmatrix}\lambda^{(2)}\\ \lambda^{(1)}\end{pmatrix} \to \begin{pmatrix}\mu^{(2)}\\ \mu^{(1)}\end{pmatrix}\right) := P^{\uparrow}_{\lambda^{(1)}\to\mu^{(1)}}(\rho_1;\rho') \cdot \frac{P^{\uparrow}_{\lambda^{(2)}\to\mu^{(2)}}(\rho_1,\rho_2;\rho') \cdot P^{\downarrow}_{\mu^{(2)}\to\mu^{(1)}}(\rho_1;\rho_2)}{\sum_{\mu\in\mathbb{Y}} P^{\uparrow}_{\lambda^{(2)}\to\mu}(\rho_1,\rho_2;\rho') \cdot P^{\downarrow}_{\mu\to\mu^{(1)}}(\rho_1;\rho_2)}$$



$\mathbb{Y} \supset \lambda^{(2)}$
$\mathbb{Y} \supset \lambda^{(1)}$
$P^{\downarrow}_{\lambda^{(2)} \to \lambda^{(1)}}(s_1; s_2) > 0$
$P^{\uparrow}_{\lambda^{(2)} \to \mu^{(2)}}(s_1, s_2; s')$
$P^{\uparrow}_{\lambda^{(1)} \to \mu^{(1)}}(s_1; s')$
$\mu^{(2)}$
$\mu^{(1)}$
$P^{\downarrow}_{\mu^{(2)} \to \mu^{(1)}}(s_1; s_2)$
$\begin{pmatrix} \lambda^{(2)} \\ \lambda^{(1)} \end{pmatrix} \in \mathbb{Y}^{(2)}$
P_{Δ}
$\begin{pmatrix} \mu^{(2)} \\ \mu^{(1)} \end{pmatrix} \in \mathbb{Y}^{(2)}$

Remark: • P_Δ does the following:

(1) $\lambda^{(1)}$ evolves to $\mu^{(1)}$ according to $P^\uparrow_{\lambda^{(1)}\to\mu^{(1)}}(s_1;s')$

(2) Given $\lambda^{(2)},\mu^{(1)}$, the distribution of $\mu^{(2)}$ is the one of the middle point in the 2-step M.C. with transitions $P^\uparrow_{\lambda^{(2)}\to\mu}(s_1,s_2;s')$ and $P^\downarrow_{\mu\to\mu^{(1)}}(s_1;s_2)$

• The transition P_Δ has the nice property that, due to the commutation of Prop 3, the following form of measures is preserved:

Prop. 5: We have

$$\sum_{(\lambda^{(1)},\lambda^{(2)})\in\mathbb{Y}^{(2)}} \underbrace{S_{s_1,s_2;s^-}(\lambda^{(2)})\,P^\downarrow_{\lambda^{(2)}\to\lambda^{(1)}}(s_1;s_2)}\cdot P_\Delta\left(\begin{pmatrix}\lambda^{(2)}\\ \lambda^{(1)}\end{pmatrix}\to\begin{pmatrix}\mu^{(2)}\\ \mu^{(1)}\end{pmatrix}\right)$$

$$= \underbrace{S_{s_1,s_2;s^-,s'}(\mu^{(2)})\,P^\downarrow_{\mu^{(2)}\to\mu^{(1)}}(s_1;s_2)}$$

~~Proof:~~ $\cancel{\text{l.h.s.} = \sum_{\lambda^{(2)}\in\mathbb{Y}}\sum_{\lambda^{(1)}\in\mathbb{Y}} \frac{S_{\lambda^{(2)}}(s_1,s_2)\,S_{\lambda^{(2)}}(s^-)}{H(s_1,s_2;s^-)}\,\frac{S_{\lambda^{(1)}}(s_1)\,S_{\lambda^{(2)}/\lambda^{(1)}}(s_2)}{S_{\lambda^{(2)}}(s_1,s_2)}}$

Proof:

$$\text{l.h.s.} = \sum_{\lambda^{(2)}\in\mathbb{Y}}\sum_{\lambda^{(1)}\in\mathbb{Y}} S_{s_1,s_2;s^-}(\lambda^{(2)})\,P^\downarrow_{\lambda^{(2)}\to\lambda^{(1)}}(s_1;s_2)\cdot P^\uparrow_{\lambda^{(1)}\to\mu^{(1)}}(s_1;s')\,\frac{P^\uparrow_{\lambda^{(2)}\to\mu^{(2)}}(s_1,s_2;s')\,P^\downarrow_{\mu^{(2)}\to\mu^{(1)}}(s_1;s_2)}{\sum_{\mu\in\mathbb{Y}} P^\uparrow_{\lambda^{(2)}\to\mu}(s_1,s_2;s')\,P^\downarrow_{\mu\to\mu^{(1)}}(s_1;s_2)}$$

Using Prop 3 we have:

$$\sum_{\lambda^{(1)}\in\mathbb{Y}} P^{\downarrow}_{\lambda^{(2)}\to\lambda^{(1)}}(g_1;g_2)\, P^{\uparrow}_{\lambda^{(1)}\to\mu^{(1)}}(g_1;g')$$

$$= \sum_{\mu\in\mathbb{Y}} P^{\uparrow}_{\lambda^{(2)}\to\mu}(g_1,g_2;g')\cdot P^{\downarrow}_{\mu\to\mu^{(1)}}(g_1;g_2)$$

$\Rightarrow$ the denominator simplifies leading to:

$$\underbrace{\sum_{\lambda^{(2)}\in\mathbb{Y}} \mathcal{S}_{g_1,g_2;g^-}(\lambda^{(2)})\, P^{\uparrow}_{\lambda^{(2)}\to\mu^{(2)}}(g_1,g_2;g')}\, P^{\downarrow}_{\mu^{(2)}\to\mu^{(1)}}(g_1;g_2)$$

$$= \mathcal{S}_{g_1,g_2;g^-,g'}(\mu^{(2)}) \qquad \#$$

Plan: $\begin{cases} \to \text{Generalize to } N\text{-tuples of Young diagrams.} \\ \to \text{Look at a concrete example.} \end{cases}$

Markov chain on N-tuples of Young ~~tableaux~~ diagrams, $\mathbb{Y}^{(N)}$.

Def 6: Let $\mathbb{Y}^{(N)}$ the space of N-tuples of Young diagrams

Let $g_1,\dots,g_N$ be Schur-positive specializations

~~...~~ and $(\lambda^{(1)},\dots,\lambda^{(N)})\in\mathbb{Y}^{(N)}$.

Assume, $P^{\downarrow}_{\lambda^{(k)}\to\lambda^{(k-1)}}(g_1,\dots,g_{k-1};g_k)>0$, $k=2,\dots,N$,

then define

$$P_\Lambda\big((\lambda^{(1)},\dots,\lambda^{(N)})\to(\mu^{(1)},\dots,\mu^{(N)})\big) :=$$

$$= P^{\uparrow}_{\lambda^{(1)}\to\mu^{(1)}}(g_1;g')\cdot\prod_{k=2}^{N}\frac{P^{\uparrow}_{\lambda^{(k)}\to\mu^{(k)}}(g_1,\dots,g_k;g')\,P^{\downarrow}_{\mu^{(k)}\to\mu^{(k-1)}}(g_1,\dots,g_{k-1};g_k)}{\sum_{\mu\in\mathbb{Y}}P^{\uparrow}_{\lambda^{(k)}\to\mu}(g_1,\dots,g_k;g')\,P^{\downarrow}_{\mu\to\mu^{(k-1)}}(g_1,\dots,g_{k-1};g_k)}$$

defines a Markov chain on $\mathbb{Y}^{(N)}$ (with sequential update: first $\mu^{(1)}$, then $\mu^{(2)}, \mu^{(3)}, \ldots, \mu^{(N)}$).

The generalisation of Prop 5 is immediate.

Prop. 7: ~~We have~~ Define

$$\underbrace{\mathcal{M}_{\mathcal{S}_1 \ldots \mathcal{S}_N; \mathcal{S}^-}((\vec{\lambda})) := \mathcal{S}_{\mathcal{S}_1 \ldots \mathcal{S}_N; \mathcal{S}^-}(\lambda^{(N)}) \cdot P^{\downarrow}_{\lambda^{(N)} \to \lambda^{(N-1)}}(\mathcal{S}_1 \ldots \mathcal{S}_{N-1}; \mathcal{S}_N) \cdots P^{\downarrow}_{\lambda^{(2)} \to \lambda^{(1)}}(\mathcal{S}_1; \mathcal{S}_2)}$$

Then,

$$\sum_{(\vec{\lambda}) \in \mathbb{Y}^{(N)}} \mathcal{M}_{\mathcal{S}_1 \ldots \mathcal{S}_N; \mathcal{S}^-}(\vec{\lambda}) \cdot P_\Lambda(\vec{\lambda} \to \vec{\mu})$$

$$= \mathcal{M}_{\mathcal{S}_1 \ldots \mathcal{S}_N; \mathcal{S}^-, \mathcal{S}'}(\vec{\mu}).$$

Proof: An easy generalisation of the one of Prop. 5 #

Remark: The measure $\mathcal{M}_{\mathcal{S}_1 \ldots \mathcal{S}_N; \mathcal{S}}$ is the one of a Schur process and thus we know ~~how~~ that it has determinantal correlation functions (and we know ~~in prin~~ how to compute its correlation kernel).

Application:

- Let us consider the simple case where each $\rho_k = ((1,0,\dots);0;0)$ and $\rho' = (0;(b,0,\dots);0)$. $b \in (0,1)$

- Consider the discrete time (homogeneous) Markov chain $\vec{\lambda}(t) = (\lambda^{(1)}(t), \dots, \lambda^{(N)}(t))$ starting from the Schur process $\mathcal{M}_{\rho_1 \dots \rho_N; \rho^-}(\vec{\lambda}(0))$ with $\rho^- = (0;0;0)$.

Q1 What is the measure at $t=0$?

~~$\mathcal{M}_{\rho_1 \dots \rho_N; 0}(\vec{\lambda}) = S_{((\alpha_1, \dots, \alpha_N); 0; 0); 0}(\lambda^{(N)}) P^{\downarrow}_{\lambda^{(N)} \to \lambda^{(N-1)}}((\alpha_N;0;0),(\alpha);0)$~~

. Since $S_{((1,\dots,1);0;0);0}(\lambda^{(N)}) = S_{\lambda^{(N)}}((0;0;0)) S_{\lambda^{(N)}}((1,\dots,1);0;0)$ (with N entries 1)

$$= \delta_{\lambda^{(N)}, \phi}$$

(this can be easily seen from the line ensembles representation

. Then $P^{\downarrow}_{\phi \to \lambda^{(N-1)}}(*;*) > 0$ implies that $\lambda^{(N-1)} = \phi$ and so on.

$\Rightarrow$ $\boxed{\text{at } t=0, \text{ the measure is all concentrated on the configuration } \vec{\lambda}(0) = (\phi, \dots, \phi).}$

Q: Measure at time t?

By Proposition 7, the measure at time t is given by

$$\underline{\mathcal{M}_{((\underbrace{1,\dots,1}_{N});0;0),(0;(\underbrace{b,\dots,b}_{t});0)}(\vec{\lambda}(t)) =}$$

$$= \mathcal{S}_{((\underbrace{1,\dots,1}_{N});0;0),(0;(\underbrace{b,\dots,b}_{t});0)}(\lambda^{(N)}(t)) \cdot$$

$$\cdot \prod_{k=2}^{N} P^{\downarrow}_{\lambda^{(k)}\to\lambda^{(k-1)}}\big(((\underbrace{1,\dots,1}_{k-1});0;0);((1);0;0)\big)$$

<u>Consequence</u>: ~~The last products implies that~~

- $\lambda^{(k)}(t)$ has at <u>most k non-empty rows</u> and their coordinates satisfy the following <u>interlacing conditions</u>:

$$\lambda^{(k)}_1 \geq \lambda^{(k-1)}_1 \geq \lambda^{(k)}_2 \geq \dots \geq \lambda^{(k-1)}_{k-1} \geq \lambda^{(k)}_k.$$

<u>Indeed</u>: . One can see by iteration that the measure on $\lambda^{(k)}(t)$ is (using Prop 2 (b))

$$\underbrace{\mathcal{S}_{((\underbrace{1,\dots,1}_{k});0;0);(0;(\underbrace{b,\dots,b}_{t});0)}(\lambda^{(k)}(t))}\cdot \prod_{\ell=2}^{k} P^{\downarrow}\dots$$

$$= \underbrace{S_{\lambda^{(k)}(t)}\big((\overbrace{1,\dots,1}^{k});0;0\big)}, t\dots$$

$$= 0 \text{ if } \lambda^{(k)}_{k+1}(t) \neq 0.$$

- Interlacing comes from the factors $P^{\downarrow}_{\lambda^{(k)}\to\lambda^{(k-1)}}\dots$

Q: Transition probabilities.

• $\lambda^{(1)}$ has a single row with transition probabilities:

$$P^{\uparrow}_{\lambda^{(1)}\to\mu^{(1)}}\big((1;0;0);(0;b;0)\big) =$$

$$= \frac{1}{1+b}\cdot \frac{\overbrace{S_{\mu^{(1)}}((1;0;0))}^{=1^{|\mu^{(1)}|}}}{\underbrace{S_{\lambda^{(1)}}((1;0;0))}_{=1^{|\lambda^{(1)}|}}} \cdot \boxed{S_{\mu^{(1)}/\lambda^{(1)}}((0;b;0))}$$

$$= \begin{cases} b, & \text{if } \mu^{(1)} = \lambda^{(1)}+1 \\ 1, & \text{if } \mu^{(1)} = \lambda^{(1)} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{b}{b+1}, & \text{if } \mu^{(1)} = \lambda^{(1)}+1, \\ \frac{1}{b+1}, & \text{if } \mu^{(1)} = \lambda^{(1)}, \\ 0, & \text{otherwise.} \end{cases}$$

• More generally,

$$\mathbb{P}\big(\lambda^{(k)}(t+1)=\nu \,\big|\, \lambda^{(k)}(t)=\lambda,\ \lambda^{(k-1)}(t+1)=\mu\big) =$$

$$= \frac{S_{\nu/\lambda}(0;b;0)\, S_{\nu/\mu}(1;0;0)}{\sum_{\eta\in\mathbb{Y}} S_{\eta/\lambda}(0;b;0)\, S_{\eta/\mu}(1;0;0)} \sim b^{|\nu|-|\lambda|}$$

given that $0 \le \nu_i - \lambda_i \le 1$ and μ, ν are interlacing.

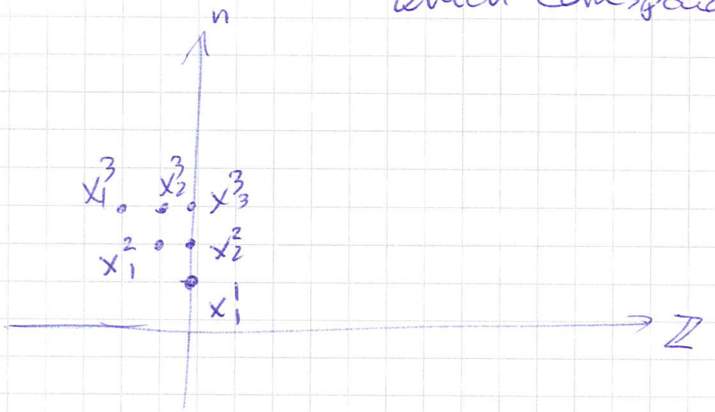
• This means that the length of each row of $\lambda^{(k)}$ indep. increases by 1 with proba $\frac{b}{b+1}$ unless this contradicts interlacing. In the latter case, the length either stays the same or increases by 1 with proba. 1).

Visualization as an interacting particle system.

- Let us introduce the $\frac{N(N+1)}{2}$ particles with integer coordinates

$$x_i^n := \lambda^{(n)}_{n+1-i} - n + i, \quad n = 1, \dots, N, \ i = 1, \dots, n$$

- In particular if all $\lambda_k^{(n)}$ are equal to 0, which corresponds to the initial condition.



- The interlacing for the particle system becomes:

$$\boxed{x_1^n < x_1^{n-1} \le x_2^n < x_2^{n-1} \le \dots \le x_n^n}, \quad n = 2, \dots, N.$$

- **Dynamics:** At each time t, each particles ~~flips a coin with~~ would like to jump to its right with proba $\frac{b}{1+b}$.

(a) $x_1^1(t+1) = \begin{cases} x_1^1(t)+1 & \text{with proba } \frac{b}{b+1} \\ x_1^1(t) & \text{with proba } \frac{1}{b+1} \end{cases}$

(b) Recursively from bottom to top:

→ If $x_k^{n-1}(t+1)$, $1 \le k \le n-1$ are updated, then

(1) If $k > 1$ and $x_k^n(t) = x_{k-1}^{n-1}(t+1) - 1 \Rightarrow x_k^n(t+1) = x_k^n(t) + 1$
(pushed)

otherwise: (2) If $x_k^{n-1}(t+1) = x_k^n(t) + 1 \Rightarrow x_k^n(t+1) = x_k^n(t)$
(blocked)

otherwise: (3) $X_k^n(t+1) = \begin{cases} X_k^n(t)+1 & \text{with proba } \frac{b}{1+b}, \\ X_k^n(t) & \text{" " } \frac{1}{b+1}. \end{cases}$

. Remarks: • One can easily take the continuous limit ($b \to 0$ and $t = \tau/b$)

In this case the dynamics is simpler:

. each particle ~~tries~~ tries to jump to its right with rate 1 and

(a) If $X_k^n(t)$ tries to jump and

"X_k^{n-1} block X_k^n" $\quad X_k^n(t) = X_k^{n-1}(t) - 1 \Rightarrow$ jump suppressed

(b) If $X_k^n(t)$ jumps and ~~after~~ before the jump $X_k^n(t) = X_{k+1}^{n+1}(t) = \dots = X_{k+c}^{n+c}(t)$

for some $c \Rightarrow$ all these particles are moved to the right by 1

("X_k^n pushes X_{k+1}^{n+1}, which pushes X_{k+2}^{n+2} ...").

• One can also take the diffusive scaling limit. The result is a system of reflected Brownian motions.

Projection onto $(X_1^{(1)}(t), \dots, X_1^{(N)}(t))$.

. The projection to $(X_1^{(1)}(t), \dots, X_1^{(N)}(t))$ is called TASEP with sequential update (in discrete time).

↑

Totally Asymmetric Simple Exclusion Process.

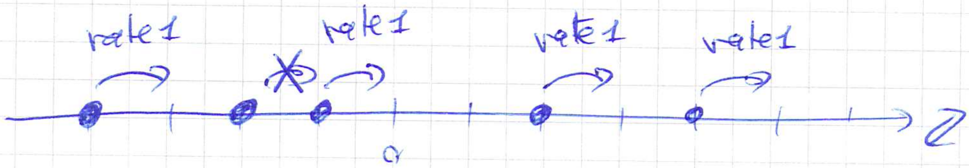
TASEP in continuous time.

- The TASEP = Totally Asymmetric Simple Exclusion in continuous time is an interacting particle system on $\mathbb{Z}$:

  - State space: $\Omega = \{0,1\}^{\mathbb{Z}}$, where
  $$\eta(x) = \begin{cases} 1, & \text{if } \exists \text{ particle at } x \\ 0, & \text{if } x \text{ is empty.} \end{cases}$$

  - Dynamics: In words, each particle tries to jump to its right with rate 1, provided that the site is empty.

  - Mathematically, it is a Markov process with generator
  $$(Lf)(\eta) = \sum_{x\in\mathbb{Z}} \eta_x(1-\eta_{x+1})\left[f(\eta^{x,x+1}) - f(\eta)\right]$$
  for f local functions; $\eta^{x,x+1}(z) = \begin{cases} \eta(x), & \text{if } z = x+1 \\ \eta(x+1), & \text{if } z = x \\ \eta(z), & \text{otherwise} \end{cases}$



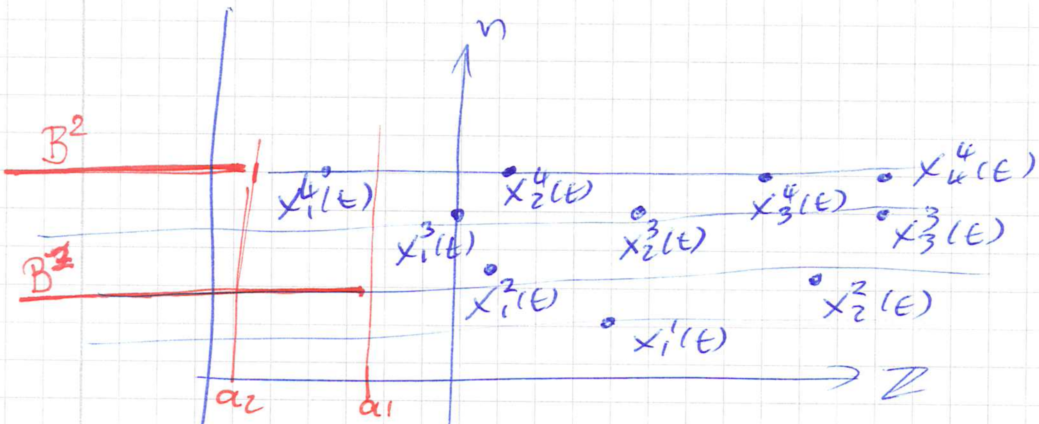
- Consider the initial condition corresponding to $\lambda^{(n)} = \phi$, $\forall n$:

$$x_i^n(0) = -n+i, \quad 1 \le i \le n \le N.$$

- TASEP particles are $(x_1^1(t), x_1^2(t), \dots, x_1^N(t))$.

Q.: Can we determine the joint distributions of ~~all~~ TASEP particles? For any $1 \le n_1 < n_2 < \dots < n_m$,

what is $\mathbb{P}\left(\bigcap_{k=1}^{m} \{ x_1^{n_k}(t) \ge a_k \} \right) = ?$



Ex.: $m=2$; $n_1=2$; $n_2=4$

$$= \mathbb{P}\left(\bigcap_{k=1}^{m} \{ \text{Set } \{(x, n_k),\ x < a_k\} \text{ is empty} \} \right)$$

- Since the measure on the point process

$$\xi = \sum_{n \ge 1} \sum_{k=1}^{n} \delta_{(x_k^n(t), n)}$$

is determinantal, the probability we are looking for is a Fredholm determinant.

- Before going into computations, remark that this model is equivalent to the growth model described in the introduction (page 3), see also Thm 1.

- Indeed, one can define a height function

$$h(x,t)=\begin{cases} 2N(t)+\sum_{y=1}^{x}(1-2\eta_y(t)), & \text{if } x\geq 1,\\ 2N(t), & \text{if } x=0,\\ 2N(t)-\sum_{y=-x+1}^{0}(1-2\eta_y(t)), & \text{if } x\leq -1.\end{cases}$$

where $N(t)=$ # of jumps from site 0 to site 1 during the time-span $[0,t]$.

In particular,

$$\boxed{\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\underbrace{X_{n_k}(t)}_{=x_1^{n_k}(t)}\geq m_k-n_k\}\Big)=\mathbb{P}\Big(\bigcap_{k=1}^{m}\{h(m_k-n_k,t)\geq m_k+n_k\}\Big).}$$

- **Correlation kernel of the measure at time t:**

- In the discrete time, the measure at time t is given by

$$S_{(\underbrace{1,\dots,1}_{N};0;0);(0;(\underbrace{b,\dots,b}_{t});0)}(\lambda^{(N)})\cdot$$

$$\cdot\prod_{k=2}^{N}P^{\downarrow}_{\lambda^{(k)}\to\lambda^{(k-1)}}((\underbrace{1,\dots,1}_{k-1};0;0);(1;0;0))$$

See page 113

$$= \frac{1}{Z} S_{\lambda^{(1)}}(\rho_0^+) S_{\lambda^{(2)}/\lambda^{(1)}}(\rho_1^+) \cdots S_{\lambda^{(N)}/\lambda^{(N-1)}}(\rho_{N-1}^+) \cdot S_{\lambda^{(N)}}(\rho_-^N)$$

with $\rho_0^+ = \cdots = \rho_{N-1}^+ = (1;0;0)$, for which $H(\rho_k^+) = \frac{1}{1-z}$, ~~and~~ and $\rho_-^N = (0;\underbrace{(b,\ldots,b)}_{t};0)$, for which $H(\rho_-^N) = (1+bz)^N$.

- The continuous limit is obtained as the limit $b \to 0$ and $t \to \frac{t}{b}$.

$\Rightarrow$ Measure at time t is given by

$$\mathcal{M}_{(\underbrace{(1,\ldots,1)}_{N};0;0);(0;0;t)}(\lambda^{(1)},\ldots,\lambda^{(N)}) =$$

see page 113

$$= \frac{1}{Z} S_{\lambda^{(1)}}((1;0;0)) \cdots S_{\lambda^{(N)}/\lambda^{(N-1)}}((1;0;0)) \cdot S_{\lambda^{(N)}}((0;0;t))$$

- We already computed the correlation kernel of this Schur process, see page 101.
  - In the general formula, we need to specialize as follows: $H(\rho_0^+,\ldots,\rho_{k-1}^+)(w) = \left(\frac{1}{1-w}\right)^k$

$$H(\rho_{k,\ldots}^-,\rho_N^-)(w^{-1}) = e^{tw^{-1}}$$

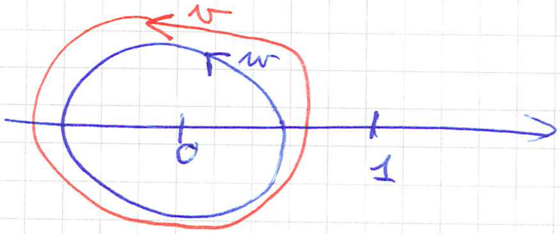
- The result of the Thm specialized reads:

Thm: Let $\xi := \sum_{n=1}^{N} \sum_{k=1}^{n} \delta_{(n, x_k^n(t))}$. ξ is a determinantal point process on $\{1,\dots,N\} \times \mathbb{Z}$ with correlation kernel:

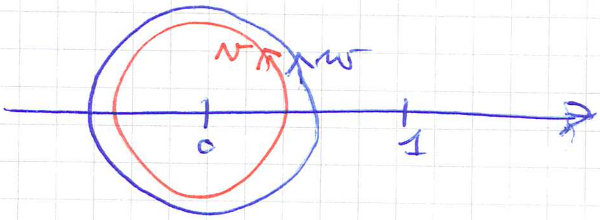
$$K(n_1,x_1;n_2,x_2) = \frac{1}{(2\pi i)^2} \oint dw \oint dv \, \frac{(1-w)^{n_1}}{(1-v)^{n_2}} \, \frac{e^{t/w}}{e^{t/v}} \, \frac{w^{x_1-1}}{v^{x_2}} \, \frac{1}{v-w}$$

where the contours are as follows:

- For $n_1 \geq n_2$:



- For $n_1 < n_2$:



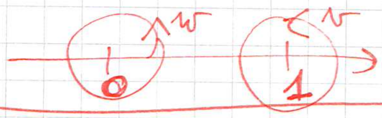
Remark: • The poles of v for $n_1 \geq n_2$ are at $v = 1$ & $v = w$.

$\Rightarrow$ we can rewrite as $\oint_{\Gamma_0} dw \oint_{\Gamma_1} dv$

• For $n_1 < n_2$, we can ~~[illegible]~~ exchange the contours and get the pole at $w = v$ as well.

• This leads to the following formula:

$$\boxed{\begin{aligned} K(n_1,x_1;n_2,x_2) = & -\frac{1}{2\pi i} \oint_{\Gamma_0} dv \, \frac{1}{(1-v)^{n_2-n_1}} \cdot \frac{1}{v^{x_2-x_1+1}} \, \mathbb{1}_{[n_1<n_2]} \\ & + \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_1} dv \, \frac{e^{t/w}}{e^{t/v}} \cdot \frac{(1-w)^{n_1}}{(1-v)^{n_2}} \cdot \frac{w^{x_1-1}}{v^{x_2}} \cdot \frac{1}{w-v} \end{aligned}}$$



. Large time asymptotics:

. We want to determine the large time asymptotics of $x_n(t)$ for $n = \mathcal{O}(t)$.

. Let $j(x,t) = \mathbb{E}\big(\eta_x(t)\,(1-\eta_{x+1}(t))\big)$ and $\rho(x,t) := \mathbb{E}(\eta_x(t))$.

j: average current of particles from x to $x+1$

ρ: particle density at x.

. **Lemma 9**: The conservation of particles implies:

$$\frac{d}{dt}\rho(x,t) + \nabla_x j(x,t) = 0$$

where $\nabla_x j(x,t) := j(x,t) - j(x-1,t)$.

. **Proof**: Let $T(t)$ be the semigroup generated by the TASEP generator $\mathcal{L}$.

The forward eq. is: $\frac{d}{dt} T(t) f = T(t)\mathcal{L} f$

Taking $f(\eta) = \eta(x)$ and integrating w.r.t. the initial condition μ we get ($\mu_t := \mu T(t)$)

$$\frac{d}{dt}\mu_t f = \frac{d}{dt}\rho(x,t).$$

. Further: $(\mathcal{L}f)(\eta) = \sum_{y\in\mathbb{Z}} \eta_y(1-\eta_{y+1})\left[f(\eta^{y,y+1}) - f(\eta)\right]$

$$= -\eta_x(1-\eta_{x+1}) + \eta_{x-1}(1-\eta_x)$$

$$\Rightarrow \mu_t\mathcal{L}f = \mathbb{E}\big(\eta_{x-1}(t)(1-\eta_x(t))\big) - \mathbb{E}\big(\eta_x(t)(1-\eta_{x+1}(t))\big)$$

$$= j(x-1,t) - j(x,t).$$

#

- **Large scale**: Now consider $X = \varepsilon x$, $x \in \mathbb{Z}$ $\Leftrightarrow x = \lfloor \varepsilon^{-1} X \rfloor$

and set $\tilde{j}(X,t) := j(\lfloor \varepsilon^{-1} X \rfloor, t)$.

$$\Rightarrow \nabla_x j(x,t) = \varepsilon \frac{\partial}{\partial X} \tilde{j}(X, t) + O(\varepsilon^2)$$

$\Rightarrow$ To have a non-trivial limit one need to take $t = \varepsilon^{-1} T$.

$(x,t) \longmapsto (\varepsilon^{-1} X, \varepsilon^{-1} T)$

is called the hydrodynamic scaling.

- let $J(X,T) := \lim\limits_{\varepsilon \to 0} \tilde{j}(\lfloor X \varepsilon^{-1} \rfloor, T\varepsilon^{-1})$ and

$\rho_{ma}(X,T) := \lim\limits_{\varepsilon \to 0} \rho(\lfloor X \varepsilon^{-1} \rfloor, T\varepsilon^{-1})$. Then

one can show (heuristically follows from the discrete version of the eq. in Lemma 8):

$$\boxed{\partial_T \rho_{ma}(X,T) + \partial_X J(X,T) = 0}$$

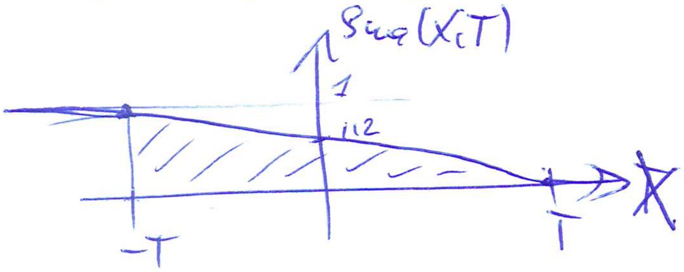
- In our case $J(X,T) = \rho_{ma}(X,T)\,(1 - \rho_{ma}(X,T))$ (*)

(*) is also known as Burgers equation.

- Solution of (*) in our case:

- We have $\rho_{ma}(X,0) = \begin{cases} 1, & X < 0, \\ 0, & X > 0. \end{cases}$

- One can get (see also any PDE lectures, method of characteristics) $\rho_{ma}(X,T) = \begin{cases} 1, & X < -T, \\ \frac{T-X}{2}, & X \in [-T,T], \\ 0, & X > T. \end{cases}$



- From this we can estimate which particles are around the origin at time T, namely particle $\# \approx \frac{T}{4}$, since the $\#$ of particles which jumped over the origin is $\approx \int_0^\infty dx\, \rho_{ma}(x,T) = T/4$.

Scaling limit:

- Let us consider $\boxed{n_i = \frac{t}{4} + u_i \left(\frac{t}{2}\right)^{2/3}}$
- From the macroscopic picture $\left(\int_{\xi_i}^{\infty} dx\, \rho_{ma}(x,t) = n_i \Rightarrow \xi_i = \dots \right)$

we expect $x_{n_i}(t)$ to be roughly around
$$\boxed{\xi_i = -2u_i \left(\frac{t}{2}\right)^{2/3} + u_i^2 \left(\frac{t}{2}\right)^{1/3}}.$$

- Due to the $\left(\frac{2}{3}, \frac{1}{3}\right)$ KPZ scaling, we set
$$\boxed{a_i = \xi_i - s_i \left(\frac{t}{2}\right)^{1/3}}.$$

Then, we can prove the following result:

Thm. 10: $\forall$ fixed $u_1 < u_2 < \dots < u_m$, $s_1, s_2, \dots, s_m \in \mathbb{R}$,
$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{ x_{n_k}(t) \geq a_k \} \right) = \mathbb{P}\left(\bigcap_{k=1}^{m} \{ \mathcal{A}_2(u_i) \leq s_i \} \right).$$

- Let us explain some of the steps to prove Thm. 10. By the formula of page 128 (Thm 9):
$$\mathbb{P}\left(\bigcap_{k=1}^{m} \{ x_{n_k}(t) \geq a_k \} \right) = \det\left(\mathbb{1} - \chi_a K_t \chi_a \right)_{\ell^2(\mathbb{Z} \times \{n_1, \dots, n_m\})}$$
with $\chi_a(n_k, x) = \mathbb{1}(x < a_k)$.

Thus we need to consider the rescaled Kernel:

$$(u_1-u_2-1)\left(\frac{t}{2}\right)^{1/3} K_t\Big(\frac{t}{4}+u_1\left(\frac{t}{2}\right)^{2/3}, -2u_1\left(\frac{t}{2}\right)^{2/3}+u_1^2\left(\frac{t}{2}\right)^{1/3}-s_1\left(\frac{t}{2}\right)^{1/3};$$

$$\frac{t}{4}+u_2\left(\frac{t}{2}\right)^{2/3}, -2u_2\left(\frac{t}{2}\right)^{2/3}+u_2^2\left(\frac{t}{2}\right)^{1/3}-s_2\left(\frac{t}{2}\right)^{1/3}\Big) =: K_t^{resc}(u_1,s_1;u_2,s_2).$$

- The ~~main part of the~~ Kernel (~~the one without the ...~~) is given by:

$$K_t^{resc}(u_1 s_1; u_2 s_2) := \left(\frac{t}{2}\right)^{1/3} \oint dw \oint dv \, \frac{e^{t f_0(w)}}{e^{t f_0(v)}} \frac{e^{\left(\frac{t}{2}\right)^{2/3} f_1(u_1,w)}}{e^{\left(\frac{t}{2}\right)^{2/3} f_1(u_2,v)}} \frac{e^{\left(\frac{t}{2}\right)^{1/3} f_2(u_1,s_1,w)}}{e^{\left(\frac{t}{2}\right)^{1/3} f_2(u_2,s_2,v)}} \cdot \frac{w^{-1}}{w-v}.$$

with
$$\begin{cases} f_0(w) = \dfrac{1}{w} + \dfrac{1}{4}\ln(w-1), \\ f_1(u_1,w) = u_1 \ln(w-1) - 2u_1 \ln(w), \\ f_2(u_1,s_1,w) = (u_1^2 - s_1)\ln(w) \end{cases}$$

Step 1: Understand from which region the integral gets the dominant contribution $\Rightarrow$ $\boxed{\text{consider only } f_0}$

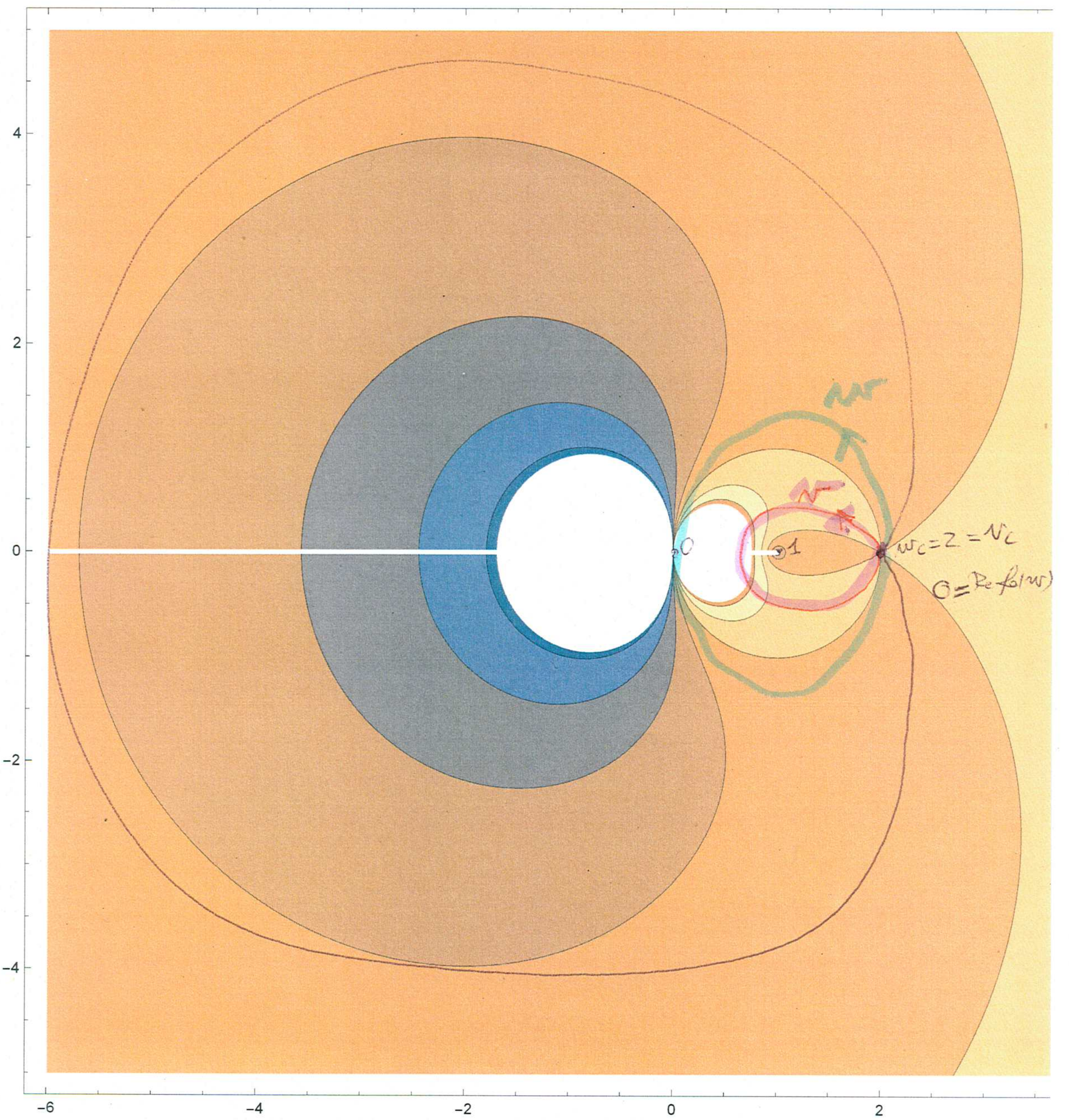
(a) Critical points: $\dfrac{df_0}{dw} = -\dfrac{1}{w^2} + \dfrac{+1}{4(w-1)} = \dfrac{-4+4w-w^2}{4w^2(1-w)}$

$$= -\frac{(2-w)^2}{4w^2(1-w)}.$$

$$\Rightarrow \frac{df_0(w)}{dw} = 0 \quad \text{for } \boxed{w = 2 =: w_c}.$$

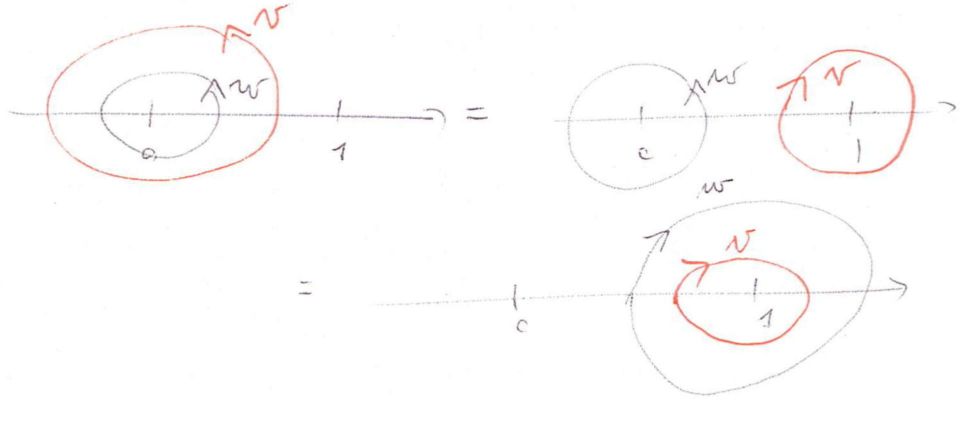
$\Rightarrow$ Close to $w_c = 2$, $f_0(w) = \dfrac{1}{2} + \dfrac{1}{48}(w-2)^3 + O((w-2)^4)$.

$$\begin{cases} f_1(u_1,w) = f_1(u_1,2) - \frac{1}{4}u_1(w-2)^2 + O((w-2)^3) \\ f_2(u_1,s_1,w) = f_2(u_1,s_1,2) + (u_1^2-s_1)\frac{w-2}{2} + O((w-2)^2). \end{cases}$$



$\mathrm{Re}\, f_0(w) = \text{constant}$ are the lines

- On the path for w, $\mathrm{Re}(f_0(w)) < \mathrm{Re}(f_0(w_c))$ except at $w = w_c = 2$
- " v, $-\mathrm{Re}(f_0(v)) < -\mathrm{Re}(f_0(v_c))$ " " $v = v_c = 2$



(b) . One shows that the leading contribution comes from a neighborhood of the critical point $w_c = v_c = 2$, by so-called steep descent analysis.

Step 2: Contribution around the critical points:
Using Taylor expansions, define the new variables

$$w := 2 + 2W\left(\frac{t}{2}\right)^{-1/3}, \quad v = 2 + 2V\left(\frac{t}{2}\right)^{-1/3}$$

$$\Rightarrow K_t^{\mathrm{resc}}(u_1, s_1; u_2, s_2) \simeq \frac{\cancel{\left(\frac{t}{2}\right)^{1/3}}}{(2\pi i)^2} \int dW \int dV\, \cancel{4}\cancel{\left(\frac{t}{2}\right)^{-2/3}} \cdot \underbrace{\frac{e^{t f_0(2) + \left(\frac{t}{2}\right)^{2/3} f_1(u_1, 2) + \left(\frac{t}{2}\right)^{1/3} f_2 \cdots}}{e^{t f_0(2) + \left(\frac{t}{2}\right)^{2/3} f_1(u_2, 2) + \left(\frac{t}{2}\right)^{1/3} f_2 \cdots}}}_{\text{it is a conjugation}}$$

$$\cdot\, e^{\frac{t}{48} \cdot \frac{2}{t} \cdot 8 \frac{W^3}{3} + O(W^4/t^{1/3})} \cdot e^{-\left(\frac{t}{2}\right)^{2/3} \cdot \frac{u_1}{4} \cdot 4W^2 \left(\frac{2}{t}\right)^{2/3} + O\left(\frac{W^3}{t^{1/3}}\right)}$$

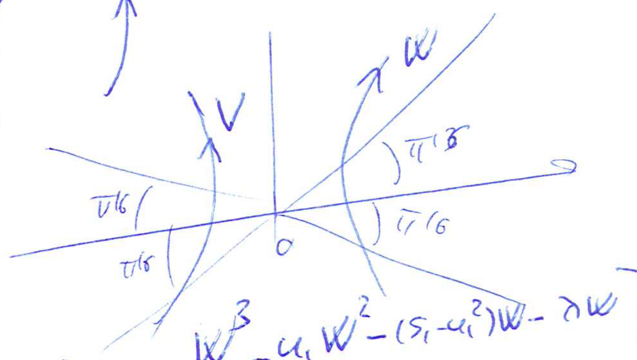
$$\cdot\, e^{(u_1^2 - s_1)\left(\frac{t}{2}\right)^{1/3} \frac{2W}{2}\left(\frac{t}{2}\right)^{-1/3} + O\left(\frac{W^2}{t^{1/3}}\right)}$$

$$\overline{e^{\frac{V^3}{3} + O(V^4/t^{1/3})}\, e^{-u_2 V^2 + O(V^3/t^{1/3})} \cdot e^{(u_2^2 - s_2)W + O \cdots}}$$

$$\cdot\, \cancel{\frac{1}{2}} \cdot \frac{1}{\cancel{2} \cdot (W - V) \cancel{\left(\frac{t}{2}\right)^{-1/3}}}$$

$$\simeq \frac{1}{(2\pi i)^2} \int dW \int dV \frac{e^{\frac{W^3}{3} - u_1 W^2 - (s_1 - u_1^2) W}}{e^{\frac{V^3}{3} - u_2 V^2 - (s_2 - u_2^2) V}} \frac{1}{W - V}$$

One shows that $O(\cdots)$ are irrelevant + conjugation



$$\frac{1}{W - V} = \int_0^\infty d\lambda\, e^{-\lambda(W - V)}$$

$$= \int_0^\infty d\lambda \left[\frac{1}{2\pi i} \int dW\, e^{\frac{W^3}{3} - u_1 W^2 - (s_1 - u_1^2) W - \lambda W}\right] \cdot \left[\frac{1}{2\pi i} \int dV\, e^{-\left[\frac{V^3}{3} - u_2 V^2 - (s_2 - u_2^2) V - \lambda V\right]}\right]$$

Change of variables:

$$\begin{cases} W = w + u_1 \\ V = v + u_2 \end{cases} \Rightarrow \quad = \int_0^\infty d\lambda \left[\frac{1}{2\pi i}\int_{\Gamma} dw\, e^{\frac{w^3}{3} - s_1 w - \lambda w}\right]$$

$$\frac{e^{u_1^3/3 - s_1 u_1}}{e^{u_2^3/3 - s_2 u_2}} \cdot \frac{e^{-\lambda u_1}}{e^{-\lambda u_2}} \cdot \left(\frac{1}{2\pi i}\int_{\Gamma} dv\, e^{-\left[\frac{v^3}{3} - (s_2+\lambda) v\right]}\right)$$

conjugation

$$\underset{(\text{conj})}{\equiv} \int_0^\infty d\lambda\, e^{-\lambda(u_1-u_2)}\, Ai(s_1+\lambda)\, Ai(s_2+\lambda),$$

which is the extended Airy kernel for $u_1 \geq u_2$.
By working out the details one proves that

$$\lim_{t\to\infty} K_t^{resc}(u_1,s_1;u_2,s_2) \underset{(\text{conj})}{=} K_{A_2}(u_1,s_1;u_2,s_2)$$

and also obtains with a little bit more work also:

$$\left|K_t^{resc}(u_1,s_1;u_2,s_2)\right| \underset{(\text{conj})}{\leq} C e^{-(s_1+s_2)} + \tilde{C}\, \mathbb{1}_{(u_1<u_2)} \cdot e^{-|s_1-s_2|},$$

for some $C, \tilde{C}$ indep. of t.
This bounds are then enough to prove Thm. 10.

. Using the correlation kernel, and doing asymptotic analysis we can obtain other results.
Here is a short description of the most important ones, obtained in my paper with A. Borodin "Anisotropic growth of random surfaces in 2+1 dimensions".

. First we define a <u>height function</u>:

~~$h_t(x,n)$~~:

$h(x,n,t) :=$ # of particles at level n on the right of position x at time t.

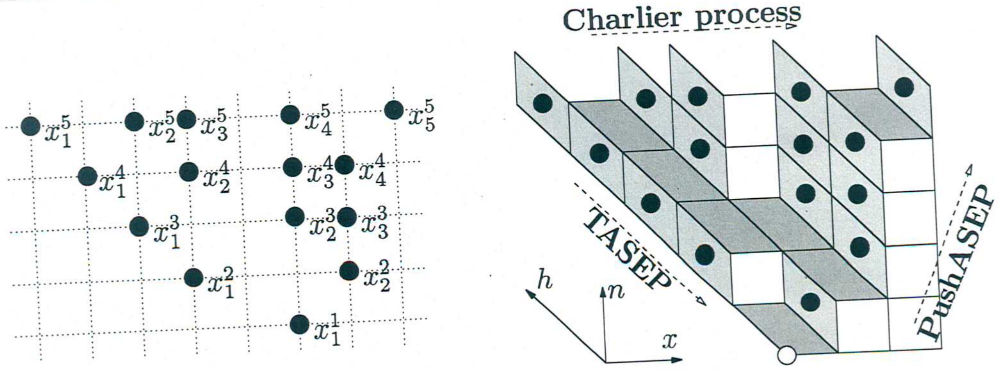


Figure 1.2: From particle configurations (left) to 3d visualization via lozenge tilings (right). The corner with the white circle has coordinates $(x, n, h) = (-1/2, 0, 0)$.

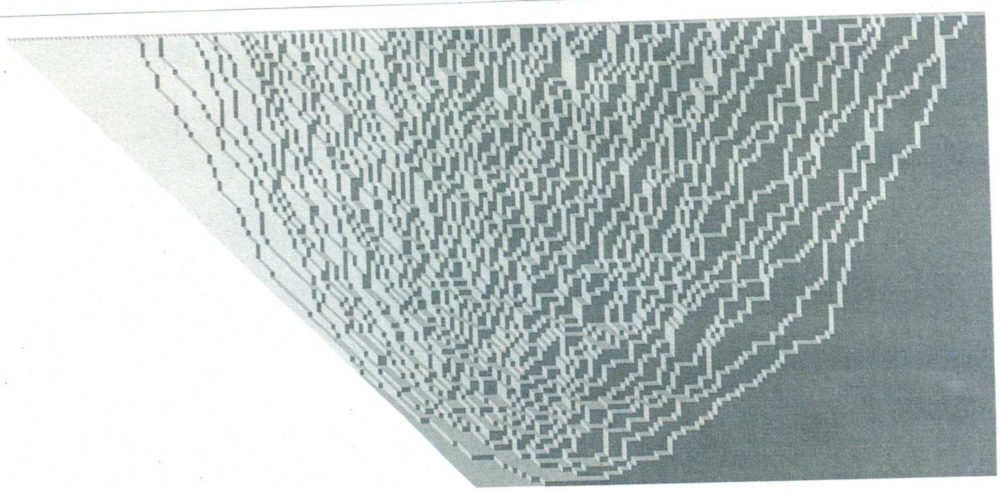


Figure 1.3: A configuration of the model analyzed with $N = 100$ particles at time $t = 25$, using the same representation as in Figure 1.2. In [38] there is a Java animation of the model.

(a) Description of the limit shape border:

- let $n_i := \eta_i L$, $x_i = -\eta_i L + \nu_i L$, $t_i = \tau_i L$.
  We want to describe the $L \to \infty$ limit.

- The random region, where there is not only one type of color/lozenge, is the following:

$$\boxed{\mathbb{D} := \left\{ (\nu, \eta, \tau) \in \mathbb{R}_+^3 \;\middle|\; \left(\sqrt{\eta} - \sqrt{\tau}\right)^2 < \nu < \left(\sqrt{\eta} + \sqrt{\tau}\right)^2 \right\}}$$

- This is obtained by computing the density of particle, which is given by $K_t(n, x; n, x)$ and see when this equals to 0 or 1.

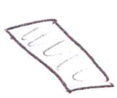
(b) Fluctuations in the disordered phase.

Thm 12: $\forall\, (\nu, \eta, \tau) \in \mathbb{D}$, with $\kappa_0 := (2\pi^2)^{-1}$

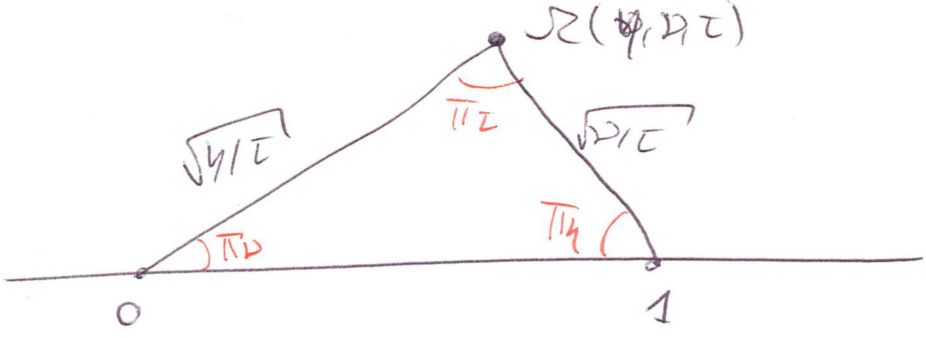
$$\lim_{L \to \infty} \frac{h((\nu-\eta)L, \eta L, \tau L) - \mathbb{E}\left(h((\nu-\eta)L, \eta L, \tau L)\right)}{\sqrt{\kappa_0 \ln L}} = \xi \sim \mathcal{N}(0,1).$$

This means that the fluctuations of the height functions are asymptotically Gaussian in the $\sqrt{\ln L}$ scale.

(c) Densities of the three types of lozenges.

-  = type I $\leftrightarrow$ angle $\pi_\eta \Leftrightarrow$ frequency $\frac{\pi_\eta}{\pi}$
-  = type II $\leftrightarrow$ angle $\pi_\tau \Leftrightarrow$ frequency $\frac{\pi_\tau}{\pi}$
-  = type III $\leftrightarrow$ angle $\pi_\nu \Leftrightarrow$ frequency $\frac{\pi_\nu}{\pi}$

- Given $(\eta,\nu,\tau)\in D$, define the map $\Omega: D \to \mathbb{H}$, where $\mathbb{H}=\{z\in\mathbb{C} \mid \mathrm{Im}(z)>0\}$ as follows:



Prop. 13 (Limit shape)

$$\lim_{L\to\infty} \frac{\mathbb{E}\left(h((\nu-\eta)L, \eta L, \tau L)\right)}{L} = h_{ma}(\nu,\eta,\tau)$$

$$= \frac{1}{\pi}\left\{ -\nu\pi_\eta + \eta(\pi-\pi_\nu) + \tau\cdot\frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\sin(\pi_\tau)} \right\}$$

Furthermore: Slopes:

- $\dfrac{\partial h_{ma}(\nu,\eta,1)}{\partial \nu} = -\dfrac{\pi_\eta}{\pi}$
- $\dfrac{\partial h_{ma}(\nu,\eta,1)}{\partial \eta} = 1-\dfrac{\pi_\nu}{\pi}$

Speed of growth:

$$\boxed{\frac{\partial h_{ma}(\nu,\eta,\tau)}{\partial \tau} = \frac{1}{\pi}\frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\sin(\pi_\tau)} = \frac{\mathrm{Im}(\Omega)}{\pi}}$$

(d) Random field in the "bulk"

- Let us introduce the notation

$$\mathcal{G}(z,w) := \frac{-1}{2\pi}\ln\left|\frac{z-w}{z-\bar{w}}\right|$$

This is the Green function of the Laplace operator on $\mathbb{H}$ with Dirichlet boundary condition on $\partial\mathbb{H}$.

Thm 14: Let $\varkappa_i = (\nu_i, y_i, \tau = 1) \in D$ be N distincts triples.

Let $H_L(\nu, y) := \sqrt{\pi}\left[h((\nu - y)L, yL, L) - \mathbb{E}(\ldots)\right]$

and $\Omega_k := \Omega(\nu_k, y_k, 1)$.

Then,

$$\lim_{L\to\infty} \mathbb{E}\left(H_L(\varkappa_1)\cdots H_L(\varkappa_N)\right) =$$

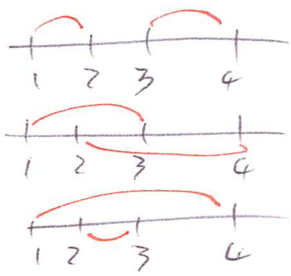
$$= \begin{cases} \sum_{\sigma \in \mathcal{F}_N} \prod_{j=1}^{N/2} \mathcal{G}(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & \text{if } N \text{ even} \\ 0, & \text{if } N \text{ odd}, \end{cases}$$

where $\mathcal{F}_N$ is the set of point free involutions on $\{1, \ldots, N\}$, also known as pairings.

Example: For $N = 2$: we have $\mathcal{G}(\Omega_1, \Omega_2)$

For $N = 4$: we have

$$\mathcal{G}(\Omega_1, \Omega_2)\,\mathcal{G}(\Omega_3, \Omega_4) + \mathcal{G}(\Omega_1, \Omega_3)\,\mathcal{G}(\Omega_2, \Omega_4) + \mathcal{G}(\Omega_1, \Omega_4)\,\mathcal{G}(\Omega_2, \Omega_3)$$



Remarks:

- Thm 12 & Thm 14 uses different normalisations! In Thm 12 we divide by $\sqrt{\varepsilon_2 L}$, in Thm 14 we do not divide!
  The reason is that the limit of H_L is not a smooth random field, but a singular one.
  In particular, H_L converges to a distribution, not to a function.
- The correlations in Thm 14 are the ones of the so-called Gaussian Free Field on $\mathbb{H}$.