13.10.2017

2) The discrete polynuclear growth model

2.1) The model

- The discrete polynuclear growth (PNG) model is a growth model with space and time discrete: $x \in \mathbb{Z}$, $t \in \mathbb{Z}_+$.

- The height-function h is integer-valued, i.e.,

$$h(x,t) \in \mathbb{Z}, \quad \forall x \in \mathbb{Z}, t \in \mathbb{Z}_+.$$

- We consider the PNG model in the "droplet geometry" (or "corner growth", "wedge initial cond."):

- Initial condition: $h(x,0) = 0, \quad \forall x \in \mathbb{Z}$

- Dynamics: $h(x,t+1) = \max\{h(x-1,t), h(x,t), h(x+1,t)\} + \omega(x,t+1)$

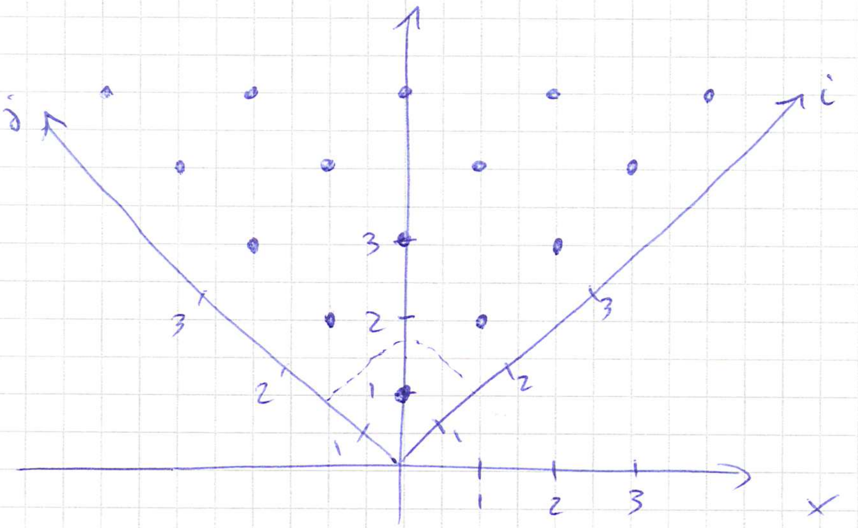
  where we choose the $\omega(x,t)$ to be independent random variables with the following distributions:

  - $\omega(x,t) = 0$ if $t-x$ is even or $|x| > t$

  - $\omega(i-j, i+j-1) = w(i,j)$, $i,j \in \mathbb{Z}_+^2$ with $\mathbb{P}(w(i,j)=k) = (1-a_ib_j)(a_ib_j)^k$, $k \geq 0$

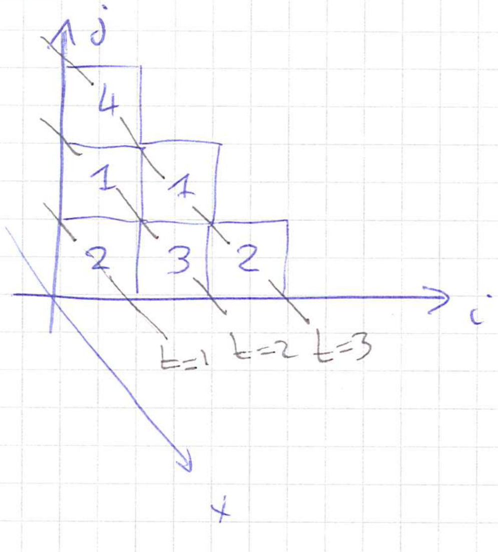
- Clearly, $a_ib_j \in (0,1)$, $\forall i,j \in \mathbb{Z}_+^2$.

. Later we will consider $a_i = b_j = \sqrt{q}$ as well, but to see the structure it is better to keep the full set of parameter.



• $= (x,t)$ s.t. $\omega(x,t)$ is not $\equiv 0$.

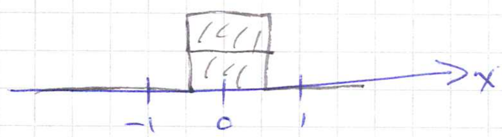
A few iterations:



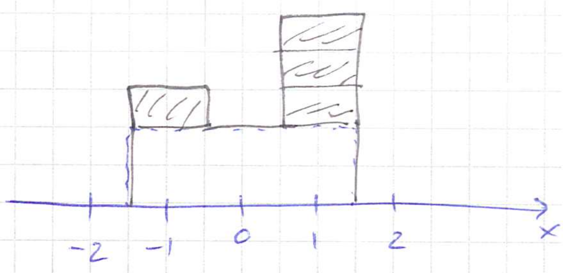
Height function:

$t=0$:

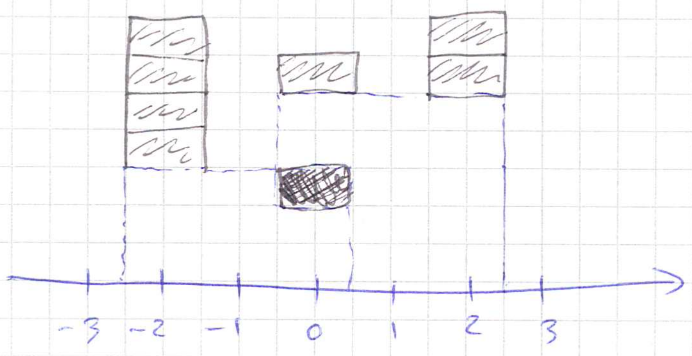
$t=1$:



$t=2$:



$t=3$:



. One sees that at $t=3$, two "islands" meet and if we look at $\{h(x,t=3), x\in\mathbb{Z}\}$ we can not reconstruct the values of the $\omega(i,j)$'s used so-far.

2.2) The multilayer PNG.

. One consequence of the above observation is that even a nice measure on the $\omega(i,j)$'s (e.g., iid), will not translate to a simple measure on $\{h(x,t), x\in\mathbb{Z}\}$.

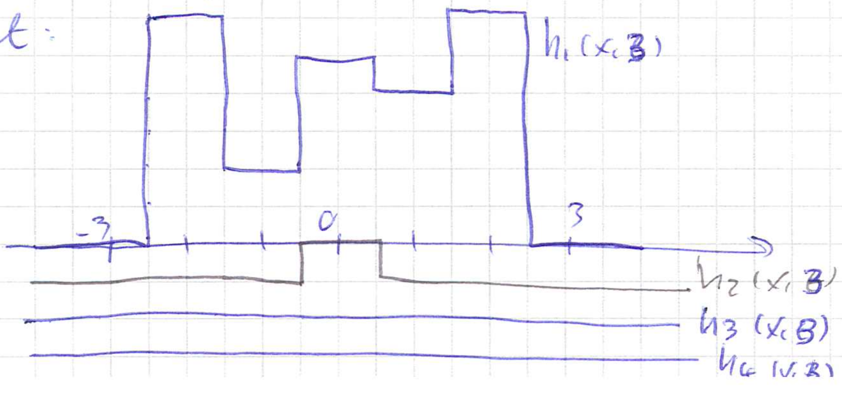
. The idea is to extend the model to a set of height functions, which are 1:1 (bijection) with the $\omega(i,j)$'s. This is as follows:

• Let $h_1(x,t) := h(x,t)$, the height function we actually want to study.

. Let $h_\ell(x,0) = -\ell+1$, $\forall \ell\geq 1, x\in\mathbb{Z}$ (initial condition)

. The dynamics of h_ℓ for $\ell\geq 2$ is as the one of h_1, with the only difference being in the $\omega(x,t)$'s: The $\omega_\ell(x,t)$ are given by the blocks at level $\ell-1$ which annihilate (the overlaps).

In the previous example, at time $t=3$ we would get:



· By construction the lines are non-intersecting.

· **Q:** How do the measure on the $w(i,j)$'s translate to a measure on the multilayer PNG?

After time we have the Measure (weight):

t=1: $(a_1 b_1)^{w(1,1)}$

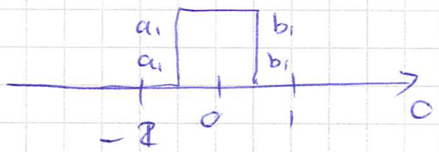
t=2: $(a_1 b_1)^{w(1,1)} \, (a_1 b_2)^{w(1,2)} \, (a_2 b_1)^{w(2,1)}$

t=3: ——— '' ——— $\cdot\, (a_1 b_3)^{w(1,3)} \, (a_2 b_2)^{w(2,2)} \, (a_3 b_1)^{w(3,1)}$

· On the lines, the weight of a configuration is the product of the weights on each vertical segment as follows:

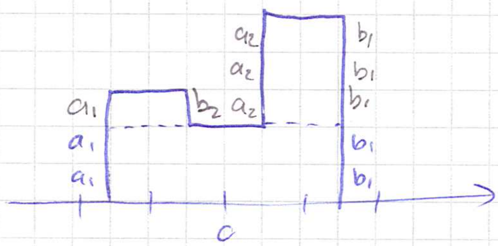
(new nucleations: left the a's, right the b's)

t=1:



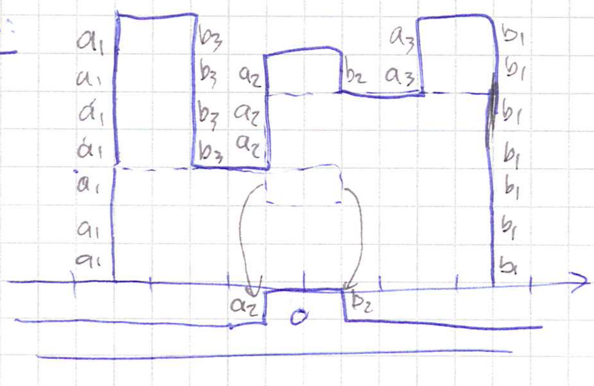
$: (a_1 b_1)^2$

t=2:



$: (a_1 b_1)^2 \cdot (a_1 b_2)^1 (a_2 b_1)^3$

t=3:



$: (a_1 b_1)^2 \, (a_1 b_2)^1 \, (a_2 b_1)^3 \cdot (a_1 b_3)^4 \, (a_2 b_2)^1 (a_3 b_1)^2$

$\Rightarrow$. We have seen that there is a mapping Φ from the set $\{w(i,j),\ i+j \le t+1\}$ to the set of non-intersecting line ensembles (with $h_e(x,t) = 0$, $|x| \geq t$).

. Further the weight ~~of~~ induced by Φ on a configuration of lines is simple.

. Remark: ~~The mapping~~ Φ is actually a bijection between

$\{w(i,j),\ i+j \le t+1,\ i,j \geq 1\}$

and $\{h_e(x,t),\ e \geq 1,$ ~~...~~ $h_e(x,t) = 0$ at $|x| \geq t$ & the lines do not intersect$\}$.

. To see this, one simply take any set of non-$\cap$ lines and runs the dynamics backwards. This allows to recover the values of $\{w(i,j),\ i+j \le t+1,\ i,j \geq 1\}$.

[. Details can be found in Johansson's original paper "discrete polynuclear growth and determinantal processes" arXiv:math/0206208, section 3.

2.3) Measure at a fixed x & LGV theorem.

- Consider the multilayer PNG droplet at time t.
  At each $x \in \mathbb{Z}$ and $|x| < t$, the heights $\{h_\ell(x,t), \ell \geq 1\}$ are not all deterministic.

Q: We would like to answer the question

$$\mathbb{P}(h_1(x,t) \leq a) = ?$$

or more generally

$$\mathbb{P}(h_1(x_1,t) \leq a_1, \ldots, h_1(x_m,t) \leq a_m) = ?$$

- Consider now a fixed x, say $x=0$.
  What is the probability of seeing a configuration of lines $\{h_1(0,t) > h_2(0,t) > h_3(0,t) > \ldots\}$, i.e.,

~~Remark: Since the~~

$$\mathbb{P}\Big(\bigcap_{\ell \geq 1} \{h_\ell(0,t) = H_\ell\}\Big) = ?$$

- To answer to the question, we use a theorem of Lindström-Gessel-Viennot (LGV).

- Let (V,E) be a directed graph $\begin{pmatrix} V = \text{Vertices,} \\ E = \text{Edges} \end{pmatrix}$.

- Assume: $\nexists$ cycles in the graph.

- A path π is a sequence of vertices joined by directed edges.

- $\mathcal{P}(u,v) :=$ set of all path from $u \in V$ to $v \in V$.

- We say that π & π' intersects if they have a common vertex.

$\Rightarrow$ We assign a weight $w(e) > 0$ to each edge $e \in E$ and the weight of a path π is defined by

$$w(\pi) := \prod_{e \in \pi \cap E} w(e)$$

- Let $h(u,v) := \sum_{\pi \in \mathcal{P}(u,v)} w(\pi)$.

- Finally, take m initial points $\vec{u} = (u_1, \dots, u_m)$ and m end points $\vec{v} = (v_1, \dots, v_m)$.

. Assumption: $\exists$ at most one permutation $\sigma \in S_m$ s.t. we can connect u_i to $v_{\sigma(i)}$, $i = 1, \dots, m$, by a set of m non-intersecting paths.

Under this assumption, we can wlog choose a labeling s.t. $\sigma = \mathbb{1}$ for to happen.

Theorem (LGV): Denote by $\mathcal{P}^{\text{non-}\cap}(\vec{u}, \vec{v})$ the set of all m non-$\cap$ paths from $\vec{u}$ to $\vec{v}$.

Then,

$$w(\mathcal{P}^{\text{non-}\cap}(\vec{u}, \vec{v})) := \sum_{(\pi_1, \dots, \pi_m) \in \mathcal{P}^{\text{non-}\cap}(\vec{u}, \vec{v})} w(\pi_1) \cdots w(\pi_m)$$

$$= \det \left[h(u_i, v_j) \right]_{1 \le i,j \le m}$$

Proof: $\det\big(h(u_i,v_j)\big)_{1\le i,j\le m} =$

$$= \sum_{\sigma\in S_m} (-1)^{|\sigma|}\prod_{k=1}^{m} h(u_k, v_{\sigma_k})$$

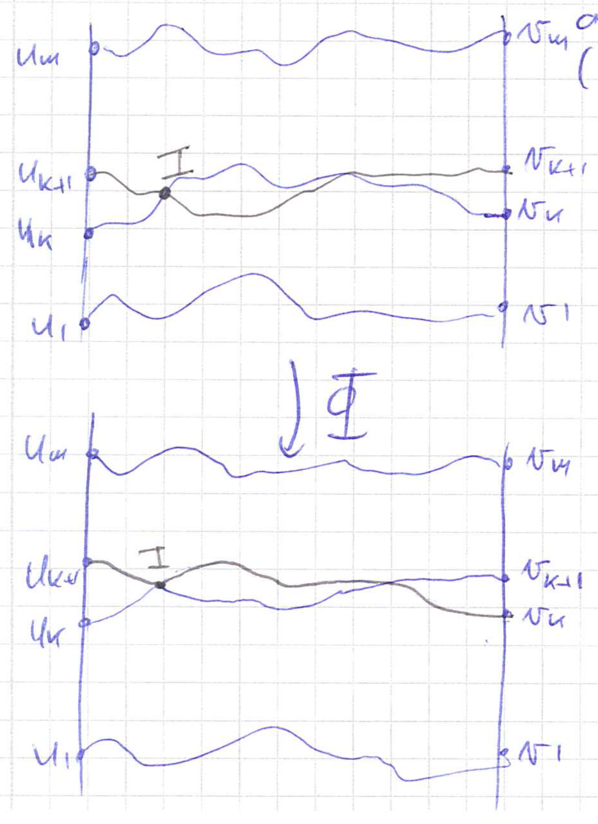
$$= \sum_{\sigma\in S_m}(-1)^{|\sigma|}\prod_{k=1}^{m} \sum_{\pi_k: u_k\to v_{\sigma_k}} w(\pi_k)$$

$$= \sum_{\sigma\in S_m} \Bigg[\sum_{\substack{\pi_1:u_1\to v_{\sigma_1}\\ \vdots \\ \pi_m: u_m\to v_{\sigma_m}\\ \&\ \text{non-}\cap}} (-1)^{|\sigma|}\cdot\prod_{k=1}^{m} w(\pi_k) \Bigg] \equiv w\big(\mathcal{P}^{\text{non-}\cap}_{(\vec{u},\vec{v})}\big)$$

$$+ \sum_{\sigma\in S_m} \Bigg[\sum_{\substack{\pi_1:u_1\to v_{\sigma_1}\\ \vdots \\ \pi_m: u_m\to v_{\sigma_m}\\ \text{with intersections}}} (-1)^{|\sigma|}\cdot\prod_{k=1}^{m} w(\pi_k) \Bigg] = (*)$$

To prove: $(*) = 0$.

- Let I be the first intersection of two consecutive paths (the definition is not unique), say π_k, π_{k+1} for some k.
- Consider the mapping Φ which exchanges the part of the paths π_k, π_{k+1} after I.



$\Rightarrow$ $w(\pi_1)\cdots w(\pi_m) =$

$$= w\big(\Phi_1(\pi_1,\dots,\pi_m)\big)\cdots w\big(\Phi_m(\pi_1,\dots,\pi_m)\big)$$

since the weights are on the edges.

. Further, notice that $\Phi \circ \Phi = \mathbb{1}$, i.e., Φ is a bijection.

$$\Rightarrow (*) = \sum_{\substack{\sigma, \pi_1, \dots, \pi_n \\ \text{with } \cap}} (-1)^{|\sigma|} \cdot \prod_{k=1}^{n} w(\pi_k)$$

$$= \sum_{\substack{\tilde{\sigma}, \Phi_1(\vec{\pi}), \dots, \Phi_n(\vec{\pi}) \\ \text{with } \cap}} (-1)^{|\tilde{\sigma}|} \cdot \prod_{k=1}^{n} w(\Phi_k(\vec{\pi}))$$

where $\tilde{\sigma}$ is the permutation s.t.

$$\Phi_k(\vec{\pi}) : u_k \to v_{\tilde{\sigma}_k}.$$

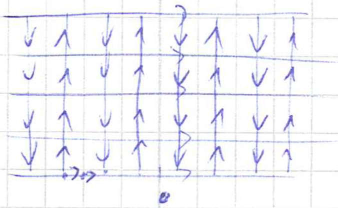
. Clearly $|\tilde{\sigma}| = |\sigma| \pm 1 \Rightarrow \quad = 0.$ #

. Now we can easily see that our non-$\cap$ line ensembles fits in the LGV scheme:

. $V = (\frac{1}{2}\mathbb{Z}) \times \mathbb{Z}$, Square graph with

. horizontal edges $\rightarrow$, weight 1,

. vertical edges: ~~directed ↑ & ↓ alternating~~ with weights:



$\uparrow$. a_{k+1} at horiz. position $-t+\frac{1}{2}+2k$, $k=1,\dots,t$

$\downarrow$. b_{k+1} at " " $t-\frac{1}{2}-2k$, $k=1,\dots,t$.

. <u>Consequence:</u> Consider a set of N non-$\cap$ lines starting from $(-t, -\ell+1)_{1\le \ell \le N}$ and ending at $(t, -\ell+1)_{1\le \ell\le N}$, i.e., $u_1 = (-t, 0), \dots, u_N = (-t, -N+1)$ & $v_1 = (t, 0), \dots, v_N = (t, -N+1)$

(19)

. Also restrict the graph to

$$V = \left(\mathbb{Z}+\tfrac{1}{2}\right) \times \{-N+1, -N+2, \dots\}.$$

. let $P_{t_1,t_2}(x_1,x_2) = \sum\limits_{\pi \in \mathcal{P}((t_1,x_1),(t_2,x_2))} w(\pi)$.

. Then we have:

Thm: . For $t<N$:

$$\mathbb{P}\left(h_1(a,t)=H_1, \dots, h_N(a,t)=H_N\right) =$$

$$= \text{const} \cdot \det\left(P_{(-t,0)}(-i+1, H_j)\right)_{1\le i,j\le N} \cdot \det\left(P_{(0,t)}(-i+1, H_j)\right)_{1\le i,j\le N}.$$

. Plan: (Wed)

① Representation in terms of diagrams.
Def. partitions, $\lambda < \mu, \dots$

② Describe the dynamics in terms of partitions for the PNG droplet. $\emptyset < \lambda^{(1)} > \mu^{(1)} < \lambda^{(2)} > \mu^{(2)} < \dots >$

③ Schur polynomials, ...

or: ① ~~Biorthogonal theorem~~, point processes

② Corr. fct.

③ ~~Biorthogonal~~ ensembles.

$\boxed{\text{Describe in terms of Schur polynomials}}$

④ Det. p.p.

⑤ ~~Asymp~~ Limit to Plancherelle

⑥ Asymptotics. (sine / F_2).

⑦ Markov chains (extended det.

. The next question is how to determine the transition weights $P_{(t_1,t_2)}(i,j)$.

. For this purpose (as well for the generalisation) let us use a slightly different representation of the line ensembles.

2.4) Partitions and Young diagrams.

. In the line ensembles, at any position $x \in \mathbb{Z}$, the multilayer lines crosses at position (random) $H_1 > H_2 > H_3 > \dots$ with $H_n + n \xrightarrow[]{} 0$ as $n \to \infty$, and $H_i \in \mathbb{Z}$, $\forall i \geq 1$.

. Equivalently, denote by

$$\lambda_i := H_i + i - 1.$$

. Then, we have $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ with $\lambda_n \to 0$ as $n \to \infty$

and $\lambda_i \in \mathbb{Z}_+$, $\forall i \geq 1$.

. This gives the natural connection to partitions.

Def: A partition is any (~~finite or infinite~~) sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers in decreasing order: $\lambda_1 \geq \lambda_2 \geq \dots$ and containing only finitely many non-zero terms.

Rem.: We regard $(2,1)$, $(2,1,0)$, $(2,1,0,0,\dots)$ as the same partition.

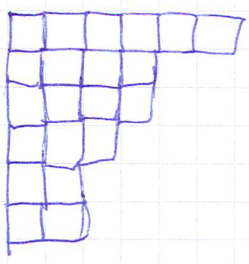
Def.:
- The non-zero λ_i of $\lambda=(\lambda_1, \lambda_2 \dots)$ are called the parts of λ.
- The number of parts is the length of λ, denoted by $\ell(\lambda)$.
- The sum of the parts is the weight of λ, denoted by $|\lambda|$: $|\lambda| = \lambda_1+\lambda_2+\dots$
- If $|\lambda|=n$, we say that λ is a partition of n.
- We denote by $\mathcal{P}_n$ the set of all partitions of n, and by $\mathcal{P}$ the set of all partitions.
- We denote by 0 or $\emptyset$ the empty partition (the only element in $\mathcal{P}_0$).

Notation: $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots)$, where
$m_i = \mathrm{Card}\{j : \lambda_j = i\}$.

- A graphical representation of a partition is provided by the Young diagrams, which we explain by an example:

Diagram

$\lambda = (6,4,4,3,2,2) \longleftrightarrow$



- We also define the <u>conjugate of λ</u>, denoted by λ', whose diagram is obtained by reflection in the main diagonal.

<u>Example:</u> $\lambda = (6,4,4,3,2,2) \to \lambda' = (6,6,4,3,1,1)$.

- In formulas: $\lambda'_i = \mathrm{Card}\{j : \lambda_j \geq i\}$.

$\Rightarrow \quad m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$.

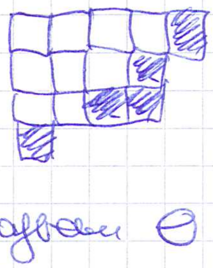
- Partial ordering:

<u>Def:</u> ~~Partial ordering~~: If λ and μ are partitions, we say that ~~$\lambda \leq \mu$ (or~~ $\lambda \subset \mu$ if $\forall i \geq 1, \quad \lambda_i \leq \mu_i$.

- <u>Skew diagram:</u> If $\lambda \subset \mu$, the set-theoretic difference $\theta = \mu - \lambda$ is called the <u>skew diagram</u>. We set $|\theta| := |\mu| - |\lambda|$.
$= \sum_i \underbrace{(\mu_i - \lambda_i)}_{=\theta_i}$

<u>Example:</u> $\mu = (5,4,4,1)$ & $\lambda = (4,3,2)$

$\Rightarrow \lambda - \mu$ is the shaded region in:



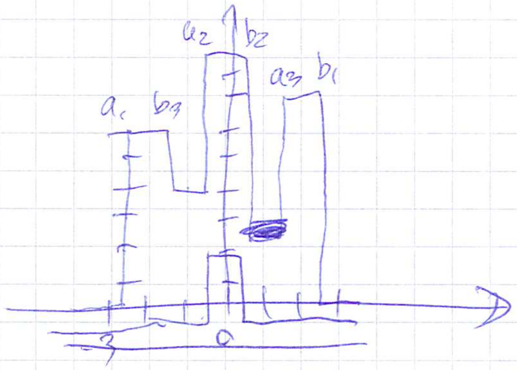
- A skew diagram θ is a <u>horizontal m-strip</u> (resp. vertical m-strip) if $|\theta| = m$ & $\theta'_i \leq 1$ (resp. $\theta_i \leq 1$), $\forall i \geq 1$.

$\Rightarrow$ A horizontal (resp. vertical) strip has at most one square in each column (resp. row).

· **Rem.:** If $\theta = \mu - \lambda$, then θ is a horizontal strip iff λ and μ interlaces ($\lambda \succ \mu$), i.e., $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots$

Notation: We denote by $\mathbb{Y}$ the set of all Young diagrams (equivalent to $\mathcal{P}$).

. To come back for a moment to the multilayer PNG model, we have ~~that~~ the following property:



Example with $t=3$.

$$\begin{cases} \lambda(t=-3) = \emptyset \\ \lambda(t=-2) = (6) \\ \lambda(t=-1) = (4) \\ \lambda(t=0) = (9,3) \\ \lambda(t=1) = (3) \\ \lambda(t=2) = (7) \\ \lambda(t=3) = \emptyset \end{cases}$$

. In terms of interlacing:

$$\emptyset \prec (6) \succ (4) \prec (9,3) \succ (3) \prec (7) \succ \emptyset.$$

2.5) Symmetric functions:

. The transition weights in this example would be $a_1^{|\lambda(-2)|-|\lambda(-3)|} = a_1^{6-0}$, from $t=-3$ to $t=-2$,

$b_3^{|\lambda(-2)|-|\lambda(-1)|} = b_3^{6-4} = b_3^2$, from $t=-2$ to $t=-1$,

$a_2^{|\lambda(0)|-|\lambda(-1)|} = a_2^{12-4} = a_2^8$, from $t=-1$ to $t=0$,

$b_2^{|\lambda(0)|-|\lambda(1)|} = b_2^{12-3} = b_2^{9}$, from $t=0$ to $t=1$,

$a_3^{|\lambda(2)|-|\lambda(1)|} = a_3^{7-3} = a_3^{4}$, from $t=1$ to $t=2$,

$b_1^{|\lambda(2)|-|\lambda(3)|} = b_1^{7-0}$, from $t=2$ to $t=3$.

These transitions can be ~~described~~ using Schur polynomials, ~~~~ which are symmetric polynomials.

- We want to define <u>the algebra Λ of symmetric functions in infinitely many variables.</u>

Way A:

(1). Let $\Lambda_N = \mathbb{C}[x_1, \dots, x_N]^{S_N}$ be the space of polynomials in $x_1, \dots, x_N$ which are <u>symmetric</u> w.r.t. permutations of the x_i's.

. Λ_N has a natural grading by the <u>total degree</u> of a polynomial:

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k,$$ where

Λ_N^k ~~~~ consists of <u>homogeneous</u> symmetric polynomials of degree k, together with the 0 polynomial.

(2) Let $\pi_{N+1} : \mathbb{C}[x_1, \dots, x_{N+1}] \to \mathbb{C}[x_1, \dots, x_N]$ obtained by setting $x_{N+1} = 0$.
(It preserves the ring of ~~~~ symmetric polyn. & gradings).

$\Rightarrow$ Tower of graded algebras.

$$\mathbb{C} \xleftarrow{\pi_1} \Lambda_1 \xleftarrow{\pi_2} \Lambda_2 \xleftarrow{\pi_3} \cdots$$

(3) $\Lambda := \varprojlim_{N \to \infty} \Lambda_N = \{(f_1, f_2, \dots) \mid f_i \in \Lambda_i, \pi_i f_i = f_{i-1}, \deg(f_i) < \infty\}$

(it is a projective limit of the tower).

Way B: Elements of Λ are formal power series $f(x_1, x_2, \dots)$ in ~~[illegible]~~ infinitely many variables $x_1, x_2 \dots$ of bounded degree, which are invariant under permutations of the x_i's.

Ex: • $x_1 + x_2 + x_3 + \dots \quad \in \Lambda$

• $(1+x_1)(1+x_2)(1+x_3)\dots \quad \notin \Lambda$. (unbounded degree!)

• Basis of Λ: ~~Def~~

Def: (a) Elementary symmetric functions e_k, $k = 1, 2 \dots$:

$$e_k := \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} \cdots x_{i_k}$$

(b) Complete homogeneous functions h_k, $k = 1, 2 \dots$:

$$h_k := \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} \cdots x_{i_k}$$

(c) Power sums p_k, $k = 1, 2, \dots$:

$$p_k = \sum_{i \geq 1} x_i^{\,k}$$

Thm: ~~Prop~~ The systems $\{e_k\}$, $\{h_k\}$, $\{p_k\}$ are algebraically independent generators of Λ.

I.e., $\Lambda = \mathbb{C}[e_1, e_2 \dots] = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[p_1, p_2 \dots]$

(Λ = algebra of polynomials in $e_1, e_2 \dots$).

Proof: See Macdonald "Symmetric functions and Hall polynomials", Oxford Univ. Press, 1995 (Chap 1, Sect. 2: (2.3), (2.4), (2.8), (2.12)).

Generating functions:

Def: We define the generating functions:

- $H(z) := \sum_{k\geq 0} h_k z^k$ with $h_0 = 1$,
- $E(z) := \sum_{k\geq 0} e_k z^k$, with $e_0 = 1$,
- $P(z) := \sum_{k\geq 1} p_k z^{k-1}$

Prop.: ⓐ $H(z) = \prod_{i\geq 1} \frac{1}{1-x_i z}$

ⓑ $E(z) = \prod_{i\geq 1} (1+x_i z)$

ⓒ $P(z) = \frac{d}{dz} \sum_{i\geq 1} \ln\left(\frac{1}{1-x_i z}\right)$

and:

ⓓ $H(z) = \frac{1}{E(-z)} = \exp\left(\sum_{k\geq 1} \frac{z^k}{k} p_k\right)$

Proof: ⓑ follows directly by opening the parenthesis since $E(z) = (1+\cancel{x_1 z})(1+x_2 z)(1+x_3 z)\cdots$

$$= 1 + z\cdot(x_1 + x_2 + \cdots) + z^2\cdot(x_1x_2 + x_1x_3 + \cdots + x_2x_3 + \cdots)$$
$$+ \cdots$$
$$= 1 + z e_1 + z^2 e_2 + \cdots.$$

ⓐ First we use $\frac{1}{1-x_i z} = 1 + x_i z + x_i^2 z^2 + \cdots$

This then gives:

$\prod \frac{1}{\ } = \prod (1 + x_i z + x_i^2 z^2 + \cdots)$

$= \cancel{1 + z(x_1 + x_2 + \dots) + z^2(}$

$$= (1 + x_1 z + x_1^2 z^2 + x_1^3 z^3 + \dots)(1 + x_2 z + x_2^2 z^2 + x_2^3 z^3 + \dots)$$
$$(1 + x_3 z + x_3^2 z^2 + x_3^2 z^3 + \dots) \cdots$$

$$= 1 + z(x_1 + x_2 + \dots) + z^2(x_1^2 + x_1 x_2 + x_1 x_3 + \dots$$
$$+ x_2^2 + x_2 x_3 + \dots)$$

$$= 1 + z \cdot h_1 + z^2 h_2 + \dots .$$

(c) By the power series expansion of the log:

$$\frac{d}{dz} \sum_{i \geq 1} \ln\left(\frac{1}{1 - x_i z}\right) = \frac{d}{dz} \sum_{i \geq 1} \left(x_i z + \frac{x_i^2 z^2}{2} + \frac{x_i^3 z^3}{3} + \dots\right)$$

$$= \sum_{i \geq 1} (x_i + x_i^2 z + x_i^3 z^2 + \dots)$$

$$= (x_1 + x_2 + \dots) + z \cdot (x_1^2 + x_2^2 + \dots) + z^2 (x_1^3 + x_2^3 + \dots)$$

$$= p_1 + z p_2 + z^2 p_3 + \dots .$$

(d) Easy (notice: $\frac{d}{dz}\left(\sum_{k \geq 1} \frac{z^k}{k} p_k\right) = P(z)$

$$\Rightarrow \frac{d}{dz}(\ln H(z)) = P(z)). \qquad \#$$

. Now we are ready to define the key object:

Def.: The Schur polynomial $S_\lambda(x_1, \dots, x_N)$ is a symmetric polynomial in N variables parameterized by Young diagram λ with $\ell(\lambda) \leq N$ and it is given by:

$$S_\lambda(x_1, \dots, x_N) = \frac{\det\left[x_i^{\lambda_j + N - j}\right]_{1 \leq i, j \leq N}}{\prod_{1 \leq i < j \leq N} (x_i - x_j)}.$$

- let us see that $S_\lambda(x_1,\dots,x_N)$ is a symm. polynomial.
- First of all, notice that $\det\left[x_i^{\lambda_j+N-j}\right]_{1\le i,j\le N}=0$

  whenever $x_i=x_j$ for some $i\neq j$.
- This means that the polynomial $(*)$ is divisible by x_i-x_j, $\forall i\neq j$.

$$\Rightarrow \quad S_\lambda'(x_1,\dots,x_N)=\frac{\det\left[x_i^{\lambda_j+N-j}\right]_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(x_i-x_j)}$$

is a polynomial.

- Further, it is an exercise to verify:

$$\prod_{1\le i<j\le N}(x_i-x_j)=\det\left[x_i^{N-j}\right]_{1\le i,j\le N}.$$

- Since both the numerator and the denominator are antisymmetric over permutations of x_i's, S_λ' is <u>symmetric</u>.

<u>Lemma:</u> let $\ell(\lambda)\le N$, then

$$\pi_{N+1}S_\lambda(x_1,\dots,x_N,x_{N+1}):=S_\lambda(x_1,\dots,x_N,0)=S_\lambda(x_1,\dots,x_N).$$

and $\pi_{\ell(\lambda)}S_\lambda(x_1,\dots,x_{\ell(\lambda)})=0$.

<u>Proof.:</u> $\ell(\lambda)\le N$ implies $\lambda_{N+1}=0$. Thus

$$S_\lambda(x_1,\dots,x_N,x_{N+1})=\frac{\begin{vmatrix} x_1^{\lambda_1+N} & \cdots & x_1^{\lambda_N+1} & x_1^{\lambda_{N+1}} \\ \vdots & & \vdots & \vdots \\ x_N^{\lambda_N+N} & \cdots & x_N^{\lambda_N+1} & x_N^{\lambda_{N+1}} \\ x_{N+1}^{\lambda_N+N} & \cdots & x_{N+1}^{\lambda_N+1} & x_N^{\lambda_{N+1}} \end{vmatrix}}{\begin{vmatrix} x_1^{N} & \cdots & x_1 & 1 \\ \vdots & & \vdots & \vdots \\ x_N^{N} & \cdots & x_N^{1} & 1 \\ x_{N+1}^{N} & \cdots & x_{N+1}^{1} & 1 \end{vmatrix}}$$

$$=\begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

Setting $x_{N+1}=0$ leads to the last rows $[0 \dots 0, 1]$, i.e, we get

$$S_\lambda(x_1,\dots,x_N,0) = S_\lambda(x_1,\dots,x_N).$$

. Next, $S_\lambda(x_1,\dots,x_{\ell(\lambda)-1},0) = \dfrac{\begin{vmatrix} x_1^{\lambda_1+\ell-1} & \dots & x_1^{\lambda_\ell} \\ \vdots & & \vdots \\ x_{\ell-1}^{\lambda_1+\ell-1} & \dots & x_{\ell-1}^{\lambda_\ell} \\ 0 & \dots & 0 \end{vmatrix}}{\begin{vmatrix} x_1^{\ell-1} & \dots & 1 \\ \vdots & & \vdots \\ x_{\ell-1}^{\ell-1} & \dots & \vdots \\ 0 & \dots\ 0 & 1 \end{vmatrix}} = 0.$

$\ell = \ell(\lambda)$

#

. Consequence: the sequence of symm. polyn. $S_\lambda(x_1 \dots x_N)$ with λ fixed and varying number of variables $N \geq \ell(\lambda)$, together with 0 for $N < \ell(\lambda)$, defines an element of Λ.
This is called Schur symmetric function S_λ, where by def. one sets $S_\emptyset(x) \equiv 1$.

. An important property is that $\{S_\lambda, \lambda \in \mathbb{Y}\}$ form a basis of Λ:

Prop.: The Schur functions $\{S_\lambda, \lambda \in \mathbb{Y}\}$, form a linear basis of Λ. Their relations with the generators e_k, h_k are given by the (Jacobi-Trudi) formulas:

$$S_\lambda = \det\left[h_{\lambda_i - i + j}\right]_{1 \leq i,j \leq \ell(\lambda)} = \det\left[e_{\lambda'_i - i + j}\right]_{1 \leq i,j \leq \ell(\lambda')}$$

where by def. $h_k \equiv e_k \equiv 0$ for $k < 0$.

Proof can be found in Macdonald book (Chap I, Sect 3).

• If we have a measure giving to a configuration $\lambda \in \mathbb{Y}$ a weight

$$S_\lambda(\vec{x}) \cdot S_\lambda(\vec{y}),$$

where $\vec{x} = (x_1, x_2, \dots)$ and $\vec{y} = (y_1, y_2, \dots)$ are two sets of variables (finite or infinite), then it is natural to look for the normalisation constant, which would turn the weight into a probability measure (provided the weight is positive), i.e., we look

• for $\sum_{\lambda \in \mathbb{Y}} S_\lambda(x) S_\lambda(y)$.

Thm (Cauchy identity): We have:

(a) $$\sum_{Y \in \mathbb{Y}} S_\lambda(x_1, x_2, \dots) S_\lambda(y_1, y_2, \dots) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}$$

(b) $$\sum_{Y \in \mathbb{Y}} \frac{P_\lambda(x_1, x_2, \dots) P_\lambda(y_1, y_2, \dots)}{z_\lambda} = \exp\left(\sum_{k=1}^{\infty} \frac{P_k(x_1, x_2, \dots) P_k(y_1, y_2, \dots)}{k} \right) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}$$

where: $P_\lambda := P_{\lambda_1} \cdot P_{\lambda_2} \cdots P_{\lambda_{\ell(\lambda)}}$,

$z_\lambda := \prod_{i \geq 1} (i^{m_i} m_i!)$

The rhs should be viewed as formal power series using $\frac{1}{1 - x_i y_j} \equiv 1 + x_i y_j + (x_i y_j)^2 + (x_i y_j)^3 + \dots$

. Proof in Macdonald book, chap. 1, sect. 4.

Q: ① The weights in PNG model at $x=0$, how are related with S_λ?

② What are the most general cases where $S_\lambda(\vec{x})$ is positive, $\forall \lambda$, i.e., it can define a probab. model?

In order to prove the Cauchy identity, (@),
we use another identity, that we will use again,
namely the Cauchy-Binet identity:

Thm (Cauchy-Binet):

$$\det\left[\int_\Lambda d\lambda(x)\,\Phi_i(x)\,\Psi_j(x)\right]_{1\le i,j\le N} =$$

$$= \frac{1}{N!}\int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\,\det\left[\Phi_i(x_j)\right]_{1\le i,j\le N}\det\left[\Psi_i(x_j)\right]_{1\le i,j\le N}$$

Proof: $\det\left[\int_\Lambda d\lambda(x)\,\Phi_i(x)\,\Psi_j(x)\right] =$

$$\overset{\text{linearity}}{=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\,\det\left[\Phi_i(x_i)\,\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\text{lin.}}{=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\prod_{i=1}^N \Phi_i(x_i)\cdot\det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\forall \sigma\in S_N}{=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\cdot\det\left[\Psi_j(x_{\sigma(i)})\right]_{1\le i,j\le N}$$

$$\overset{\text{antisym. of det}}{=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\,\mathrm{sgn}(\sigma)\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\cdot\det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\int \text{indep. of } \sigma}{=} \frac{1}{N!}\int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\underbrace{\left(\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\right)}_{=\det\left[\Phi_i(x_j)\right]_{1\le i,j\le N}}\det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

#.

Proof of Cauchy (a): Consider $X=(x_1,x_2,\dots,x_n)$, $Y=(y_1,\dots,y_n)$
[If some $y_n=0 \Rightarrow$ fine as well]. The ∞ variable case as limits, as limit as well.

$$\sum_{Y\in\mathbb{Y}} s_\lambda(x_1,\dots,x_n)\, s_\lambda(y_1,\dots,y_n) = ?$$

$$= \sum_{\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n\geq 0} \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}\cdot\det\left(y_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}}{\Delta_n(x_1,\dots,x_n)\,\Delta_n(y_1,\dots,y_n)}$$

$s_\lambda(x_1,\dots,x_n)=0$ if $\ell(\lambda)>n$

Vandermonde det.

$\Rightarrow$ Compute: $\Delta_n^{-1}(x_1,\dots,x_n)\,\Delta_n^{-1}(y_1,\dots,y_n)\sum_{\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n\geq 0}\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}\cdot\det\left(y_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n} =$

$\xi_i := \lambda_i+n-i$

$$= \Delta_n^{-1}(\vec{x})\,\Delta_n^{-1}(\vec{y})\cdot\underbrace{\sum_{\xi_1>\dots>\xi_n\geq 0}\det\left(x_i^{\xi_j}\right)_{1\leq i,j\leq n}\det\left(y_i^{\xi_j}\right)_{1\leq i,j\leq n}}_{!}$$

~~$\Delta_n(\vec{x})\Delta_n(\vec{y})$ Cauchy-Binet~~

$$= \frac{1}{n!}\sum_{\xi_1,\dots,\xi_n\geq 0}\det\left(x_i^{\xi_j}\right)_{1\leq i,j\leq n}\det\left(y_i^{\xi_j}\right)_{1\leq i,j\leq n}$$

$$\overset{\text{Cauchy-Binet}}{=} \Delta_n^{-1}(\vec{x})\,\Delta_n^{-1}(\vec{y})\det\left(\sum_{\xi\geq 0}(x_iy_j)^{\xi}\right)_{1\leq i,j\leq n}$$

$$= \Delta_n^{-1}(\vec{x})\,\Delta_n^{-1}(\vec{y})\cdot\det\left(\frac{1}{1-x_iy_j}\right)_{1\leq i,j\leq n}$$

$$= \prod_{i,j=1}^{n}\frac{1}{1-x_iy_j} \quad \text{by the following lemma. } \#$$

Lemma:
$$\left(\prod_{1\leq i<j\leq n}(x_j-x_i)(y_j-y_i)\right)^{-1}\det\left(\frac{1}{1-x_iy_j}\right)_{1\leq i,j\leq n} = \prod_{i,j=1}^{n}\frac{1}{1-x_iy_j}$$

Proof: There are several proof of this identity. We illustrate here how a direct computation leads to the proof. For clarity we ~~[illegible]~~ do it for $n=3$. The general case uses the same ideas.

$$\begin{vmatrix} \frac{1}{1-x_1y_1} & \frac{1}{1-x_1y_2} & \frac{1}{1-x_1y_3} \\ \frac{1}{1-x_2y_1} & \frac{1}{1-x_2y_2} & \frac{1}{1-x_2y_3} \\ \frac{1}{1-x_3y_1} & \frac{1}{1-x_3y_2} & \frac{1}{1-x_3y_3} \end{vmatrix} = \left(\prod_{i,j=1}^{3} \frac{1}{1-x_iy_j} \right) \cdot (*)$$

$$(*) = \det \begin{bmatrix} (1-x_iy_2)(1-x_iy_3) & (1-x_iy_1)(1-x_iy_3) & (1-x_iy_1)(1-x_iy_2) \end{bmatrix}_{1\le i\le 3}$$

$$\underset{\substack{Col2 \to Col_2 - Col1 \\ \& Col3 \to Col_3 - Col_1}}{=} \det \begin{bmatrix} (1-x_iy_2)(1-x_iy_3) & (1-x_iy_3)(y_2-y_1)x_i & (1-x_iy_1)(y_3-y_2)x_i \end{bmatrix}_{1\le i\le 3}$$

$$\underset{\text{linearity}}{=} (y_3-y_2)(y_2-y_1) \det \begin{bmatrix} (1-x_iy_2)(1-x_iy_3) & (1-x_iy_3)x_i & (1-x_iy_1)x_i \end{bmatrix}_{1\le i\le 3}$$

$$\underset{Col3 \to Col3 - Col2}{=} (y_3-y_2)(y_2-y_1) \cdot \det \begin{bmatrix} (1-x_iy_2)(1-x_iy_3) & (1-x_iy_3)x_i & (y_3-y_1)x_i^2 \end{bmatrix}_{1\le i\le 3}$$

$$\underset{\text{lin.}}{=} (y_3-y_2)(y_2-y_1)(y_3-y_1) \cdot \det \begin{bmatrix} 1-(y_2+y_3)x_i+y_2y_3x_i^2 & x_i - y_3x_i^2 & x_i^2 \end{bmatrix}_{1\le i\le 3}$$

can be deleted by lin. comb. with col. 3.

afterwards, this linear term in x_i can be deleted by lin. comb. with col. 2.

$$= (y_3-y_2)(y_2-y_1)(y_3-y_1) \cdot \det \left\{ x_i^{j-1} \right\}_{1\le i,j\le 3} = \Delta_3(\vec{y})\, \Delta_3(\vec{x})$$

#.

- The next object we want to introduce is the one, which for the PNG droplet, will give us the transition from $\lambda(t)$ to $\lambda(t+1)$. It will not be directly evident at this stage that there is a relation.

- Consider two sets of variables $x=(x_1, x_2, \ldots)$ and $y=(y_1, y_2, \ldots)$. Let (x,y) be the unions of sets of vars. $x \& y$. For a symmetric function $f \in \Lambda$, $f(x,y)$ is a function in x_i, y_j, symmetric w.r.t. all permutations of variables.

$\Rightarrow$ $f(x,y)$ is symmetric in x_i's but also in y_i's.

$$\Rightarrow f(x,y) = \sum_k f_k(x) g_k(y) \quad \text{for some } f_k, g_k$$

symmetric functions.

Ex.: $$P_k(x,y) = \sum_{i\geq 1} x_i^k + \sum_{i \geq 1} y_i^k = P_k(x) + P_k(y)$$

Q.: What is $S_\lambda(x,y)$?

Def.: Let λ be a Young tableau.

$$\Rightarrow S_\lambda(x,y) = \sum_{\mu \in \mathbb{Y}} S_{\lambda/\mu}(x) S_\mu(y)$$

where the coefficients $S_{\lambda/\mu}(x)$ are called skew Schur functions (which are symmetric functions in x_i's).

- We need to study some properties of $S_{\lambda/\mu}$.

Here are two simple properties:

Lemma:

(a) $s_{\lambda/\phi}(x) = s_\lambda(x)$,

(b) $s_{\phi/\nu}(x) = \begin{cases} 1 & \text{if } \nu=\phi, \\ 0 & \text{otherwise.} \end{cases}$

Proof: To see it, one verify on the schur polynomials.

(a) One has: $s_\lambda(x_1,\dots,x_N;0) = s_\lambda(x_1,\dots,x_N)$ $(\forall\, \ell(\lambda)\le N)$

$$\Vert \sum_{\mu\in\mathbb{Y}} s_{\lambda/\mu}(x_1,\dots,x_N)\cdot s_\mu(0)$$

But $s_\mu(0) = \begin{cases} 1 & \text{if } \mu=\phi \\ 0 & \text{if } \mu\neq\phi \end{cases}$

$$\Rightarrow\ = s_{\lambda/\phi}(x_1,\dots,x_N).$$

(b) $s_\phi(x_1,\dots,x_N;y_1,\dots,y_M) = 1$

$$\Vert \sum_{\mu\in\mathbb{Y}} s_{\phi/\mu}(x_1,\dots,x_N)\cdot s_\mu(y_1,\dots,y_M) = \overbrace{s_{\phi/\phi}(x_1,\dots,x_N)}^{=s_\phi(x_1,\dots,x_N)=1}\cdot \overbrace{s_\phi(y_1,\dots,y_M)}^{=1}$$

$$+ \underbrace{\sum_{\substack{\mu\in\mathbb{Y};\\ \mu\neq\phi}} s_{\phi/\mu}(x_1,\dots,x_N)\cdot s_\mu(y_1,\dots,y_M)}_{(*)}$$

$\Rightarrow$ $(*) = 0$, $\forall$ choice of y_i

$\Rightarrow$ $s_{\phi/\mu}(x_1,\dots,x_N)=0$ for $\mu\neq\phi$.

. The skew-Schur functions can also be expressed in terms of the complete homogeneous functions as follows:

Prop: We define $h_k=0$ for $k<0$. Then,

$$s_{\lambda/\mu} = \det\left[h_{\lambda_i-\mu_j-i+j}\right]_{1\le i,j\le \max\{\ell(\lambda),\ell(\mu)\}}$$

~~It follows, $s_{\lambda/\mu}=0$ unless $\mu\subset\lambda$, i.e., $\mu_i\le\lambda_i\ \forall i\geq 1$.~~

the details of the proof are in Macdonald, Chap 1, Sect 5 (eq. (5.4)).

Let us see the consequences of this formula.

Corollary:
- (a) $s_{\lambda/\mu}=0$ unless $\mu\subset\lambda$, i.e., $\mu_i\le\lambda_i,\ \forall i\ge 1$
- (b) If $\mu\subset\lambda$, let $\theta=\lambda-\mu$ the skew diagram. Then $s_{\lambda/\mu}=\prod_{i\ge1} s_{\theta_i}$.
- ~~(c) Finally, $s_{\lambda/\mu}(x_1,\dots,x_n)=0$ unless $0\le\lambda_i'-\mu_i'\le n$, $\forall i\ge 1$.~~

Proof: (a) If $\lambda_k<\mu_k$ for some k, then

$$\forall j\le k\le i:\quad \lambda_i\le\lambda_k<\mu_k\le\mu_j$$

$$\Rightarrow \lambda_i-\mu_j-i+j<0$$

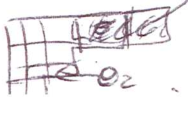
$$\Rightarrow \begin{vmatrix} * & * \\ \underbrace{0}_{(n-k+1)\times k\ \text{block}} & * \end{vmatrix} = 0$$

Since $\det\underbrace{\begin{pmatrix} A & B \\ 0 & C\end{pmatrix}}_{n\times n}=0$ unless $A=\frac{n}{2}\times\frac{n}{2}$, $C=\frac{n}{2}\times\frac{n}{2}$ matrices $\Rightarrow \begin{cases} k=\frac{n}{2}+\frac{1}{2} \\ k=\frac{n}{2}-1\end{cases}$ $\Rightarrow$ impossible.

(b) If $\mu_k\ge\lambda_{k+1}$ for some $k<n=\ell(\lambda)$.

$\Rightarrow$ we have a matrix

$$\det\begin{vmatrix} \underset{r\times r}{A} & B \\ 0 & \underset{(n-r)\times(n-r)}{C}\end{vmatrix} = \det(A)\cdot\det(C)$$

$\Rightarrow$ If $\lambda-\mu=$  $\Rightarrow \det(h_{\dots})=s_{\theta_1}s_{\theta_2}$.

· This extends to a general $\mu \subset \lambda$ similarly. #

· A generalisation of the Cauchy identity is the following.

Prop. (Skew Cauchy identity)

$\forall \lambda, \nu \in \mathbb{Y}$,
$$\sum_{\mu \in \mathbb{Y}} s_{\mu/\lambda}(\vec{x}) \cdot s_{\mu/\nu}(\vec{y}) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} \cdot \sum_{\kappa \in \mathbb{Y}} s_{\nu/\kappa}(\vec{x}) \, s_{\lambda/\kappa}(\vec{y})$$

· The most important identity that we will need, it is a sort of "consistency relation".

Let x, y two sets of variables. Then,

Prop.

$\forall \lambda, \mu \in \mathbb{Y}$,
$$s_{\lambda/\mu}(x, y) = \sum_{\nu \in \mathbb{Y}} s_{\lambda/\nu}(x) \, s_{\nu/\mu}(y).$$

Proof: By def of the skew schur fct:

$$s_\lambda(x, y, z) = \sum_\mu s_{\lambda/\mu}(x, y) \, s_\mu(z)$$

$$\Big\| \text{ also by def.}$$

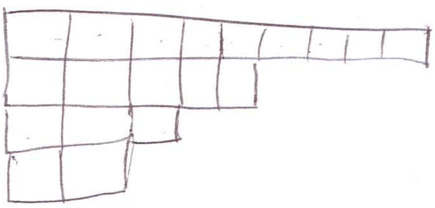
$$\sum_\nu s_{\lambda/\nu}(x) \, s_\nu(y, z)$$

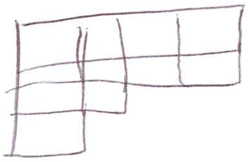
$$= \sum_{\mu, \nu} s_{\lambda/\nu}(x) \, s_{\nu/\mu}(y) \, s_\mu(z)$$

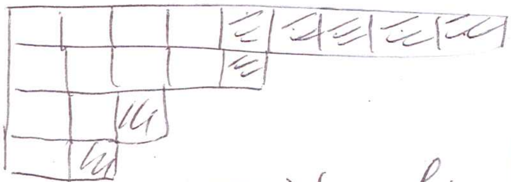
$$\forall s_\mu(z) \Rightarrow \quad s_{\lambda/\mu}(x, y) = \sum_\nu s_{\lambda/\nu}(x) \, s_{\nu/\mu}(y).$$

#

Example of Corollary (b)

$\lambda =$  $= (9, 5, 3, 2)$

$\mu = (4, 4, 2, 1).$ 

III: $\lambda - \mu$  $\Rightarrow \theta_1 = (5, 1)$

$\theta_2 = (1)$; $\theta_3 = (1)$

Table of: $h_{\lambda_i - \mu_j - i + j}$

$\Rightarrow$

$j \backslash i$	1	2	3	4
1	5	0	-3	-5
2	6	1	-2	-4
3	9	4	1	-1
4	11	6	3	1

$$S_{\lambda/\mu} = \det \begin{bmatrix} h_5 & \overset{=1}{h_0} & 0 & 0 \\ h_6 & h_1 & 0 & 0 \\ h_9 & h_4 & h_1 & 0 \\ h_{11} & h_6 & h_3 & h_1 \end{bmatrix}^{\text{transpose}}$$

$$= \det \begin{bmatrix} h_5 & h_0 \\ h_6 & h_1 \end{bmatrix} \cdot \det [h_1] \cdot \det [h_1] = S_{\theta_1} \cdot S_{\theta_2} \cdot S_{\theta_3} \checkmark$$

Recall: $s_\lambda = \det [h_{\lambda_i - i + j}]_{1 \leq i, j \leq \ell(\lambda)}$

$$\Rightarrow S_{\theta_1} = \det \begin{bmatrix} h_5 & h_0 \\ h_6 & h_1 \end{bmatrix}; \quad S_{\theta_2} = \det [h_1]$$

$$S_{\theta_3} = \det [h_1]$$

Generalization:

let $X^{(1)}, \dots, X^{(n)}$ be n sets of variables and λ, μ two partitions.

$$\Rightarrow S_{\lambda/\mu}(X^{(1)}, \dots, X^{(n)}) = \sum_{\mu = \nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n)} = \lambda} \prod_{i=1}^{n} S_{\nu^{(i)}/\nu^{(i-1)}}(X^{(i)})$$

In particular, for $\mu = \phi$,

$$S_\lambda(X^{(1)}, \dots, X^{(n)}) = \sum_{\phi \subset \nu^{(1)} \subset \dots \subset \nu^{(n)} = \lambda} S_{\nu^{(1)}}(X^{(1)}) \dots S_{\lambda/\nu^{(n-1)}}(X^{(n)}).$$

Remark that everything do not depends on the order of $X^{(1)}, \dots, X^{(n)}$, being S_λ symmetric.

We want to prove the skew Cauchy identity, which will be a key identity as well.

. For that we need first another relation.

Lemma: Let us set the coefficients $c^{\lambda}_{\mu\nu}$ s.t.

$$s_{\lambda/\mu}(x) = \sum_{\nu\in\mathbb{Y}} c^{\lambda}_{\mu\nu}\, s_{\nu}(x).$$

$$\Rightarrow s_{\mu}(x)\, s_{\nu}(x) = \sum_{\lambda\in\mathbb{Y}} c^{\lambda}_{\mu\nu}\, s_{\lambda}(x).$$

Proof: Let $P(x,y) := \prod_{i,j\geq 1} \frac{1}{1-x_i y_j}$; let $d^{\lambda}_{\mu\nu}$ s.t. $s_\mu(z)s_\nu(z) = \sum_{\lambda\in\mathbb{Y}} d^{\lambda}_{\mu\nu} s_\lambda(z)$.

We can write, for x, y, z three sequences of variables,

$$P(x,z)P(y,z) = \sum_{\lambda\in\mathbb{Y}} s_\lambda(x,y)\, s_\lambda(z) = \sum_{\lambda,\mu\in\mathbb{Y}} s_{\lambda/\mu}(x)\, s_\mu(y)\, s_\lambda(z)$$

$$= \sum_{\nu,\lambda,\mu\in\mathbb{Y}} c^{\lambda}_{\mu\nu}\, s_\lambda(z)\cdot s_\nu(x)\, s_\mu(y).$$

$$= \sum_{\mu,\nu\in\mathbb{Y}} s_\mu(y)\, s_\nu(x)\cdot \boxed{\sum_{\lambda\in\mathbb{Y}} c^{\lambda}_{\mu\nu}\, s_\lambda(z).}$$

$$= \sum_{\mu,\nu\in\mathbb{Y}} s_\mu(y)\, s_\mu(z)\cdot s_\nu(x)\, s_\nu(z)$$

$$= \sum_{\mu,\nu\in\mathbb{Y}} s_\mu(y)\, s_\nu(x)\cdot \boxed{\sum_{\lambda\in\mathbb{Y}} d^{\lambda}_{\mu\nu}\, s_\lambda(z).}$$

$\forall x, y, z \Rightarrow d^{\lambda}_{\mu\nu} = c^{\lambda}_{\mu\nu}$. #

Proof of the skew Cauchy identity:

let $P(x;y)$ as before.

$$\Rightarrow P = P(x;y)P(x;u)P(z;y)P(z;u)$$

$$= \sum_{\rho\in\mathbb{Y}} s_\rho(x,z)\, s_\rho(y,u) = \sum_{\rho,\lambda,\mu\in\mathbb{Y}} s_{\rho/\lambda}(x)\, s_\lambda(z)\, s_{\rho/\mu}(y)\, s_\mu(u)$$

$$= P(x;y)\cdot \sum_{\sigma,\nu,\tau\in\mathbb{Y}} s_\sigma(x)\, s_\sigma(y)\, s_\nu(y)\, s_\nu(z)\, s_\tau(z)\, s_\tau(u)$$

$$\underset{\substack{\Uparrow\\ \text{Lemma}}}{=} P(x;y) \sum_{\sigma,\nu,\tau\in\mathbb{Y}} \sum_{\lambda,\mu\in\mathbb{Y}} c^{\mu}_{\sigma\tau}\, s_\mu(u)\, c^{\lambda}_{\nu\tau}\, s_\lambda(z)\, s_\sigma(x)\, s_\nu(y)$$

$$= P(x;y)\cdot \sum_{\tau,\lambda,\mu\in\mathbb{Y}} s_{\mu/\tau}(x)\, s_{\lambda/\tau}(y)\, s_\lambda(z)\, s_\mu(u)$$

$$\forall z,u \Rightarrow \Big(\sum_{\tau\in\mathbb{Y}} s_{\mu/\tau}(x)\, s_{\lambda/\tau}(y)\Big)\cdot P(x;y) = \sum_{\rho\in\mathbb{Y}} s_{\rho/\lambda}(x)\, s_{\rho/\mu}(y).$$

#

First special case:

Consider the case $x=(x_1,x_2,\dots)=(\alpha,0,0,\dots)$

From the corollary (b) of $s_{\lambda/\mu} = \det\left[h_{\lambda_i-\mu_j-i+j}\right]_{1\le i,j\le \max\{\ell(\lambda),\ell(\mu)\}}$.

$$s_{\lambda/\mu}(x) = \prod_{i\geq 1} s_{\theta_i}(x) \quad \text{with } \theta_i=\lambda_i-\mu_i.$$

~~Lemma~~

Prop.: For $x=(\alpha,0,0,\dots)$, $s_\lambda(x)=0$ unless λ is a one-row Young diagram (i.e., $\ell(\lambda)=1$), and, more generally, $s_{\lambda/\mu}(x)=0$ unless λ/μ is a horizontal strip. In the latter case,

$$s_{\lambda/\mu}(x) = \alpha^{|\lambda|-|\mu|}.$$

Proof.: We have $S_{\lambda/\mu}((\alpha,0,\dots)) = \prod_{i\geq 1} S_{\theta_i}((\alpha,0,\dots))$.

. If θ_i is not a one-row Young diagram, then for example, $S_{\theta}((\alpha,0,\dots)) = \det \begin{bmatrix} h_{\theta(1)} & h_{\theta(2)-1} & \dots \\ h_{\theta(1)+1} & h_{\theta(2)} & \\ \vdots & & \ddots & h_{\theta(n)} \end{bmatrix}$

$\theta = (\theta_1,\dots,\theta_n)$

Now, $h_k((\alpha,0,\dots)) = \alpha^k$

$$\Longrightarrow = \det \begin{bmatrix} \alpha^{\theta(1)} & \alpha^{\theta(2)-1} & \dots \\ \alpha^{\theta(2)+1} & \alpha^{\theta(2)} & \\ \vdots & \vdots & \ddots \\ & & & \alpha^{\theta(n)} \end{bmatrix} \overset{n\geq 2}{=} 0.$$

#

. Now consider the PNG model at time $t=n$. In terms of partitions we have:

$$\Rightarrow \phi \xrightarrow{a_1} \lambda^{(1)} \xrightarrow{b_n} \mu^{(1)} \xrightarrow{a_2} \lambda^{(2)} \xrightarrow{b_{n-1}} \mu^{(2)} \xrightarrow{a_3} \lambda^{(3)} \dots \xrightarrow{a_n} \lambda^{(n)} \xrightarrow{b_n} \phi.$$

. ~~In particular~~ <u>Normalisation constant:</u>

$$\Rightarrow Z(\vec{a};\vec{b}) := \sum_{\substack{\lambda^{(1)},\dots,\lambda^{(n)} \\ \mu^{(1)},\dots,\mu^{(n-1)}}} S_{\lambda^{(1)}/\phi}(a_1)\, S_{\lambda^{(1)}/\mu^{(1)}}(b_n)\, S_{\lambda^{(2)}/\mu^{(1)}}(a_2) \dots\dots S_{\lambda^{(n)}/\mu^{(n-1)}}(a_n)\, S_{\lambda^{(n)}/\phi}(b_n)$$

Skew-Cauchy identity on $\sum_{\lambda^{(1)}}$

$$\Longrightarrow Z = \sum_{\substack{\lambda^{(2)},\dots,\lambda^{(n)} \\ \mu^{(1)},\dots,\mu^{(n-1)}}} H(a_1;b_n) \sum_{\nu} \underbrace{S_{\phi/\nu}(b_n)}_{=\delta_{\nu,\phi}} S_{\mu^{(1)}/\nu}(a_1) \cdot S_{\lambda^{(2)}/\mu^{(1)}}(a_2) \dots S_{\lambda^{(n)}/\phi}(b_n)$$

$$= H(a_1;b_n) \sum_{\substack{\lambda^{(2)},\dots,\lambda^{(n)} \\ \mu^{(1)},\dots,\mu^{(n-1)}}} S_{\mu^{(1)}/\phi}(a_1)\, S_{\lambda^{(2)}/\mu^{(1)}}(a_2) \dots$$

where $\boxed{H(\vec{a};\vec{b}) := \prod_{i,j\geq 1} \frac{1}{1-a_i b_j}}$

$$\overset{\text{Def } s_{\lambda^{(2)}/\mu^{(1)}}}{=} H(a_1;b_n) \sum_{\substack{\lambda^{(2)},\dots,\lambda^{(n)} \\ \mu^{(2)},\dots,\mu^{(n-1)}}} \underline{S_{\lambda^{(2)}}(a_1,a_2)\, S_{\lambda^{(2)}/\mu^{(2)}}(b_{n-1})}\, S_{\lambda^{(3)}/\mu^{(2)}}(a_3)\dots S_{\lambda^{(n)}}(b_n)$$

$$\overset{\sum_{\lambda^{(2)}}\ \text{I}}{=} H(a_1;b_n) \cdot \sum_{\substack{\lambda^{(3)},\dots,\lambda^{(n)} \\ \mu^{(2)},\dots,\mu^{(n-1)}}} H(a_1,a_2;b_{n-1}) \sum_{\nu} \overbrace{S_{\phi/\nu}(b_{n-1})}^{=\delta_{\nu,\phi}}\, S_{\mu^{(2)}/\nu}(a_1,a_2)\, S_{\lambda^{(3)}/\mu^{(2)}}(a_3)\dots S_{\lambda^{(n)}}(b_n)$$

$$\overset{\sum_{\mu^{(2)}}}{=} H(a_1;b_n) \underbrace{H(a_1,a_2;b_{n-1})}_{\substack{=H(a_1;b_{n-1})\cdot \\ \cdot H(a_2;b_{n-1})}} \sum_{\substack{\lambda^{(3)},\dots,\lambda^{(n)} \\ \mu^{(3)},\dots,\mu^{(n-1)}}} S_{\lambda^{(3)}}(a_1,a_2,a_3)\dots S_{\lambda^{(n)}}(b_n)$$

$$\overset{\text{iterate}}{=} \prod_{1\le i<j\le n} H(a_i;b_j).$$

~~We have shown~~

~~Prop: The~~ Next, consider the measure of the PNG multilayer at ~~time~~ time $n = 2m+1$ position 0

$\Rightarrow$ This is given by:

$$\frac{\displaystyle\sum_{\substack{\lambda^{(1)},\dots,\lambda^{(m)} \\ \lambda^{(m+2)},\dots,\lambda^{(n)} \\ \mu^{(1)},\dots,\mu^{(n)}}} S_{\lambda^{(1)}}(a_1)\, S_{\lambda^{(1)}/\mu^{(1)}}(b_n)\dots S_{\lambda/\mu^{(m)}}(a_{m+1}) \cdot S_{\lambda/\mu^{(m+1)}}(b_{m+1})\dots S_{\lambda^{(n)}/\mu^{(n-1)}}(a_n)\cdot S_{\lambda^{(n)}}(b_1)}{Z(\vec{a};\vec{b})}$$

By similar computations as above, one obtains:

Prop: The measure at $x=0$ of the PNG multilayer at time $n=2m+1$ is given by:

$$\frac{S_\lambda(a_1,a_2,\dots,a_{m+1})\cdot S_\lambda(b_1,b_2,\dots,b_{m+1})}{\prod_{i,j=1}^{m+1} \frac{1}{1-a_ib_j}}$$

The case considered here is just one example of "specialization".

<u>Definition:</u> Any algebra homomorphism

$$\rho: \Lambda \to \mathbb{C}$$
$$f \mapsto f(\rho)$$

is called a <u>specialization</u>, i.e., it has to satisfy:

$$(f+g)(\rho) = f(\rho) + g(\rho) \qquad (a)$$

$$(fg)(\rho) = f(\rho)\, g(\rho) \qquad (b)$$

$$(\theta f)(\rho) = \theta \cdot f(\rho), \quad \forall \theta \in \mathbb{C}. \qquad (c)$$

<u>Example:</u> Let $u_1, u_2, \ldots$ a sequence of complex numbers s.t. $\sum_{i \geq 1} |u_i| < \infty$.

Then the <u>substitution map</u> $\Lambda \to \mathbb{C}$, $x_i \mapsto u_i$

is a specialization.

. Any specialization is uniquely determined by its values on any sets of generators of Λ.

. If they are algebraic indep. $\Rightarrow$ the values can be any numbe

In other words, defining ρ is equivalent to setting the numbers $\{p_1(\rho), p_2(\rho), \ldots\}$

or $\{h_1(\rho), h_2(\rho), \ldots\}$

Example: For $\rho \equiv$ substitution by $(u_1, u_2, \dots)$ (

$\Rightarrow$ $p_k \mapsto p_k(\rho) = \sum_{i \geq 1} (u_i)^k$. (well-defined since $\sum_{i\geq 1} |u_i| < \infty$).

- As in the PNG model considered before, to have a probability model, we need to know if $\forall \lambda \in \mathbb{Y}$,

$$s_\lambda(\rho) \geq 0.$$

These are called <u>Schur positive specializations</u>.

- There is a characterization of these (for the proba. models it will be automatically the case).

<u>**Thm.**</u> The Schur positive specializations are parameterized by two sequences of non-negative real numbers
$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0)$
s.t. $\sum_{i \geq 1} (\alpha_i + \beta_i) < \infty$, and a parameter $\gamma \geq 0$.

- The specialization with parameters $(\alpha; \beta; \gamma)$ are given by its values on power sums:

$$p_1 \mapsto p_1(\alpha;\beta;\gamma) := \gamma + \sum_{i\geq 1} (\alpha_i + \beta_i)$$

$$p_k \mapsto p_k(\alpha;\beta;\gamma) := \sum_{i \geq 1} (\alpha_i^k + (-1)^{k-1} \beta_i^k)$$
for $k \geq 2$

- Equivalently: $\sum_{k=0}^{\infty} h_k(\alpha;\beta;\gamma) z^k = e^{\gamma z} \prod_{i \geq 1} \frac{1+\beta_i z}{1-\alpha_i z}.$

. Let us verify first the equivalence

$$.\ H(z) = \sum_{k=0}^{\infty} h_k z^k = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} p_k\right)$$

$$= \exp\left(z p_1 + \sum_{k=2}^{\infty} \frac{z^k}{k} p_k\right)$$

Specialization $(\alpha;\beta;\gamma)$

$$= \exp\left(\gamma z + \sum_{k\geq 1} \frac{z^k}{k} \cdot \sum_{i\geq 2} (\alpha_i^k + (-1)^{k-1} \beta_i^k)\right)$$

$$= \exp\left(\gamma z + \sum_{i\geq 2} \left[\underbrace{\sum_{k\geq 1} \frac{z^k}{k} \alpha_i^k}_{= -\ln(1 - z\alpha_i)} - \underbrace{\sum_{k\geq 1} \frac{z^k(-\beta_i)^k}{k}}_{= -\ln(1+z\beta_i)}\right]\right)$$

$$= e^{\gamma z} \prod_{i\geq 1} \frac{1+\beta_i z}{1-\alpha_i z}.$$

Remark: The case of γ can be seen as limit of β's; since $\lim_{M\to\infty} \left(1+\frac{\gamma z}{M}\right)^M = e^{\gamma z}$.

. The special case considered for the PNG, was $H(z) = \frac{1}{1-\alpha z}$

. In that case, let $\alpha = (a, 0, 0 \ldots)$, $\beta = 0$, $\gamma = 0$

$$\Rightarrow S_{\lambda/\mu}(\alpha;\beta;\gamma) = \begin{cases} a^{|\lambda|-|\mu|}, & \text{if } \lambda/\mu = \text{horiz-strip}, \\ 0, & \text{otherwise}. \end{cases}$$

. The second case is the following:

Prop.: Let $\alpha = 0$, $\beta = (b, 0, 0, \ldots)$, $\gamma = 0$.

$\Rightarrow S_\lambda(\alpha;\beta;\gamma) = 0$ unless λ is a one-column diagram

Further, $S_{\lambda/\mu}(\alpha;\beta;\gamma) = 0$ unless λ/μ is a vertical strip. In that case, $S_{\lambda/\mu}(\alpha;\beta;\gamma) = b^{|\lambda|-|\mu|}$.

Proof.: $H(z) = \sum_{k=0}^{\infty} h_k z^k = 1 + h_1 z + h_2 z^2 + \dots$

$$= 1+\beta z \Rightarrow h_1(\beta) = \beta,\ h_k(\beta) = 0,\ \forall k \geq 2.$$

$$\Rightarrow S_{\lambda/\mu}(\beta) = \det \left[h_{\lambda_i - \mu_j - i + j}(\beta) \right]_{i,j = 1,\dots,\max(\ell(\lambda),\ell(\mu))}$$

let $\theta_i = \lambda_i - \mu_i,\ i \geq 1$

$$= \det \begin{bmatrix} h_{\lambda_1-\mu_1}(\beta) & h_{\lambda_1-\mu_2+1}(\beta) & \cdots \\ h_{\lambda_2-\mu_1-1}(\beta) & h_{\lambda_2-\mu_2}(\beta) & \\ & & \end{bmatrix}$$

If $\lambda_i - \mu_i \geq 2$ for some i, then $\lambda_j - \mu_i - i + j \geq 2,\ \forall j \leq i$

and also $\lambda_i - \mu_j - i + j \geq 2,\ \forall j \geq i$.

This, together with $h_k(\beta) = 0$, for $k \geq 2$, implies that

$$S_{\lambda/\mu}(\beta) = \begin{array}{c|c} & i \\ \hline & \begin{bmatrix} h_{\lambda_1-\mu_1}(\beta) & \cdots & 0 & & \\ \vdots & & 0 & & 0 \\ & & 0 & & \\ & \cdots & 0 & 0 \ 0 \ 0 \ 0 & \\ & & & & \end{bmatrix} \end{array} = 0$$

with row i the row of zeros.

$\Rightarrow \lambda_i - \mu_i \in \{0,1\},\ \forall i$. If $\lambda_i - \mu_i = 1 \Rightarrow$ the column above (i,i) has $h_k(\beta)$ for $k \geq 2 \Rightarrow$ zero. & the same to its right.

If $\lambda_i - \mu_i = 0 \Rightarrow$ the row to the left of (i,i) has $h_k(\beta)$ for $k \leq 0$ ~~$\Rightarrow$ zero~~.

but also the column below (i,i) is zero.

$$\Rightarrow S_{\lambda/\mu}(\rho) = \begin{bmatrix} h_{\lambda_1-\mu_1}(\rho) & * & & 0 \\ * & h_{\lambda_2-\mu_2}(\rho) & * & \\ & * & \ddots & * \\ 0 & & * & h_{\lambda_n-\mu_n}(\rho) \end{bmatrix}$$

$\Rightarrow h_{\lambda_i-\mu_i}(\rho)$ has zeros either on its right or below it, $\forall i$

$$\Rightarrow S_{\lambda/\mu}(\rho) = h_{\lambda_1-\mu_1}(\rho)\cdot h_{\lambda_2-\mu_2}(\rho)\cdots h_{\lambda_n-\mu_n}(\rho)$$

$$= \prod_{i=1}^{n} b^{\lambda_i-\mu_i} = b^{|\lambda|-|\mu|},$$ provided of course that λ/μ is a ~~[illegible]~~ vertical strip. #

- Finally we want to see how a composition of two Schur positive specializations.

- <u>Def.</u>: Given two specializations ρ_1, ρ_2, their <u>union</u> (ρ_1, ρ_2) is defined through its values on power sums p_k:

$$p_k(\rho_1, \rho_2) := p_k(\rho_1) + p_k(\rho_2).$$

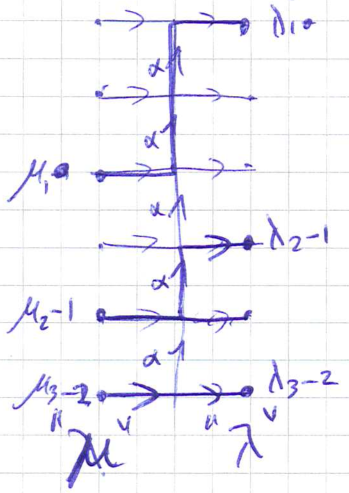
<u>Prop.</u>: If ρ_1 is a Schur-positive spec. with parameters $(\alpha^{(1)}; \beta^{(1)}; \gamma^{(1)})$ and ρ_2 one with parameters $(\alpha^{(2)}, \beta^{(2)}; \gamma^{(2)})$, then (ρ_1, ρ_2) is a Schur-positive spec. with parameter ~~([illegible]~~ $(\alpha^{(1)} \cup \alpha^{(2)}; \beta^{(1)} \cup \beta^{(2)}; \gamma^{(1)} + \gamma^{(2)})$.

Proof: It is a direct consequence of the Thm on Schur-positive spec. #

Rem.: In particular, if ρ_1 & ρ_2 specialize symm. fcts by substituting sets of variables $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ (which is the case when $\beta_i = 0$, $\gamma = 0$), then (ρ_1, ρ_2) substitutes all the variables $(x_1, x_2, \dots, y_1, y_2, \dots)$.

Relation with transition weights in line ensembles.

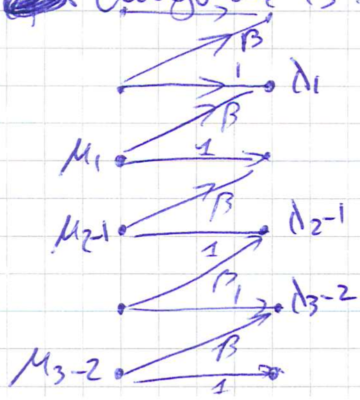
ⓐ We already know that an α-specialization corresponds to the following LGV picture:



$S_{\lambda/\mu}((\alpha, 0, \dots); 0; 0)$

↳ For one particle: Geometric jumps

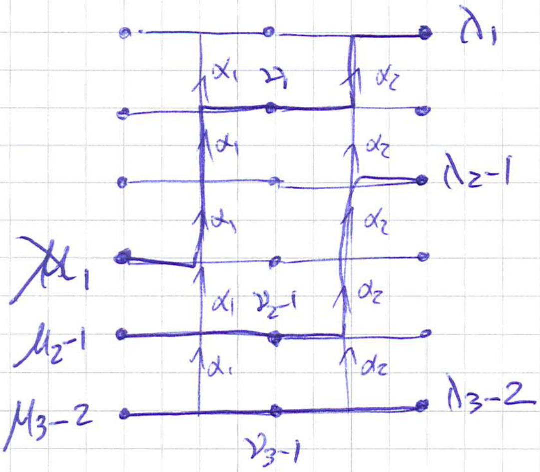
ⓑ What about β-specialization? The corresponding LGV diagram is:



$S_{\lambda/\mu}(0; (\beta, 0, \dots); 0)$.

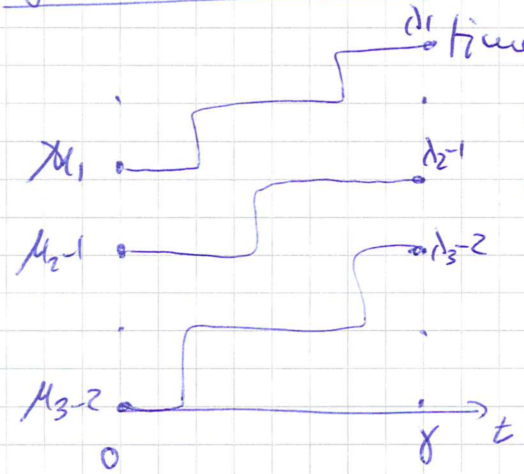
↳ One-sided "random walk" for one particle

(c) Compositions: $\rho=((\alpha_1,\alpha_2,0,\dots);0;0)$.



$$S_{\lambda/\mu}((\alpha_1,\alpha_2,0,\dots);0;0) = \sum_{\nu\in\mathbb{Y}} S_{\lambda/\nu}((\alpha_2,0,\dots);0;0)\cdot S_{\nu/\mu}((\alpha_1,0,\dots);0;0)$$

(d) γ-specialization. It can be seen as "continuous time limit" of β-specialization.



Let the one-particles do one-sided simple random walks with rate 1.

$\Rightarrow$ $S_{\lambda/\mu}(0;0;\gamma)$ gives the transition weights for non-$\cap$ onesided r.w. during time $[0,\gamma]$.

- A final combinatorial ~~~~ formulation of S_λ for γ-spec. (we might come back later to this model).

Prop.: Let $\alpha\equiv 0$, $\beta\equiv 0$ and $\gamma>0$. Then,

$$S_\lambda(\alpha;\beta;\gamma)=\frac{\gamma^{|\lambda|}}{|\lambda|!}\cdot \dim(\lambda), \quad \lambda\in\mathbb{Y},$$

where $\dim(\lambda)=\#$ standard Young tableaux of shape λ, i.e., # of ways of putting numbers $\{1,2,\dots,N=|\lambda|\}$ inside the Young diagram λ s.t. the numbers strictly increases both along columns & rows.