Advanced topics in Applied Probability:

(Introduction to integrable probability).

• Discuss about possible extra time.

1) Introduction.

1.1) Universality

• We will describe in this lecture probabilistic models, which can be analyzed by using some "integrable structure" and uses algebraic methods.

• In particular, we will be interested in statements about the systems investigated in the large size / time limits.

• It is also of interest to distinguish the model-dependent quantities and the universal ones as one writes the limit laws.

CLT: • Just to mention the simplest model of universality, let $X_1, X_2, \ldots$ a sequence of i.i.d. random variables with $\mathbb{E}(X_i)=\mu$, $Var(X_i)=\sigma^2<\infty$.

• Then from Probability I we know that

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} X_i}{N} = \mu \quad \text{a.s.} \qquad (\text{L.L.N.})$$

and
$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} X_i - \mu N}{\sqrt{N\sigma^2}} \overset{(D)}{=} \mathcal{N}(0,1) \qquad (\text{C.L.T.})$$

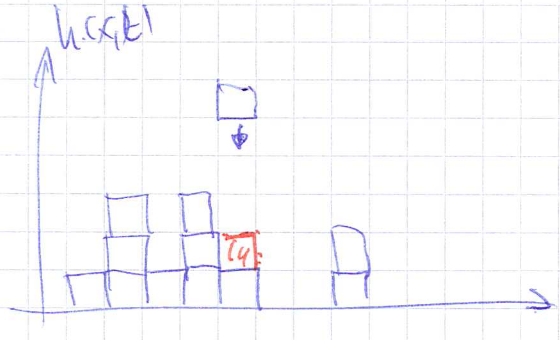
- In this case, the model-dependent quantities are "μ", "σ^2", while the universal quantities are:
  - $\rightarrow$ the normal distribution $\mathcal{N}(0,1)$
  - $\rightarrow$ the $N^{1/2}$ scaling in the fluctuations of $\sum_{i=1}^{N} x_i$.

- In this lecture we are going to consider **models in the so-called Kardar-Parisi-Zhang (KPZ) universality class.**

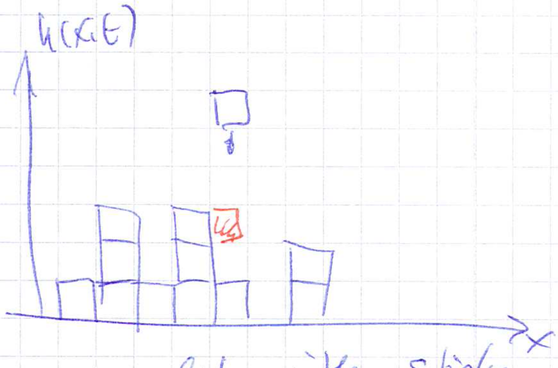
- As we shall see, ~~we are going~~ the models ~~have some~~ can (mostly) be described in terms of iid random variables, but the random variables under investigation are not independent, but strongly correlated.

- As a consequence, ~~both~~ the scaling of the fluctuations and the limiting distributions differs from the CLT case.

- **An example:** Growth models based on falling boxes



- model without interaction



- model with sticky boxes (ballistic deposition).

- In the model without interaction, clearly for any x, $h(x,t)$ will have Gaussian fluctuations in the $\sqrt{t}$ scale.

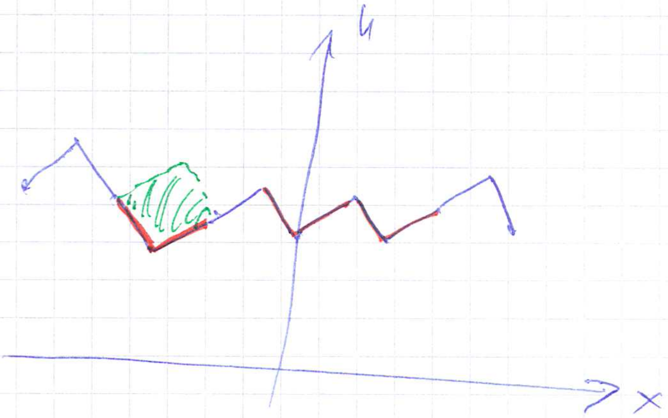
- This changes radically for the ballistic deposition model. Simulations shows that

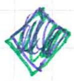
$$\sqrt{\mathrm{Var}(h(x,t))} \simeq t^{1/3}$$

with non-Gaussian fluctuations.

- This model is too difficult for analytic treatment, but another similar model can be fully analyzed:

- <u>Model:</u> - <u>Configurations:</u> $x \mapsto h(x)$ with $h(x+1) - h(x) \in \{\pm 1\}$, $\forall x \in \mathbb{Z}$

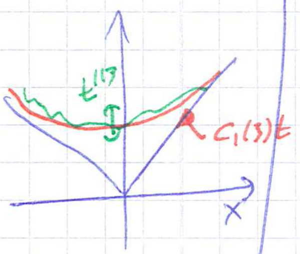


- <u>Dynamics:</u> ~~At~~ each "valley" (local minimum), one adds <u>independently</u> with <u>rate 1</u> a block , i.e., the local minimum turns into a local maximum.

- <u>Results:</u> let $h(x,t)$ denote the height function at position x and time t.

Thm 1) (Johansson '00) **Wedge initial condition**

Let $h(x,0)=|x|$, $x\in\mathbb{Z}$. Then, $\forall \xi\in(-1,1)$,



$$\lim_{t\to\infty}\mathbb{P}\left(\frac{h(\xi t,t)-c_1(\xi)t}{-c_2(\xi)\,t^{1/3}}\leq s\right)$$

$$=F_2(s)$$

for some $c_1(\xi)$, $c_2(\xi)>0$ explicit functions of ξ.

. F_2 is called the GUE Tracy-Widom distribution function.

. Interestingly, the limiting distribution **depends** on the **initial condition**. For instance:

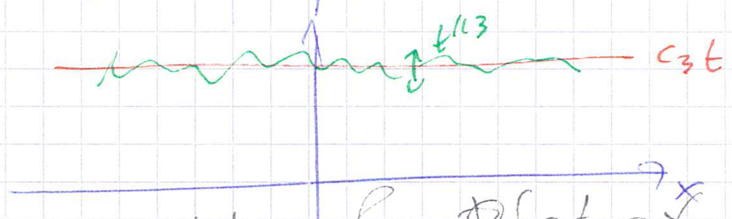
Thm 2) (Sasamoto '05; Borodin, Ferrari, Prähofer, Sasamoto '07)

Flat initial condition: let $h(x,0)=\begin{cases}0, & \text{even } x\\ 1, & \text{odd } x\end{cases}$

Then, $\forall \xi\in\mathbb{R}$, $\lim_{t\to\infty}\mathbb{P}\left(\frac{h(\xi t,t)-c_3 t}{-c_4\,t^{1/3}}\leq s\right)=F_1(s)$

for some $c_3, c_4>0$ explicit constants.

. F_1 is called the GOE Tracy-Widom distribution function.



. Show simulation & plot of the distr.

- We will come back to these distributions during the lecture. I just mention that they were first observed (derived) in random matrix theory.
- In short, they describe the limiting distribution of the largest eigenvalue of random matrices H with the probability measures having densities $\sim \exp(-\mathrm{Tr}(H^2))$; one for real symmetric matrices (GOE), the other for complex hermitian matrices (GUE).

  → Sketch of $F_1(s)$ & $F_2(s)$. Also, decay properties.
- It is conjectured that such theorems should hold for a broad class of models, the ones in the KPZ universality class.

1.2) KPZ class:

- Models in the KPZ class of stochastic growth models in 1 dimension are described by a height function

$$x \longmapsto h(x,t), \quad \begin{cases} t = \text{time} \subseteq \mathbb{R} \\ x = \text{space} \subseteq \mathbb{R} \end{cases}$$

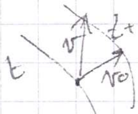
- The dynamics is: random, local and has to be such that there is a non-random limit shape:

$$h_{ma}(\xi) := \lim_{t\to\infty} \frac{h(\xi t, t)}{t}.$$

- Further, let $v = v(\nabla h)$ the macroscopic speed of growth as a function of the slope $u = \nabla h$, Then, the model is in the KPZ class if

$v''(u) \neq 0$.

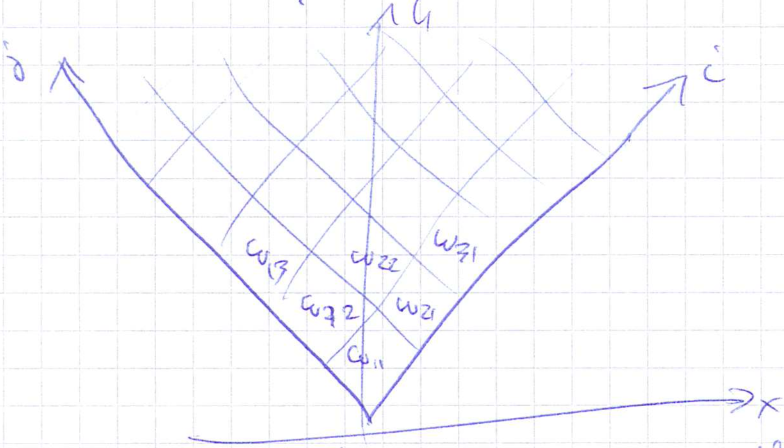
→ Example of isotropic growth:



$v(\nabla h) = v_0 \sqrt{1+(\nabla h)^2}$

1.21) Directed polymers / Last passage percolation

. Consider the wedge initial condition. Then we can reformulate the model as follows.



. Let ω_{ij} the "waiting time" for the local valley at (i,j) to become a local mountain.

. Then, for the model we consider,

ω_{ij} are iid $\exp(1)$ random variables.

One has: $T(i,j) = \max_{\pi:(1,1)\to(i,j)} \overbrace{\sum_{(u,v)\in\pi} \omega_{(u,v)}}^{=H(\pi)}$

where π are all paths made by consecutive steps of $(1,0)$ or $(0,1)$.

What is $T(i,j)$ is the time a given box (i,j) is absorbed by the growing interface?

Note: ⓐ $\omega_{ij} \sim \exp(1)$ is the limit of

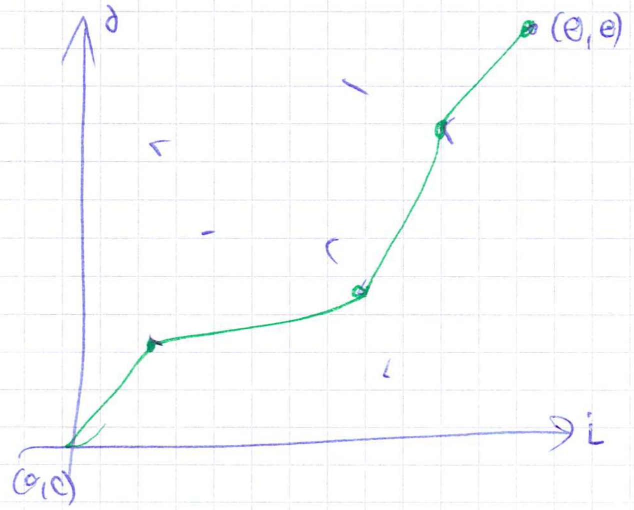
$\omega_{ij} \sim \mathrm{geom}(q)$, i.e., $\mathbb{P}(\omega_{ij} = k) = (1-q)q^k$, $k \geq 0$,

as $q \to 1$.

ⓑ The other limit is when $q \to 0$. In that situation, to see a number of $\omega_{ij} \neq 0$ of order 1, we need to look at coordinates $(i,j) \sim \frac{1}{\sqrt{q}}$.

- In this case, the model degenerates to the following:

- let us consider a (homogeneous) Poisson point process with density 1 in $\mathbb{R}_+$



let $L(\theta) =$ max # points along a path from $(0,0)$ to (θ,θ) with increasing i & j directions

Then: Theorem (Baik, Deift, Johansson '99):

$$\lim_{\theta \to \infty} \mathbb{P}\left(\frac{L(\theta) - 2\theta}{\theta^{1/3}} \leq s \right) = F_2(s)$$

this is equivalent to:

Thm (BDJ '99): let σ be a uniformly distributed permutation of $\{1, \ldots, n\}$ and $\ell_n(\sigma)$ = length of the longest (increasing) subsequence of σ.

$$\Rightarrow \lim_{n \to \infty} \mathbb{P}\left(\frac{\ell_n - 2\sqrt{n}}{n^{1/6}} \leq s \right) = F_2(s).$$

- The models mentioned so far will be solved by putting them in the framework of "Schur Processes".

- However, there are variations of them which have still an integrable structure.

Two examples:

(ASEP) (a) Interface  with dynamics changed as follows:

$$\begin{cases} \cdot\ \vee \to \wedge & \text{rate } 1 \\ \cdot\ \wedge \to \vee & \text{rate } p \in (0,1) \end{cases}$$

(For $p=1$, one would have $N^q(n)=0$).

(b) Positive temperature polymers:

- Instead of maximising the "energy" $H(\pi)$, one can consider the Gibbs ensemble where the probability of having a configuration π is proportional to $\exp(\beta H(\pi))$ with β called inverse-temperature.

- Then, instead of looking at $\max_\pi H(\pi)$ one looks at the free energy

$$\frac{1}{\beta} \ln Z_\beta(n,n)$$

with $Z_\beta^{(n,n)} = \sum_{\pi:(1,1)\to(n,n)} \exp(\beta H(\pi))$.

- For a special distribution of w_{ij}'s, this model is solvable and one has:

Thm: (Borodin–Corwin–Remenik)

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Z(n,n) - c_1 n}{c_2 n^{1/3}} \le s \right) = F_2(s)$$

for some $c_1, c_2 > 0$ and under the assumption w_{ij} have density $\frac{1}{\Gamma(\theta)} x^{-\theta-1} \exp\left(-\frac{1}{x}\right)$. ($\theta > 0$ is a param

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1.2) Plan: (a) Consider the model with wedge initial condition and $w_{ij} \sim \text{geom}(a_i \cdot b_j)$

→ describe this model and make the connection with partitions (which is a Markov chain are natural for the permutation model).

. This gives us an example of Schur processes.

(b) We will study Schur processes in a ~~more~~ general framework

and see/mention other applications which are not ~~necessarily~~ growth models (e.g. random tiling).

(c) We will discuss how to get some of the already mentioned results.

(d) We will see what can be done for some models beyond Schur processes.