Lecture notes of the course

V3F2/F4F1 - Foundations in Stochastic Analysis

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Contents

0 Topics for the oral examination

1. Stopping time, optional sampling
2. Semimartingales, quadratic variation
3. Construction of the Itô integral, Itô-Isometry
4. Itô-Formula
5. Exponential local martingales, Levy char.
6. Strong solutions of SDE
7. Time change, Dubins-Schwarz Theorem
8. Change of measure, Girsanov Theorem

Important.

1 Introduction to Stochastical Analysis

Plan:

(a) Brownian Motion: the fil rouge of the lecture

(b) Filtration & Martingales in continuous time

(c) Continuous semimartingales

(d) Stochastic Integrals and the Itô Formula

(e) Stochastic Differential Equations (SDE)

(f) Brownian Martingale

Examples

1. Population Dynamics
   Let S_t the size of a population at time t (if $S_t >> 1$: a continuous approximation is ok) and let R_t the growth rate at time t
   $$\frac{dS_t}{dt} = R_t S_t \tag{1.1}$$
   If $R_t = \bar{R}$, where $\bar{R}$ is a constant, then $S_t = S_0 e^{\bar{R}t}$. If R_t is random, e.g.
   $$R_t = \underbrace{\bar{R}}_{average} + \underbrace{N_t}_{noiseterm} . \tag{1.2}$$
   Question: What is the law of S_t? What is a good choice for N_t?

2. Langevin Equation
   $$m\frac{dv_t}{dt} = -\underbrace{\eta}_{viscosity} v_t + \underbrace{N_t}_{noiseterm} \tag{1.3}$$

3. Stocks
   If S_t = Stockprice at time t and evolves as
   $$\frac{dS_t}{dt} = (R + N_t)S_t \tag{1.4}$$
   and if $\tilde{R}$ is the bond rate let C_0 be the portfolio at time $t = 0$ made by A_0 stocks and B_0 bonds. $\Rightarrow C_t = A_t S_t + B_t e^{\tilde{R}t}$. For a self financing portfolio
   $$\Rightarrow dC_t = A_t dS_t + B_t d\left(e^{\tilde{R}t}\right) \tag{1.5}$$
   Question: How much is the fair price of an European Call Option?
   Answer: Black Scholes Formula
   But: 1.4 ist not necessarily satisfied by the market.

4. Dirichlet Problems
   Let f be an harmonic function on D (bounded and regular) and $f(x) = 0$ on ∂D.
$$\Rightarrow f(x) = E[f(B_t^x)] \tag{1.6}$$
   where $B_t^x = x + \int_0^t N_s ds$ and τ is the time t when B_t^x reaches ∂D.
   Goals:
   - Understand what is N_t & B_t
   - Work with them

1. Trial N_t should be the contiuous analogue of a sequence of iid random variables. We would like to have:

1. N_t should be independent of N_s for $s \neq t$.
2. $N_t, t \geq 0$ should all have the same distribution μ.
3. $E[N_t] = 0$.

$t \equiv$ time is in $\mathbb{R}$. Problem (if $N_t \neq 0$): Such an object is not well defined (e.g. N_t is not measurable (in t)).

2. Trial In examples (1), (2) & (4) we are actually interested in the integral of N_t. Denote by
$$B_t = \int_0^s N_s ds. \tag{1.7}$$
The 3 conditions become:

(BM1) *Independent increments* For $0 \leq t_0 < t_1 < \cdots < t_n$: the variables $B_{t_{k+1}} - B_{t_k}$, for $k = 0, ..., n-1$ are independent.

(BM2) B_t has *stationary increment*, i.e. the joint distribution of $(B_{t_1+s} - B_{u_1+s}, \dots, B_{t_n+s} - B_{u_n+s})$ for $u_k < t_k, k = 1, \dots, n$ is independent of $s > 0$.

(BM3) $E[B_t] = 0$

(BM4) And a normalization $Var[B_1] = E[B_1^2] = 1$.

But: (BM1)-(BM4) are not enough to determine the process B_t uniquely. Thus we add:

(BM5) $t \mapsto B_t$ is continuous (almost surely).

B_t is called the *Wiener Process* or *Brownian Motion*.

Lemma 1.1.
It holds:
$$\forall \varepsilon > 0 \lim_{n\to\infty} nP(|B_{t+\frac{1}{n}} - B_t| > \epsilon) = 0 \tag{1.8}$$

Proof. Let $H_n := \sup_{1\leq k\leq n} \left|B_{\frac{k}{n}} - B_{\frac{k-1}{n}}\right|$. By (BM5) H_n is almost surely continuous on $[0, 1]$.
$$\Rightarrow \forall \varepsilon > 0 \lim_{n\to\infty} P(H_n > \varepsilon) = 0 \tag{1.9}$$

But:

$$
\begin{aligned}
P(H_n > \varepsilon) &= 1 - P(H_n < \varepsilon) && (1.10)\\
&\overset{BM1}{=} 1 - \prod_{k=1}^{n} P(|B_{\frac{k}{n}} - B_{\frac{k-1}{n}}| \leq \varepsilon) && (1.11)\\
&\underset{B_0=0}{\overset{BM2}{=}} 1 - (P(|B_{\frac{1}{n}}| \leq \varepsilon))^n && (1.12)\\
&= 1 - (1 - P(|B_{\frac{1}{n}}| > \varepsilon))^n && (1.13)\\
&\geq 1 - \underbrace{e^{-nP(|B_{\frac{1}{n}}|>\varepsilon)}}_{\leq 1} && (1.14)
\end{aligned}
$$

because $1 - x \leq e^{-x}$. As we take $n \to \infty$ we get

$$\lim_{n\to\infty} nP(|B_{\frac{1}{n}}| > \varepsilon) = 0 \tag{1.15}$$

Using (BM2) we get the general result by seeing that

$$P(|B_{t+\frac{1}{n}} - B_t| > \varepsilon) = P(|B_{\frac{1}{n}}| > \varepsilon) \tag{1.16}$$

□

What is the distribution of B_t?

Lemma 1.2.
It holds:

$$\forall t, s \geq 0 : P(B_{t+s} - B_t \in A) = \frac{1}{\sqrt{2\pi s}} \int_A e^{\frac{-x^2}{2s}} dx \quad \forall A \in \mathcal{B}(\mathbb{R}) \tag{1.17}$$

Proof. Without loss of generality we can assume $t = 0$ (because of BM2). Define

$$B_s := \sum_{k=1}^{n} X_{n,k} \tag{1.18}$$

with $X_{n,k} = B_{\frac{sk}{n}} - B_{\frac{s(k-1)}{n}}$ are iid R.V. From BM3 it follows $E[X_{n,k}] = 0$ and from BM4 $Var[B_s] = s$. As we use the CLT we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} X_{n,k} \sim \mathcal{N}(0, s) \tag{1.19}$$

□

New condition:

$(\widetilde{BM2})$ $\forall s, t \geq 0 \forall A \in \mathcal{B}(\mathbb{R})$

$$P(B_{s+t} - B_s \in A) = \frac{1}{\sqrt{2\pi t}} \int_A e^{\frac{-x^2}{2t}} dx \tag{1.20}$$

and $B_0 = 0$.

Definition 1.3.
A one-dimensional (standard) Brownian-Motion (BM) is a real-valued process in continuous time satisfying (BM1), $(\widetilde{BM2})$, (BM5).

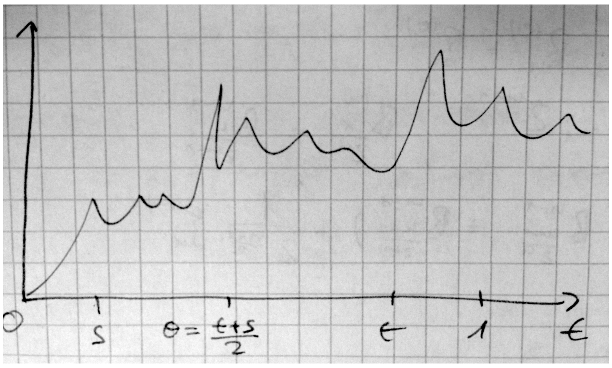
[09.10.2012]
[12.10.2012]

2 Brownian Motion

2.1 Construction of the Brownian Motion

Question: Is there an object satisfying Definition 1.3? We construct $\{B_t, t \in [0,T]\}$. WLOG $T = 1$, otherwise one has to multiply time variables by T and space variables by $\sqrt{T}$.

Remark: *Let's assume the Brownian Motion is constructed.*



Question: Given that $B_s = x$, $B_t = z$, what is the distribution of B_ϑ?
Answer: $B_\vartheta \sim \mathcal{N}(\mu = \frac{x+z}{2}, \sigma^2 = \frac{t-s}{4})$. Using BM1 ($B_s$, $B_\vartheta - B_s$ and $B_t - B_\vartheta$ are independent):

$$\mathbb{P}(B_s \in dx, B_\vartheta \in dy, B_t \in dz) = p(0,x,s)p\left(x,y,\frac{t-s}{2}\right)p\left(y,z,\frac{t-s}{2}\right)dxdydz \tag{2.1}$$

$$\underset{1}{=} p(0,x,s)p(x,z,t-s)\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dxdydz \tag{2.2}$$

with

$$p(x,y,\tau) := \frac{1}{\sqrt{2\pi\tau}}e^{-\frac{(x-y)^2}{2\tau}} \tag{2.3}$$

Also:

$$P(B_s \in dx, B_t \in dz) = p(0,x,s)p(x,z,t-s)dxdz \tag{2.4}$$

Which leads to

$$P(B_\vartheta \in dy | B_s = x, B_t = z) = \frac{P(B_\vartheta \in dy, B_s \in dx, B_t \in dz)}{P(B_s \in dx, B_t \in dz)} \tag{2.5}$$

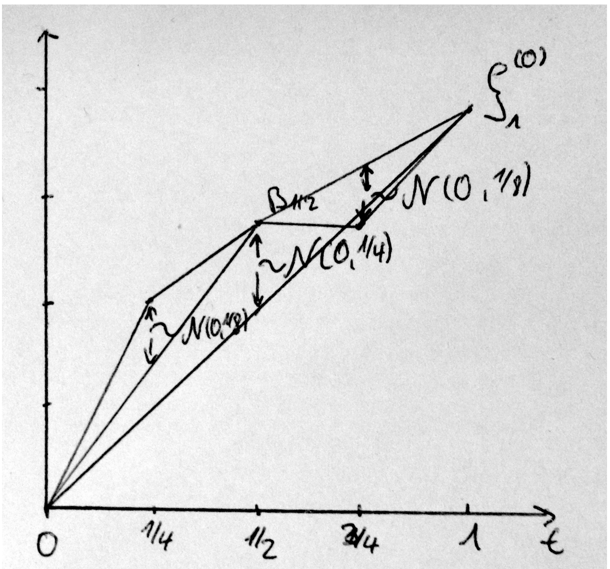
$$= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy \tag{2.6}$$

Construction: Let $\{\xi_k^{(n)}, k \in I(n), n \geq 1\}$ independent R.V.$\sim \mathcal{N}(0,1)$ where $I(n) = \{k \in \mathbb{N} : 1 \leq k \leq 2^n, k \text{ odd}\}$.

a) $B_0^{(n)} = 0, B_1^{(0)} = \xi_1^{(0)}$

b) For $k = 0, \ldots, 2^{n-1}$: $B^{(n)}_{\frac{k}{2^{n-1}}} := B^{(n-1)}_{\frac{k}{2^{n-1}}}$

[1] Algebra



c) $B^{(n)}_{\frac{k}{2^n}} = \frac{1}{2}\left(B^{(n-1)}_{\frac{k-1}{2^n}} + B^{(n-1)}_{\frac{k+1}{2^n}}\right) + \frac{1}{2^{\frac{n+1}{2}}}\xi_k^{(n)}$

Goal: Show that

$$B_t^{(n)} \overset{n\to\infty}{\longrightarrow} B_t \tag{2.7}$$

uniformly in t and that B_t is almost surely continuous. First we introduce

$$H_1^{(0)} = 1 \tag{2.8}$$

$$H_k^{(n)} = \begin{cases} 2^{\frac{k-1}{2}} & , \frac{k-1}{2^n} \le t < \frac{k}{2^n} \\ -2^{\frac{k-1}{2}} & , \frac{k}{2^n} \le t < \frac{k+1}{2^n} \\ 0 & , \text{otherwise.} \end{cases} \tag{2.9}$$

for $n \ge 1, k \in I(n)$. We set

$$S_k^n(t) = \int_0^t H_k^{(n)}(u)du \tag{2.10}$$

For $n = 0$:

$$B_t^{(0)}(\omega) = S_1^{(0)}(t)\xi_1^{(0)}(\omega) \tag{2.11}$$

For general n (e.g. by induction):

$$B_t^{(n)}(\omega) = \sum_{m=0}^{n} \sum_{n\in I(m)} S_k^{(m)}(t)\xi_k^{(m)}(\omega) \tag{2.12}$$

Lemma 2.1.
The sequence of functions

$$\left(\left\{B_t^{(n)}(\omega), 0 \le t \le 1\right\}\right)_{n\ge1} \tag{2.13}$$

converges uniformly to a continuous function $\{B_t(\omega), 0 \le t \le 1\}$ for almost every ω.

Proof. Let $b_n := \max_{k\in I(m)} \left|\xi_k^{(n)}\right|$. $\forall x > 0, k, n$ it holds

$$P\left(\left|\xi_k^{(n)}\right| > x\right) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-u^2}{2}} du \tag{2.14}$$

$$\leq \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} e^{\frac{-u^2}{2}} du \tag{2.15}$$

$$\underset{2}{=} \sqrt{\frac{2}{\pi}} \int_{\frac{x^2}{2}}^\infty \frac{\sqrt{2v}}{x} e^{-v} \sqrt{\frac{2}{v}} \frac{1}{2} dv \tag{2.16}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{x} e^{-\frac{x^2}{2}} \tag{2.17}$$

$$\Rightarrow P(b_n > n) = P\left(\bigcup_{k\in I(m)} \left\{|\xi_k^{(n)}| > n\right\}\right) \tag{2.18}$$

$$\leq \sum_{k\in I(n)} P\left(|\xi_k^{(n)}| > n\right) \tag{2.19}$$

$$\underset{3}{=} \sum_{k\in I(n)} P\left(|\xi_1^{(n)}| > n\right) \tag{2.20}$$

$$\leq \sqrt{\frac{2}{\pi}} e^{-\frac{u^2}{2}} \cdot \underbrace{2^n}_{|I(n)|\leq 2^n} \tag{2.21}$$

$\Rightarrow \sum_{n\geq 1} P(b_n > n) < \infty$. We can now use Borel-Cantelli I:

$$\exists \tilde{\Omega} \subset \Omega \text{ s.t. } P(\tilde{\Omega}) = 1 \text{ s.t. } \forall \omega \in \tilde{\Omega}\ \exists n_0(\omega) \text{ s.t. } \forall n \geq n_0(\omega)\ b_n(\omega) \leq n \tag{2.22}$$

$$\Rightarrow \sum_{n\geq n_0(\omega)} \sum_{k\in I(n)} \underbrace{S_k^{(n)}(t)}_{\leq \frac{1}{2^{\frac{n+1}{2}}}} \underbrace{|\xi_k^{(n)}(\omega)|}_{\leq n} \leq \sum_{n\geq n_0(\omega)} n \frac{1}{2^{\frac{n+1}{2}}} \tag{2.23}$$

because $\forall t$ at most one $k \in I(n)$ is s.t. $S_k^{(n)}(t) > 0$. Moreover, as $n_0 \to \infty$

$$\sum_{n\geq n_0(\omega)} \sum_{k\in I(n)} S_k^{(n)}(t)\, |\xi_k^{(n)}(\omega)| \to 0 \tag{2.24}$$

$\Rightarrow \forall \omega \in \tilde{\Omega}$ it holds: $B_t^{(n)}(\omega)$ converges uniformly in $t \in [0, 1]$ to a limit $B_t(\omega)$. Due to the uniform convergence $B_t(\omega)$ is continuous. □

Lemma 2.2.
The Haarfunctions $\left\{H_k^{(n)}, n \geq 0, k \in I(n)\right\}$ are a complete orthonormal system of $L^2([0, 1])$ with the scalarproduct

$$< f, g >= \int_0^1 f(x)g(x)dx \tag{2.25}$$

It holds the parseval equation

$$< f, g >= \sum_{n\geq 0} \sum_{k\in I(n)} < f, H_k^{(n)} >< H_k^{(n)}, g > \tag{2.26}$$

[2] $u \mapsto \sqrt{2v}$

Proof. See exercises. □

If we take $f = \mathbb{1}_{[0,t]}, g = \mathbb{1}_{[0,s]}$, (2.26) becomes

$$\min(s,t) = \underbrace{< f, g >}_{=\int_0^1 \mathbb{1}_{[0,s]}(x)\mathbb{1}_{[0,t]}(x)dx} = \sum_{n\geq 0}\sum_{k\in I(n)} S_k^{(n)}(t)S_k^{(n)}(s) \tag{2.27}$$

Lemma 2.3.
Let

$$B_t := \lim_{n\to\infty} B_t^{(n)}. \tag{2.28}$$

Then B_t is a Brownian Motion on $[0,1]$.

Proof. We have to show: $\forall 0 = t_0 < t_1 < \cdots < t_n \leq 1$ the R.V.$B_{t_j} - B_{t_{j-1}}, j = 1,\ldots,n$ are independent and $\sim \mathcal{N}(0, t_j - t_{j-1})$. We will show:

$$\underbrace{E\left[e^{-i\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)B_{t_j}}\right]}_{=E\left[e^{-i\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)B_{t_j}}\right]} = \prod_{j=1}^n e^{-\frac{1}{2}\lambda_j^2(t_j-t_{j-1})} \tag{2.29}$$

setting $\lambda_{n+1} = 0 = B_0$. Now let $M \in \mathbb{N}$.

$$E\left[\exp(-i\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)B_{t_j}^{(M)})\right] = E\left[\exp(-i\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)\cdot\sum_{m=0}^M\sum_{j\in I(m)} S_k^{(m)}(t_j)\xi_k^{(m)})\right] \tag{2.30}$$

$$\underset{4}{=} \prod_{m=0}^M\prod_{k\in I(m)} E\left[\exp(-i\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)S_j^{(m)}(t_j)\xi_k^{(m)})\right] = \Delta \tag{2.31}$$

We use $\xi \sim \mathcal{N}(0,1) \Rightarrow E\left[e^{-i\alpha\xi}\right] = e^{-\frac{1}{2}\alpha^2}$ and get

$$\Delta = \prod_{m=0}^M\prod_{k\in I(m)} \exp(-\frac{1}{2}(\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)S_k^{(m)}(t_j))^2) \tag{2.32}$$

$$= \exp[-\frac{1}{2}\sum_{m=0}^M\sum_{k\in I(m)}\sum_{j,l=1}^n(\lambda_{j+1}-\lambda_j)(\lambda_{l+1}-\lambda_l)S_k^{(m)}(t_j)S_k^{(m)}(t_l)] \tag{2.33}$$

$$= \exp\left[-\frac{1}{2}\sum_{j,l=1}^n(\lambda_{j+1}-\lambda_j)(\lambda_{l+1}-\lambda_l)\sum_{m=0}^M\sum_{k\in I(m)} S_k^{(m)}(t_j)S_k^{(m)}(t_l)\right] \tag{2.34}$$

if we reconsider (2.27) this becomes

$$\overset{M\to\infty}{\longrightarrow} \exp\left[-\frac{1}{2}\sum_{j,l=1}^n(\lambda_{j+1}-\lambda_j)(\lambda_{l+1}-\lambda_l)\min(t_j,t_l)\right] \tag{2.35}$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)^2 t_j - \sum_{j=1}^{n-1}\sum_{l=j+1}^n(\lambda_{j+1}-\lambda_j)(\lambda_{l+1}-\lambda_l)t_j\right] \tag{2.36}$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^n(\lambda_{j+1}-\lambda_j)^2 t_j - \sum_{j=1}^{n-1}(\lambda_{j+1}-\lambda_j)\sum_{l=j+1}^n(\lambda_{l+1}-\lambda_l)t_j\right] = \Delta \tag{2.37}$$

the last sum is a telescoping series (and $\lambda_{n+1} = 0$)

$$\Delta = \exp\left[-\frac{1}{2}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)^2 t_j + \sum_{j=1}^{n-1}(\lambda_{j+1}-\lambda_j)\lambda_{j+1}t_j\right] \tag{2.38}$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n} t_j(\lambda_{j+1}^2 - 2\lambda_{j+1}\lambda_j + \lambda_j^2 - 2\lambda_{j+1}^2 + 2\lambda_j\lambda_{j+1})\right] \tag{2.39}$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n} t_j\lambda_j^2\right]\cdot\exp\left[\frac{1}{2}\sum_{j=1}^{n} t_j\lambda_{j+1}^2\right] \tag{2.40}$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n} t_j\lambda_j^2\right]\cdot\exp\left[\frac{1}{2}\sum_{j=1}^{n} t_{j-1}\lambda_j^2\right] \tag{2.41}$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})\lambda_j^2\right] \tag{2.42}$$

□

[12.10.2012]
[16.10.2012]

2.2 Trajectories of Browian Motions

The BM has continuous trajectories, but they are very rough.

Theorem 2.4.
The trajectories

$$t \mapsto B_t \tag{2.43}$$

a) have an a.s. unbounded variation.

b) and so they are nowhere differentiable.

This theorem shows why the object “N_t“ is difficult to define.

Lemma 2.5.
Let $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T$ a family of partitions of $[0, T]$ s.t.

$$\lim_{n\to\infty}\max_{0\le j\le n-1}\left|t_{j+1}^{(n)} - t_j^{(n)}\right| = 0. \tag{2.44}$$

Then

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\left(B_{t_{j+1}}^{(n)} - B_{t_j}^{(n)}\right)^2 = T \text{ in } L^2. \tag{2.45}$$

[4]iid.

Proof. Define $\Delta B_j := B^{(n)}_{t_{j+1}} - B^{(n)}_{t_j}; \Delta t_j := t^{(n)}_{j+1} - t^{(n)}_j, \delta_k := \max_j \Delta t_j$. Calculate

$$\|\sum_j (\Delta B_j)^2 - T\|^2 \underset{5}{=} E\left[(\sum_j (\Delta B_j)^2 - T)^2\right] \tag{2.46}$$

$$= E\left[\sum_{i,j} (\Delta B_j)^2 (\Delta B_i)^2 - 2T \sum_i (\Delta B_j)^2 + T^2\right] \tag{2.47}$$

$$= \sum_i \underbrace{E\left[(\Delta B_i)^4\right]}_{=3(\Delta t_j)^2} + \sum_{i\neq j} \underbrace{E\left[(\Delta B_j)^2\right]}_{=\Delta t_i} E\left[(\Delta B_i)^2\right] - 2T \sum_i E\left[(\Delta B_j)^2\right] + T^2 \tag{2.48}$$

$$= 2\sum_j (\Delta t_j)^2 \tag{2.49}$$

$$\leq 2\Delta_n \sum_j \Delta t_j = 2\delta_n T \underset{n\to\infty}{\longrightarrow} 0 \tag{2.50}$$

by using in (2.48) that we know for $X \sim \mathcal{N}(0, \sigma^2) \Rightarrow E[X^2] = \sigma^2, E[X^4] = 3\sigma^4$ □

Informally Lemma 2.5 shows with $T = dt$

$$(dB_t)^2 \approx dt \tag{2.51}$$

$$\Rightarrow dB_t \approx \sqrt{dt} \gg dt \tag{2.52}$$

Therefore B_t will not be differentiable, since

$$\frac{dB_t}{dt} \to \infty. \tag{2.53}$$

Lemma 2.6.
Let $X_1, X_2, \ldots$ be a sequence of R.V. s.t

$$\lim_{n\to\infty} \mathbb{E}\left[|X_k|^2\right] = 0 \tag{2.54}$$

Then there exists a subsequence $(X_{n_k})_{k\geq 1}$ s.t. $X_{n_k} \to 0$ almost surely.

Proof. We choose a subsequence s.t. $\mathbb{E}\left[|X_{n_k}|^2\right] < \frac{1}{k^2}$. Then $\sum_{k=1}^\infty \mathbb{E}\left[|X_{n_k}|^2\right] < \infty$. By using Cebicev we get

$$\forall m \in \mathbb{N} \sum_{k=1}^\infty \mathbb{P}\left(|X_{n_k}| \geq \frac{1}{m}\right) \leq m^2 \mathbb{E}\left[|X_{n_k}|^2\right] \tag{2.55}$$

$$\Rightarrow \forall m \in \mathbb{N} \sum_{k=1}^\infty \mathbb{P}\left(|X_{n_k}| \geq \frac{1}{m}\right) \leq m^2 \sum_{k=1}^\infty \mathbb{E}\left[|X_{n_k}|^2\right] < \infty \tag{2.56}$$

$$\Rightarrow \forall m \in \mathbb{N}\, \mathbb{P}\left(|X_{n_k}| \geq \frac{1}{m} \text{ u.o.}\right) = 0 \tag{2.57}$$

$$\Rightarrow X_{n_k} \to 0 \text{ a.s.} \tag{2.58}$$

□

Proof of Theorem 2.4(a). The previous two lemmas give: $\exists$ subsequence $(n_k)_{k\geq 1}$ s.t. for almost all $\omega \in \Omega$

$$\lim_{k\to\infty} \sum_{j=1}^{n-1} \left(B_{t^{(n_k)}_{j+1}}(w) - B_{t^{(n_k)}_j}(w)\right)^2 = T. \tag{2.59}$$

[5]Nach Defintion der L^2-Norm

Let $\omega \in \Omega$ be fix s.t.(2.59) holds. Let $\varepsilon_{n_k} := \max_j |\Delta B_j| \Rightarrow \lim_{k\to\infty} \varepsilon_{n_k} = 0$ because $t \mapsto B_t$ is uniformely continuous.

$$\Rightarrow \sum_{j=0}^{n_k-1} |\Delta B_j| \geq \sum_{j=0}^{n_k-1} \frac{1}{\varepsilon_{n_k}} |\Delta B_j|^2 \approx \frac{T}{\varepsilon_{n_k}} \to \infty \text{ as } k \to \infty \tag{2.60}$$

□

Lemma 2.7.
Let $(B_t)_{0\leq t\leq T}$ be a Brownian Motion on $[0, T]$. Then, $\forall c > 0$

$$(cB_{\frac{t}{c^2}})_{0\leq t\leq T} \tag{2.61}$$

is a Brownian Motion on $[0, \frac{T}{c^2}]$.

Proof. Exercise Sheet 1. □

Proof of Theorem 2.4(b). Let

$$X_{n,k} := \max_{j=k,k+1,k+2} |B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}| \tag{2.62}$$

$$\Rightarrow \forall \varepsilon > 0 \mathbb{P}\left(X_{n,k} \leq \varepsilon\right) = \mathbb{P}\left(|B_{\frac{1}{2^n}}| \leq \varepsilon|\right)^3 \tag{2.63}$$

$$\underset{6}{=} \mathbb{P}\left(|B_1| \leq 2^{\frac{n}{2}}\varepsilon\right)^3 \tag{2.64}$$

$$\leq (2^{\frac{n}{2}}\varepsilon)^3 \tag{2.65}$$

Now let $Y_n := \min_{k\leq 2^n T} X_{n,k}$.

$$\Rightarrow \mathbb{P}\left(Y_n \leq \varepsilon\right) \leq T2^n(2^{\frac{n}{2}}\varepsilon)^3 \tag{2.66}$$

Let $A := \{\omega \in \Omega \text{ s.t. } t \mapsto B_t(\omega) \text{ is differentiable somewhere}\}$. For an $\omega \in A, t \mapsto B_t(\omega)$ is in $t_0(\omega)$ differentiable. Let D be the derivative.

$$\Rightarrow \exists\delta > 0 \text{ s.t. } \forall t \in [t_0 - \delta, t_0 + \delta] \quad |B_t - B_{t_0}| \leq (|D| + 1)|t - t_0| \tag{2.67}$$

We now choose n_0 big enough s.t.

$$\delta > \frac{1}{2^{n_0-1}}, n_0 > 2(|D| + 1), n_0 > t_0 \tag{2.68}$$

Now for $\forall n \geq n_0$ choose k s.t.

$$\frac{k}{2^n} \leq t_0 \leq \frac{k+1}{2^n}. \tag{2.69}$$

Then

$$|t_0 - \frac{j}{2^n}| < \delta \text{ for } j = k, k+1, k+2. \tag{2.70}$$

$$\Rightarrow X_{n,k}(\omega) \leq (|D| + 1)\frac{1}{2^n} \leq \frac{n}{2^n} \tag{2.71}$$

and, since $n > t_0 > \frac{k}{2^n}$, also $Y_n(\omega) \leq X_{n,k}(\omega) \leq \frac{n}{2^n}$. Therefore $A \subset A_n := \{Y_n(\omega) \leq \frac{n}{2^n}\}$ for n large enough and hence also

$$A \subset \liminf_n A_n \tag{2.72}$$

[6]Lemma 2.7

But (2.66) implies

$$\sum_{n\geq 1} \mathbb{P}(A_n) \leq \sum_{n\geq 1} n2^2(2^{\frac{n}{2}+1}n2^{-n})^3 < \infty \tag{2.73}$$

$$\Rightarrow \mathbb{P}\left(\liminf_{n\to\infty} A_n\right) = 0 \tag{2.74}$$

i.e. $t \mapsto B_t(\omega)$ is a.s. not differentiable. □

[16.10.2012]
[19.10.2012]

Definition 2.8.
Let

$$p(x,y,\tau) := \frac{1}{\sqrt{2\pi\tau}} \exp(-\frac{(x-y)^2}{2\tau}) \tag{2.75}$$

be the Heat-Kernel $\forall x, y \in \mathbb{R}, \tau > 0$. A stochastic process $(B_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^d$ is called a d-dimensional Brownian Motion if

- $B_0 = (0, \dots, 0)$
- The increments are independent and stationary with distribution

$$\mathbb{P}(B_t - B_s \in A) = \int_A p(0, x_1, t-s) \dots p(0, x_d, t-s) dx_1 \dots dx_n \tag{2.76}$$

  $\forall A \in \mathcal{B}(\mathbb{R}^d)\ \forall 0 \leq s < t \leq T$.
- The trajectories $t \mapsto B_t(\omega)$ are continuous for a.e. $\omega \in \Omega$.

2.3 Stochastic Processes

Definition 2.9.
A family $(X_t)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable space $(E, \mathcal{S})$ if $\forall t \geq 0$ X_t is a R.V.. t usually plays the role of time and E is the space where X lives (=state space). For all $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is called a trajectory.

Definition 2.10.
Let X and Y two stochastic processes (defined on the same probability space and with the same state space). Then

(a) X is a modification/version of Y if

$$\mathbb{P}(X_t = Y_t) = 1\ \forall t. \tag{2.77}$$

(b) X and Y are indistinguishable if

$$\mathbb{P}(X_t = Y_t, \forall t \geq 0) = 1. \tag{2.78}$$

It holds $b) \Rightarrow a)$ but not the other way round.

Example: $\Omega = [0,1], \mathbb{P}$ *the Lebesguemeasure. Define*

$$\begin{cases} X_t(\omega) = 0 \\ Y_t(\omega) = \mathbb{1}_{\{t=\omega\}} \end{cases} \tag{2.79}$$

Then, $\forall t \geq 0 \mathbb{P}(X_t = Y_t) = \mathbb{P}(t \neq \omega) = 1$ *but* $\mathbb{P}(X_t = Y_t, \forall t \in [0,1]) = 0$.

Lemma 2.11.
Let Y be a modification of X. If X and Y have a.s. right-continuous paths (trajectories). Then X and Y are indistinguishable.

Proof. Let $\Omega_0 \subset \Omega$ be the set where either X or Y are not right continuous. By assumption: $\mathbb{P}(\Omega_0) = 0$. For $q \in \mathbb{Q}_+$ let $N_q = \{\omega \in \Omega | X_q(\omega) \neq Y_q(\omega)\}$. Since Y is a modification of X $\mathbb{P}\left(N_q\right) = 0$. As $\mathbb{Q}_+$ is countable also $\mathbb{P}\left(\bigcup_{q\in\mathbb{Q}_+} N_q\right) = 0 \Rightarrow \mathbb{P}(\underbrace{\Omega_0 \cup \bigcup_{q\in\mathbb{Q}_+} N_q}_{=\tilde{\Omega}}) = 0$.

Therefore $\forall \omega \notin \tilde{\Omega} X_t(\omega) = Y_t(\omega) \forall t \in Q_+$ and as $X_t(\omega)$ and $Y_t(\omega)$ are rightcontinuous it holds $X_t(\omega) = Y_t(\omega) \forall t \geq 0$ and with $\mathbb{P}\left(\tilde{\Omega}^c\right) = 1$ the statement follows. □

2.4 Hölder continuity for Brownian Motion

Definition 2.12.
A function $f : \mathbb{R}_+ \to \mathbb{R}$ is called $\gamma-$Hölder continuous in $x \geq 0$ if $\exists \varepsilon > 0 C < \infty$ s.t.

$$|f(x) - f(y)| \leq C|x - y|^\gamma \ \forall y \geq 0 : |y - x| \leq \varepsilon \tag{2.80}$$

γ is called the Hölder-exponent.

Theorem 2.13 (Kolmogorov-Chentsov)**.**
Let $(X_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P}), \alpha \geq 1, \beta \geq 0, c > 0$ s.t.

$$\mathbb{E}\left[|X_t - X_s|^\alpha\right] \leq C|t - s|^{\beta+1} \tag{2.81}$$

Then there exists a version/modification $(Y_t)_{0\leq t\leq T}$ of $(X_t)_{0\leq t\leq T}$ for all $T > 0$ s.t. Y is $\gamma-$Hölder continuous $\forall \gamma \in (0, {}^\beta/_\alpha)$.

Before we proof this theorem, we will apply it on BM. We have

$$\mathbb{E}\left[|B_t - B_s|^n\right] = \frac{(2n)!}{2^n n!}|t - s|^n \tag{2.82}$$

Therefore with $\alpha = 2n, \beta + 1 = n$ there exists a $\gamma-$Hölder-continuous version $\forall \gamma < \frac{n-1}{2n} \forall n \underset{\to\infty}{\Longrightarrow}$ $\forall \gamma < {}^1/_2$.

Corollary 2.14.
Let B be a BM. Then there exists a Version $\tilde{B}$ s.t. $\tilde{B}$ is $\gamma-$Hölder-continuous forall $\gamma < \frac{1}{2}$ s.t.

$$\mathbb{P}\left(\sup_{0\leq t-s\leq h(\omega), 0\leq s,t\leq T} \frac{|B_t(\omega) - B_s(\omega)|}{|t - s|^\gamma} \leq C\right) = 1 \tag{2.83}$$

where $h(\omega)$ is a positive R.V. (a.s.).

Proof of Theorem 2.13. WLOG $T = 1$. The proof consists of 5 claims.

1. claim $X_s \xrightarrow{P} X_t$ when $s \to t$.
   Proof:

$$\forall \varepsilon > 0 \mathbb{P}\left(|X_t - X_s| \geq \varepsilon\right) \leq \frac{\mathbb{E}\left[|X_t - X_s|^\alpha\right]}{\varepsilon^\alpha} \tag{2.84}$$

$$\leq C\frac{C|t - s|^{\beta+1}}{\varepsilon^\alpha} \to 0 \tag{2.85}$$

2. claim $\exists \Omega^* \subset \Omega$ with $\mathbb{P}(\Omega^*) = 1$ s.t. $\forall \omega \in \Omega^*$

$$\max_{1 \le k \le 2^n} |X_{\frac{k}{2^n}}(\omega) - X_{\frac{k-1}{2^n}}(\omega)| < 2^{-\gamma n} \forall n > n^*(\omega), \gamma \in (0, \beta/\alpha) \tag{2.86}$$

Proof: Let $D_n = \{\frac{k}{2^n}, 0 \le k \le 2^n, k \in \mathbb{N}\}$ and $D = \cup_{n \ge 1} D_n$. Using (2.85) with $t = \frac{k}{2^n}, s = \frac{k-1}{2^n}, \varepsilon = 2^{-\gamma n}$ we get

$$\mathbb{P}\left(|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \ge 2^{-\gamma n}\right) \le C 2^{-n(\beta+1)} 2^{\alpha\gamma n} \tag{2.87}$$

$$= C 2^{-n(\beta+1-\gamma\alpha)} \tag{2.88}$$

Let $E_n = \{\omega : \max_{1 \le k \le 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \ge 2^{-\gamma n}\}$.

$$\Rightarrow \mathbb{P}(E_n) \le 2^n C 2^{-n(\beta+1-\alpha\gamma)} \tag{2.89}$$

$$\le C 2^{-n(\beta-\alpha\gamma)} \tag{2.90}$$

$$\Rightarrow \sum_{n \ge 1} \mathbb{P}(E_n) \le C \sum_{n \ge 1} \frac{C}{2^{n(\beta-\alpha\gamma)}} < \infty \tag{2.91}$$

whenever $\gamma < \beta/\alpha$. Using Borel-Cantelli we get claim 2.

3. claim: For any given $\omega \in \Omega^*, n > n^*(\omega)$, $\forall m \ge n$

$$|X_t(\omega) - X_s(\omega)| \le 2 \sum_{j=n+1}^{m} \frac{1}{2^{j\gamma}}, \forall s, t \in D_m, 0 \le t - s \le 2^{-n} \tag{2.92}$$

Proof (induction): $m = n+1 \Rightarrow t = \frac{k}{2^n}, s = \frac{k-1}{2^n}$ follows from claim 2. Now assume that claim 3 holds for $m = n+1, ..., M-1$. Choose $s, t \in D_m$, $s < t$ and define $t' = \max\{u \in D_{m-1}, u \le t\}$, $s' = \min\{u \in D_{m-1}, u \ge s\}$. Therefore $s \le s' \le t' \le t, s' - s \le 2^{-M}, t - t' \le 2^{-M}$.Claim 2 gives

$$\Rightarrow |X_{s'}(\omega) - X_s(\omega)| \le 2^{-\gamma M} \tag{2.93}$$

$$|X_{t'}(\omega) - X_t(\omega)| \le 2^{-\gamma M} \tag{2.94}$$

By the induction hypothesis:

$$|X_{t'}(\omega) - X_{s'}(\omega)| \le 2 \sum_{n+1}^{M-1} \frac{1}{2^{\gamma j}} \tag{2.95}$$

and with the triangular inequality

$$|X_s(\omega) - X_t(\omega)| \le 2 \sum_{j=n+1}^{M} \frac{1}{2^{\gamma j}} \tag{2.96}$$

4. claim: $t \mapsto X_t(\omega)$ is uniformly continuous $\forall \omega \in \Omega^*$.
Proof: Choose $s, t \in D, 0 < t - s < h(\omega) := 2^{-n^*(\omega)}$ and $n > n^*(\omega)$ s.t. $2^{-(n+1)} \le t - s \le 2^{-n}$. Then from claim 3

$$|X_t(\omega) - X_s(\omega)| \le 2 \sum_{j=n+1}^{\infty} \frac{1}{2^{\gamma j}} \tag{2.97}$$

$$= C \frac{1}{2^{\gamma n}} \le C|t - s|^\gamma \tag{2.98}$$

5. step: Define a modification:

$$\tilde{X}_t(\omega) = \begin{cases} X_t(\omega) & \text{, if } \omega \in \Omega^*, t \in D \\ 0 & \text{, if } \omega \notin \Omega^* \end{cases} \tag{2.99}$$

For $\omega \in \Omega^*, t \notin D$ choose a sequence $(s_n)_{n\geq 1}$ in D s.t. $s_n \to t$. From claim 4 we gett that X_{s_n} is a convergent sequence (cauchy-sequence). So we can define

$$\tilde{X}_t(\omega) = \lim_{n\to\infty} X_{s_n}(\omega) \tag{2.100}$$

$\Rightarrow \tilde{X}_t$ is continuous and satisfies

$$|X_t(\omega) - X_s(\omega)| < C|t-s|^\gamma \tag{2.101}$$

for $t - s$ small enough. Finally one verify that $\tilde{X}_t$ is indeed a modification of X_t.

$$\left.\begin{array}{l} X_{s_n} \xrightarrow{a.s.} \tilde{X}_t \\ X_{s_n} \xrightarrow{P} X_t \end{array}\right\} \Rightarrow X_t \overset{a.s.}{=} X_t \tag{2.102}$$

□

[19.10.2012]
[23.10.2012]

3 Filtrations and Stoppingtimes

3.1 Filtrations

From now on $(\Omega, \mathcal{F}, \mathbb{P})$ is always a probability space.

Definition 3.1 (Filtration)**.**
An increasing family $\{\mathcal{F}_t, t \geq 0\}$ of $\sigma-$algebras of $\mathcal{F}$ is called a *filtration*, i.e.

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \quad \forall\, 0 \leq s \leq t \leq \infty. \tag{3.1}$$

Intuition: $\mathcal{F}_t$ contains the information, that are known until the time $t \in [0, \infty)$.

Definition 3.2 (Filtered probability space)**.**
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called *filtered probabilty space*.

Notation: *We define*

$$\mathcal{F}_\infty := \sigma(\mathcal{F}_t, t \geq 0) \qquad \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s \tag{3.2}$$

$$\mathcal{F}_{t-} := \sigma(\mathcal{F}_s, s < t) \text{ the past} \qquad \mathcal{F}_{0-} = \{\emptyset, \Omega\} \tag{3.3}$$

Obviously it holds $\mathcal{F}_{t-} \subset \mathcal{F}_t \subset \mathcal{F}_{t+}$.

If we have a stochastic process X on $(\Omega, \mathcal{F}, \mathbb{P})$ we denote by $\mathcal{F}_t^X := \sigma(X_s, 0 \leq s \leq t)$ the natural filtration (of X)

Definition 3.3.
If $\mathcal{F}_t = \mathcal{F}_{t+} \forall t \geq 0$, then we say that $(\mathcal{F}_t)_{t\geq 0}$ is *right-continuous*.

$(\mathcal{F}_{t+})_{t\geq 0}$ is always right-continuous.

Definition 3.4.
A set A is called a $(\mathcal{F}, \mathbb{P})$-*nullset* if

$$\exists\, \tilde{A} \in \mathcal{F} \text{ s.t. } A \subset \tilde{A} \text{ and } \mathbb{P}\left(\tilde{A}\right) = 0. \tag{3.4}$$

$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is called *complete*, if all $(\mathcal{F}, \mathbb{P})$-nullsets are in $\mathcal{F}_0$

Remark:
- *If* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *is complete, then every* $(\Omega, \mathcal{F}_t, \mathbb{P})$ *is complete.*
- *The other direction does not hold!*
- *Augmentation: Let* $\mathcal{N} = \{(\mathcal{F}, \mathbb{P})$*-nullsets*$\}$. *Set* $\mathcal{F}' = \sigma(\mathcal{F} \cup \mathcal{N}), \mathcal{F}_t' = \sigma(\mathcal{F}_t \cup \mathcal{N})$. *Then* $(\Omega, \mathcal{F}', \mathcal{F}_t', \mathbb{P})$ *is complete.*

Definition 3.5.
A filtered probabilty space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is called *standard*, if it is complete and the filtration is right-continuous.

One can extend an filtration s.t. it becomes standard by

- Augmentation of $\mathcal{F}_t$ and $\mathcal{F}$, and
- using $\mathcal{F}_{t+}$ instead of $\mathcal{F}_t$.

3.2 Adapted processes

Definition 3.6.

(i) Let X be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E, \mathcal{E})$.

$$\mathcal{F}_t^X := \sigma(X_s : s \leq t) \tag{3.5}$$

is called the *filtration generated by* X.

(ii) A stochastic process $(X_t)_{t\geq 0}$ is called *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if

$$\mathcal{F}_t^X \subset \mathcal{F}_t \; \forall t \geq 0, \tag{3.6}$$

i.e. if X_t is $\mathcal{F}_t$-measurable $\forall t \geq 0$.

Example: *a) Let B_t a standard B; and $\mathcal{F}_t$ the natural filtration. Then $X_t = B_{t/2}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ but $Y_t := B_{2t}$ is not adapted to $(\mathcal{F}_t)_{t\geq 0}$.*

b) Let $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\geq 0}$ a filtration, then $X_t := \mathbb{E}[f|\mathcal{F}_t]$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$.

3.3 Progressively measurable processes

Definition 3.7 (Progressively measurable)**.**
A process $(X_t)_{t\geq 0}$ is called *progressively measurable* (or simply *progressiv*) with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if $\forall t \geq 0$ the map

$$X : [0,t] \times \Omega \to E \tag{3.7}$$

$$(s, \omega) \mapsto X_s(\omega) \tag{3.8}$$

is measurable with respect to $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

Remark:
- *It holds: progressively measurable $\Rightarrow$ adapted but not the otherway round.*
- *As one can see in Theorem 3.15 we need this property to ensure that the stopped process is again measurable.*

Proposition 3.8.
Let $(X_t)_{t\geq 0})$ be a stochastic process which is adapted to $(\mathcal{F}_t)_{t\geq 0}$. Assume that each trajectory $t \mapsto X_t(\omega)$ is right-continuous (or left-continuous). Then $(X_t)_{t\geq 0}$ is progressively measurable.

Remark: *For a BM there exists a modification that is progressively measurable.*

Proof. Let $t > 0$ fixed. We approximate X by $X^{(n)}$. So for $k = 0, 1, ..., 2^{n-1}, 0 \leq s \leq t$, set

$$X_s^{(n)}(\omega) := X_{\frac{(k+1)t}{2^n}}(\omega) \text{ for } \frac{kt}{2^n} < s \leq \frac{(k+1)t}{2^n} \tag{3.9}$$

and $X_0^{(n)}(\omega) := X_0(\omega)$. Then $X^{(n)} : (s, \omega) \mapsto X_s^{(n)}(\omega)$ is measurable w.r.t. $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$, since this map is equal to $(s, \omega) \mapsto \sum_{k=0}^{2^n-1} X_{\frac{(k+1)t}{2^n}} \mathbb{1}_{\{\frac{kt}{2^n} < s \leq \frac{(k+1)t}{2^n}\}}$. But since X is right-continuous $\lim_{r\to\infty} X_s^{(n)}(\omega) = X_s(\omega) \forall (s, \omega) \in [0,t] \times \Omega$. $\Rightarrow (s, \omega) \mapsto X_s(\omega)$ is also $\mathcal{B}([0,t] \otimes \mathcal{F}_t$ measurable. □

3.4 Stopping times

Definition 3.9 (Stopping time)**.**
A map $T : \Omega \to [0, \infty]$ is called a *(strong) stopping time* w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ if $\forall t \geq 0$

$$\{T \leq t\} = \{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t. \tag{3.10}$$

T is called a *weak stopping* time if

$$\{T < t\} \in \mathcal{F}_t. \tag{3.11}$$

If T is a (weak) stopping time, then T is measurable w.r.t. $\mathcal{F}$.

Proposition 3.10.

a) Each fixed time $T = c \geq 0$ is a stopping time.

b) Each stopping time is also a weak stopping time.

c) If $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous filtration, then a weak stopping time is a stopping time.

d) T is a stopping time $\Leftrightarrow X_t = \mathbb{1}_{[0,T)}$ is adapted to the filtration.

e) T is a weak stopping time w.r.t. $(\mathcal{F}_t) \Leftrightarrow T$ is a stopping time w.r.t. $(\mathcal{F}_{t+})$.

Proof. **ad a)** $A_t := \{\omega \in \Omega | c \leq t\}$ is either Ω or $\emptyset$. So $A_t \in \mathcal{F}_t\ \forall t$.
ad b) $\{T < t\} = \cup_{n\geq 1} \underbrace{\{T \leq t - \frac{1}{n}\}}_{\in \mathcal{F}_{t-\frac{1}{n}}} \in \mathcal{F}_t$
ad c) Let T be a weak stopping time. Recall that $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$. Then

$$\forall m \geq 1 \{T \leq t\} = (\cap_{n\geq m} \underbrace{\{T < t + \frac{1}{n}\}}_{\in \mathcal{F}_{t+\frac{1}{n}} \subset \mathcal{F}_{t+\frac{1}{m}}}) \in \mathcal{F}_{t+\frac{1}{m}} \tag{3.12}$$

$$\Rightarrow \{T \leq t\} \in \mathcal{F}_{t+\frac{1}{m}} \forall m \Rightarrow \{T \leq t\} \in \mathcal{F}_{t+} \Rightarrow \{T \leq t\} \in \mathcal{F}_t \tag{3.13}$$

ad d) $\{T \leq t\} = \{X_t = 0\} \in \mathcal{F}_t$ since X_t is adapted. □

Proposition 3.11.

a) Let T be a weak stopping time and $\vartheta > 0$ a constant $\Rightarrow T + \vartheta$ is a stopping time.

b) Let T, S be stopping times $\Rightarrow T \wedge S, T \vee S$ and $S + T$ are also stopping times.

c) Let S, T be weak stopping times $\Rightarrow S + T$ is a weak stopping time.

d) Let S, T be weak stopping times. If $T > 0$ and $S > 0$ OR if $T > 0$ and T is even a strong stopping time, then $T + S$ is a strong stopping time.

e) Let $\{T_n\}_{n\geq 0}$ be a sequence of weak stopping times. $\Rightarrow \sup_{n\geq 1} T_n, \inf_{n\geq 1} T_n, \limsup_{n\to\infty} T_n$ and $\liminf_{n\to\infty} T_n$ are also weak stopping times. If the T_n are strong stopping times, $\Rightarrow \sup_{n\geq 1} T_n$ is a strong stopping time.

Example: *Let $(X_t)_{t\geq 0}$ be right-continuous and adapted, with $X_t \in \mathbb{R}^d$. For $A \in \mathcal{B}(\mathbb{R}^d)$. Define*

$$T_A(\omega) := \inf\{t \geq 0 | X_t(\omega) \in A\} \textit{ with } \inf \emptyset = \infty \tag{3.14}$$

$$\textit{is called the first entrance time of } A. \tag{3.15}$$

$$T_A^*(\omega) := \inf\{t > 0 | X_t(\omega) \in A\} \tag{3.16}$$

$$\textit{is called the first hitting time.} \tag{3.17}$$

Remark: *Each stopping time is a first entrance time* ($X_t := \mathbb{1}_{(0,T_A)}(t)$).

Lemma 3.12.

a) If A is open $\Rightarrow T_A$ is a weak stopping time.

b) If A is closed and $X_t(\omega)$ is continuous $\Rightarrow T_A$ is a stopping time.

Proof. **ad a)**

$$\{T_A < t\} = \{X_s(\omega) \in A \text{ for some } 0 \leq s \leq t\} \tag{3.18}$$

$$\overset{\Delta}{=} \cup_{s\in\mathbb{Q},0\leq s<t}\{X_s(\omega) \in A\} \in \mathcal{F}_t \tag{3.19}$$

Regarding Δ: "$\supset$" is clear. "$\subset$" follows from the right-continuity of X_t and A open.
ad b)

$$\{T_A \leq t\}^c = \{T_A > t\} \tag{3.20}$$

$$= \{\|X_s - A\| > 0, \forall 0 \leq s \leq t\} \tag{3.21}$$

$$= \cup_{n\geq 1}\{\|X_s - A\| > \frac{1}{n}, \forall 0 \leq s \leq t\} \tag{3.22}$$

$$\overset{\text{continuity}}{=} \cup_{n\geq 1}\{\|X_s - A\| > \frac{1}{n}, \forall 0 \leq s \leq t, s \in \mathbb{Q}\} \tag{3.23}$$

$$= \cup_{n\geq 1} \cap_{s\in\mathbb{Q},0\leq s\leq t} \underbrace{\{\|X_s - A\| > \frac{1}{n}\}}_{\in\mathcal{F}_s\subset\mathcal{F}_t} \in \mathcal{F}_t \tag{3.24}$$

□

[23.10.2012]
[26.10.2012]

Definition 3.13 ($\mathcal{F}_T$).
Let T be a stopping time, then

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \forall t \geq 0\} \tag{3.25}$$

is called the *σ-algebra of events determined prior to the stopping time T*. Zu deutsch: Die $\sigma-$Algebra der T-Vergangenheit.

Lemma 3.14.
Let S and T be stopping times for a filtration $(\mathcal{F}_t)$. It holds

a) Let $A \in \mathcal{F}_s \Rightarrow A \cap \{S \leq T\} \in \mathcal{F}_T$.

b) $S \leq T \Rightarrow \mathcal{F}_S \subset \mathcal{F}_T$

c) $\mathcal{F}_{T\wedge S} = \mathcal{F}_T \cap \mathcal{F}_S$

d) $\{\{T < S\}, \{T \leq S\}, \{T = S\}, \{T \geq S\}, \{T > S\}\} \subset \mathcal{F}_T \cap \mathcal{F}_S$.

e) $\mathbb{E}[\cdot|\mathcal{F}_{T\wedge S}] = \mathbb{E}[\mathbb{E}[\cdot|\mathcal{F}_S]|\mathcal{F}_T]$.

f) $\mathbb{E}[\cdot|\mathcal{F}_T] = \mathbb{E}[\cdot|\mathcal{F}_{T\wedge S}]$ a.s. on the set $\{T \leq S\}$.

Theorem 3.15.
Let X be progressively measurable w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ and T be a stopping time. Then

1. $X_T : \{T < \infty\} \to E,\ \omega \mapsto X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable.
2. The stopped process

$$X^T : (t,\omega) \mapsto X_{T(\omega)\wedge t}(\omega) \tag{3.26}$$

is also progressively measurable w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

Proof. **ad 1)** To show (1) we have to see that $\forall B \in \mathcal{B}(E)$ and $\forall t \geq 0$ it holds

$$\{X_T \in B\} \cap \{T \leq t\} = \underbrace{\{X_{T\wedge t} \in B\}}_{\in \mathcal{F}_t \text{ if (2) holds}} \cap \underbrace{\{T \leq t\}}_{\in \mathcal{F}_t} \overset{!}{\in} \mathcal{F}_t \tag{3.27}$$

ad 2)

$$(s,\omega) \xmapsto{\text{measurable being T a r.v.}} (T(\omega) \wedge s, \omega) \longmapsto X_{T(\omega)\wedge s}(\omega) \tag{3.28}$$

$$(s,\omega) \xmapsto[\text{w.r.t } \mathcal{B}([0,t])\otimes\mathcal{F}_t]{\text{measurable}} X_s(\omega) \tag{3.29}$$

$\Rightarrow (s,\omega) \mapsto X_{T(\omega)\wedge s}(\omega)$ is also measurable w.r.t. $\mathcal{B}([0,t]) \otimes \mathcal{F}_t \forall t \geq 0$. □

Example: *Let B be a standard BM and $b > 0$ a constant. Let $T_b = \inf\{t \geq 0 | B_t = b\}$. Question:* $\mathbb{P}(T_b \leq t) = ?$.
We know that for fixed $s : B_t - B_s$ and B_s are independent (Markov property). The same holds if s is stopping time (strong markov property).

$$\mathbb{P}(T_b \leq t) = \mathbb{P}(T_b \leq t, B_t < b) + \underbrace{\mathbb{P}(T_B \leq t, B_t = b)}_{=0} + \mathbb{P}(T_b \leq t, B_t > b) \tag{3.30}$$

$$= 2\mathbb{P}(T_b \leq t, B_t > b) \tag{3.31}$$

$$= 2\mathbb{P}(B_t > b) \tag{3.32}$$

$$= 2\frac{1}{\sqrt{2\pi t}} \int_b^\infty e^{-\frac{x^2}{2t}} dx \tag{3.33}$$

$$\underset{1}{=} \frac{2}{\sqrt{2\pi}} \int_{b/\sqrt{t}}^\infty e^{-\frac{y^2}{2}} dy \tag{3.34}$$

In particular

$$\mathbb{P}(T_b \in dt) = \frac{1}{\sqrt{2\pi t^3}} e^{-b^2/2t} |b| dt \tag{3.35}$$

[1] $\frac{x}{\sqrt{t}} = y$

4 Continuous time martingales

From now on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is always a filtered probability space and we have $E = \mathbb{R}$.

4.1 Conditional expectation

Definition 4.1 (Conditional expectation)**.**
Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ a random variable. Then a random variable Y is called *conditional expectation* of X if $\forall A \in \mathcal{G}$

$$\int_A X d\mathbb{P} = \int_A Y d\mathbb{P} \quad \text{and } Y \text{ is } \mathcal{G}\text{-measurable.} \tag{4.1}$$

and it is usually denoted by

$$Y = \mathbb{E}\left[X|\mathcal{G}\right]. \tag{4.2}$$

Remark: $\mathbb{E}[X|\mathcal{G}]$ *is a.s. unique.*

Properties:

- $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$
- *If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] = X$ a.s..*
- *If Y is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$ a.s.*
- *If X is $\mathcal{G}$ independent i.e., X independent from $\mathbb{1}_A, \forall A \in \mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.*
- *If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F} \Rightarrow \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ a.s.*
- *$\mathbb{E}[\alpha X + \beta Y|\mathcal{G}] = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Y|\mathcal{G}]\ \forall X, Y$ r.v. and $\alpha, \beta \in \mathbb{R}$.*
- *If $X \leq Y$ a.s., then $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$ a.s.*
- *Jensen: Let φ be a convex function, then $\varphi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\varphi(X)|\mathcal{G}]$.*

Now let $(X_n)_{n\geq 1}$ be a sequence of r.v.

- *Fatou: If there exists a $\mathcal{F}-$measurable r.v. Y with $\mathbb{E}[Y] > -\infty$ s.t. $\forall k \geq 1, X_k \geq Y$, then $\mathbb{E}[\liminf_{n\to\infty} X_k|\mathcal{G}] \leq \liminf_{n\to\infty} \mathbb{E}[X_k|\mathcal{G}]$*
- *Monoton convergence: If $\mathbb{E}[X] > -\infty$ and $X_k \nearrow X$ a.s., then $\mathbb{E}[X_k|\mathcal{G}] \nearrow \mathbb{E}[X|\mathcal{G}]$ a.s.*
- *Dominated convergence: If there exists a $\mathcal{F}$-measurable r.v. Y s.t. $\mathbb{E}[Y] < \infty$ and $|X_k| \leq Y$ and if $X_k \to X$ a.s., then $\mathbb{E}[X_k|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$ a.s.*

4.2 Martingale

Definition 4.2 (Martingale)**.**
Let X be a stochastic process adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. X is called *submartingale*, if

- $X_t \in \mathbb{R}$ with $\mathbb{E}[X_t^+] \equiv \mathbb{E}[\max\{X_t, 0\}] < \infty$ for all $t \geq 0$.
- $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ a.s. $\forall 0 \leq s \leq t$.

X is a *supermartingale* if $-X$ is a submartingale.
X is a *martingale* if it is both a super- and a submartingale.

Properties: $\forall 0 \le s \le t$

$$\mathbb{E}[X_t] = \mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_s]] \ge \mathbb{E}[X_s] \text{ for submartingales} \tag{4.3}$$

$$\mathbb{E}[X_t] = \mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_s]] \le \mathbb{E}[X_s] \text{ for supermartingales} \tag{4.4}$$

$$\mathbb{E}[X_t] = \mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_s]] = \mathbb{E}[X_s] \text{ for martingales} \tag{4.5}$$

We will now see some examples for martingales.

Proposition 4.3.
Let B be a d-dimensional (standard) BM and $\mathcal{F}_t \equiv \mathcal{F}_t^B$ the natural filtration. Then

a) For any fixed vector $Y \in \mathbb{R}^d$

$$Y \cdot B_t = \langle Y, B_t \rangle \tag{4.6}$$

is a martingale.

b) $|B_t|^2 - t \cdot d$ is a martingale.

c) For $Y \in \mathbb{R}^d$

$$\exp\left(Y \cdot B_t - 1/2|Y|^2 t\right) \tag{4.7}$$

is a martingale.

Remark: *We will see that for any X with properties a) and b) + a.s. continuity and* $(X_0 = 0) \Rightarrow X$ *is a BM. (Levy-Martingale-Characterization)*

Proof. B is adapted, therefore the transformations are also adapted.
Integrability is easy, due to the gaussian tails of the normal distribution. We will now check $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$.
ad a) Let $0 \le s \le t$.

$$\mathbb{E}[Y \cdot B_t|\mathcal{F}_s] = \sum_{k=1}^{d} Y_k \mathbb{E}\left[B_t^{(k)}|\mathcal{F}_s\right] \tag{4.8}$$

$$= \sum_{k=1}^{d} Y_k[\mathbb{E}[\underbrace{B_t^{(k)} - B_s^{(k)}}_{\text{independent of } \mathcal{F}_s} |\mathcal{F}_s] + \mathbb{E}[\underset{\text{measurable w.r.t. } \mathcal{F}_s}{\underset{\uparrow}{B_s^{(k)}}} |\mathcal{F}_s]] \tag{4.9}$$

$$= \sum_{k=1}^{d} Y_k \left(\mathbb{E}\left[B_t^{(k)} - B_s^{(k)}\right] + B_s^{(k)}\right) \tag{4.10}$$

$$= Y \cdot B_s \tag{4.11}$$

ad b) Let $0 \le s \le t$.

$$\mathbb{E}\left[|B_t|^2|\mathcal{F}_s\right] = \mathbb{E}[\underbrace{|B_t - B_s|^2}_{\text{indep. of } \mathcal{F}_s}|\mathcal{F}_s] + \mathbb{E}[\underset{\mathcal{F}_s \text{ measurable}}{\underset{\uparrow}{|B_s|^2}}|\mathcal{F}_s] + 2\mathbb{E}[\overbrace{(B_t - B_s)}^{\text{indep. of } \mathcal{F}_s} \underset{\mathcal{F}_s \text{ measurable}}{\underset{\uparrow}{B_s}}|\mathcal{F}_s] \tag{4.12}$$

$$= \mathbb{E}\left[|B_t - B_s|^2\right] + |B_s|^2 + 2B_s \underbrace{\mathbb{E}[B_t - B_s]}_{=0} \tag{4.13}$$

$$= d(t - s) + |B_s|^2 \tag{4.14}$$

ad c) Let $0 \le s \le t$.

$$\mathbb{E}\left[e^{Y \cdot B_t}|\mathcal{F}_s\right] = \mathbb{E}\left[e^{Y(B_t - B_s)} e^{YB_s}|\mathcal{F}_s\right] \tag{4.15}$$

$$= e^{YB_s} \underbrace{\mathbb{E}\left[e^{Y(B_t - B_s)}|\mathcal{F}_s\right]}_{=\mathbb{E}\left[e^{YB_{t-s}}\right]} \tag{4.16}$$

It holds

$$\mathbb{E}\left[e^{YB_{t-s}}\right] = \prod_{k=1}^{d} \underbrace{\mathbb{E}\left[e^{Y^{(k)}B^{(k)}_{t-s}}\right]}_{=e^{\frac{(Y^{(k)})^2}{2}(t-s)}} = e^{\frac{t-s}{2}|Y|^2} \tag{4.17}$$

□

[26.10.12]
[30.10.12]

Example: *Let X be a L^1 r.v. and $(\mathcal{F}_t)_{t\geq 0}$ a filtration. $\Rightarrow Y_t := \mathbb{E}[X|F_t]$ is a martingale.*
Indeed:

- *adapted by def of the conditional expectation*
- L^1 *since* : $\mathbb{E}[|Y_t|] = \mathbb{E}[|\mathbb{E}[X|\mathcal{F}_t]|] \leq \mathbb{E}[\mathbb{E}[|X||\mathcal{F}_t]] = \mathbb{E}[|X|] < \infty$ *by using Jensen.*
- *For all* $0 \leq s \leq t : \mathbb{E}[Y_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[X|\mathcal{F}_s] = Y_s$ *a.s. because* $\mathcal{F}_s \subset \mathcal{F}_t$

4.3 Properties and inequalities

Proposition 4.4.

a) Let X, Y be two martingales, $\alpha \in \mathbb{R}$

$$X + Y, \quad X - Y, \quad \alpha X \tag{4.18}$$

are also martingales.

b) Let X, Y be two submartingales, $\alpha \geq 0$,

$$X + Y, \quad \alpha X, \quad X \vee Y, \tag{4.19}$$

are also submartingales.

c) Let X be a martingale and φ a convex funktion with $\varphi(X_t) \in L^1$ for all $t \geq 0$, then $\varphi(X)$ is a submartingale.

d) X is a Martingale $\Leftrightarrow X$ is a L^1-sub-/supermartingale and $t \mapsto \mathbb{E}[X_t]$ is constant.

Example: $|B_t|$ *with* B_t *a BM is a submartingale.*

Proof. **ad a),b)** trivial.
ad c) Let $0 \leq s \leq t$

$$\mathbb{E}[\varphi(X_t)|\mathcal{F}_s] \overset{Jensen}{\geq} \varphi(\mathbb{E}[X_t|\mathcal{F}_s]) = \varphi(X_s) \tag{4.20}$$

□

Theorem 4.5 (Doobs maximum inequality)**.**
Let $(X_t)_{t\geq 0}$ be a submartingale with

a) each trajectory is right-continuous and $I = [\sigma, \tau] \subset [0, \infty)$ ($I = [\sigma, \infty)$ also possible)

or b) $I = \{\tau_1, \tau_2, ...\}$ with $\tau_k \leq \tau_{k+1}$ and $\lim_{k\to\infty} \tau_k = \tau$

Then

1. $\lambda \cdot \mathbb{P}(\sup_{t\in I} X_t \geq \lambda) \leq \mathbb{E}[X_\tau^+]$ with $X_\tau^+ = \max\{X_\tau, 0\}, \lambda > 0$.
2. If X is even a martingale or $X \geq 0$, then

$$\mathbb{E}\left[(\sup_{t\in I} X_t)^p\right] \leq \left(\frac{p}{1-p}\right)^p \mathbb{E}[|X_\tau|^p] \; \forall p > 1 \tag{4.21}$$

Proof. **ad b)** ≡ discrete case → proven in Stochastic Processes Thm 4.3.1 and 4.3.4.
ad a) Strategy: discrete time → use the fact that the trajectories are rightcontinuous □

Definition 4.6.
The number of upcrossings of $[a, b]$ (for $a < b \in \mathbb{R}$) during the time $I = [0, T]$ is given by

$$U_I(a, b, X(\omega)) = \sup\{n \in \mathbb{N} : \exists t_1 < t_2 < ... < t_{2n} \leq T \text{ s.t. } X_{t_1}(\omega) < a, X_{t_2}(\omega) > b, X_{t_3}(\omega) < a, ...\} \tag{4.22}$$

Theorem 4.7.
Let $a < b \in \mathbb{R}$, X_t a submartingale like in Thm 4.5

$$\mathbb{E}[U_I(a, b, X)] \leq \frac{\mathbb{E}\left[X_T^+\right] + |a|}{b - a} \tag{4.23}$$

Proof. The proof is similar to the discrete case. □

4.4 Convergence

Theorem 4.8.
Let X be a right-continuous submartingale with

$$C := \sup_{t\geq 0} \mathbb{E}[X_t^+] < \infty \tag{4.24}$$

then there exists a r.v. X_∞ s.t.

$$X_\infty = \lim_{t\to\infty} X_t \text{ a.s.} \tag{4.25}$$

Corollary 4.9.
Let X be a supermartingale, right-continuous and positive.

$$X_\infty = \lim_{t\to\infty} X_s \text{ exists a.s.} \tag{4.26}$$

Proof of the Corollary. Trivial from Thm 4.8 $Y_t = -X_t \Rightarrow C = \sup_{t\geq 0} \mathbb{E}[Y_t^+] = 0$. □

Proof of the Theorem. From Thm 4.7 we know that $\forall n \geq 1, a < b$

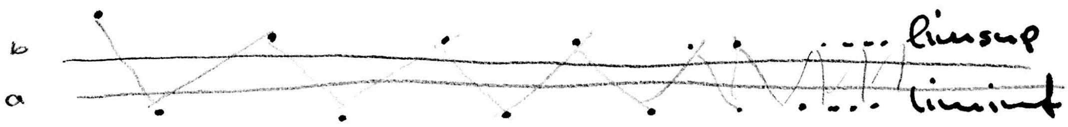
$$\mathbb{E}[U_{[0,n]}(a, b, X)] \leq \frac{\mathbb{E}[X_n^+] + |a|}{b - a} \leq \frac{C + a}{b - a} \tag{4.27}$$

Taking $n \to \infty$ gives with monoton convergence

$$E[U_{[0,\infty)}(a,b,X)] \le \frac{C+a}{b-a} < \infty \tag{4.28}$$

$\Rightarrow P(\underbrace{U_{[0,\infty)}(a,b,X) = \infty}_{\Lambda_{a,b}}) = 0 \forall a < b$

$\Rightarrow \mathbb{P}\left(\bigcup_{a<b,a,b\in\mathbb{Q}} \Lambda_{a,b}\right) = 0 \Rightarrow \mathbb{P}\left(\limsup_{t\to\infty} X_t > \liminf_{t\to\infty} X_t\right) = 0.$ □



Remark: *Finally one can also verify that X_∞ is a.s. finite.*

$$\mathbb{E}\left[|X_\infty|\right] \le \liminf_{t\to\infty} \mathbb{E}\left[|X_t|\right] \overset{?}{<} \infty \tag{4.29}$$

by using Fatou. Regarding "?":

$$\mathbb{E}\left[|X_t|\right] = 2\mathbb{E}\left[X_t^+\right] - \mathbb{E}\left[X_t\right] \le 2C - \mathbb{E}\left[X_0\right] < \infty \tag{4.30}$$

because $\mathbb{E}[X_t] \ge \mathbb{E}[X_0]$ (since X_t is a submartingale)

In the exercise we will show

Theorem 4.10.
Let X be a right-continuous, positive submartingale (resp. martingale). Then we have 3 equivalent statements

1. $\lim_{t\to\infty} X_t$ exists in L^1.
2. $\{X_t, t \in [0,\infty)\}$ is uniformely integrable
3. $\exists X_\infty \in L^1$ s.t. $X_\infty = \lim_{t\to\infty} X_t$ a.s. and $(X_t)_{t\in[0,\infty]}$ is a submartingale (resp. martingale) w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty]}$.

Remark: *For the case of a martingale, $\exists X_\infty \in L^1$ s.t. $X_t = \mathbb{E}[X_\infty|\mathcal{F}_t]$ a.s.*

Remark (So nicht in der Vorlesung)**:** *Es gilt:*

$$\{X_t : t \in [0,\infty)\} \textit{ unif. integ.} \Leftrightarrow \begin{cases} \{X_t : t \in [0,\infty)\} \textit{ unif. bounded in } L^1 \textit{ and} \\ \forall \varepsilon > 0 \exists \delta > 0 : \forall A \in \mathcal{F} : \mathbb{P}(A) < \delta \Rightarrow \sup_t \mathbb{E}\left[|X_t| \mathbb{1}_A\right] < \varepsilon \end{cases} \tag{4.31}$$

Angenommen $\sup_t \mathbb{E}[|X_t|^p] \le C < \infty$ für ein $p > 1$. Dann sind die beiden rechten Bedingungen erfüllt.

$$\sup_t \mathbb{E}\left[|X_t|\right]^p \le \sup_t \mathbb{E}\left[|X_t|^p\right] < \infty \Rightarrow \sup_t \mathbb{E}\left[|X_t|\right] < \infty \tag{4.32}$$

$$\mathbb{E}\left[|X_t| \mathbb{1}_A\right] \overset{\text{Hölder}}{\le} \mathbb{E}\left[|X_t|^p\right]^{1/p} \mathbb{E}\left[|\mathbb{1}_A|^{p'}\right]^{1/p'} \le C \cdot \mathbb{P}(A)^{1/p'} \overset{\mathbb{P}(A)\to 0}{\longrightarrow} 0 \tag{4.33}$$

Somit sind die Vorraussetzungen für das obige Theorem erfüllt! Tatsächlich gilt sogar $X_t \to X_\infty$ in L^p.

4.5 Optional Sampling

For a submartingale X it holds

$$X_s \leq \mathbb{E}[X_t|\mathcal{F}_s] \text{ a.s.} \tag{4.34}$$

We now want a generalisation for s, t two stopping times.

Theorem 4.11 (Optional Sampling)**.**
Let X be a right-continuous submartingale w.r.t $(\mathcal{F}_t)_{t\geq 0}$ and S, T two bounded stopping times satisfying $S \leq T$.

$$\Rightarrow X_S \leq \mathbb{E}[X_T|\mathcal{F}_S] \text{ a.s.} \tag{4.35}$$

Remark: *To verify $X_S \leq \mathbb{E}[X_T|\mathcal{F}_S]$ a.s. we have to show that $\forall A \in \mathcal{F}_S$*

$$\int_A X_S \, d\mathbb{P} \leq \int_A X_T \, d\mathbb{P} \overset{def}{\equiv} \int_A \mathbb{E}[X_T|\mathcal{F}_S] \, d\mathbb{P} \tag{4.36}$$

Proof. $\exists t_0$ s.t. $S \leq T \leq t_0$. Assume that $X_S \leq \mathbb{E}[X_T|\mathcal{F}_S]$ holds for $X_t \geq 0$. $\Rightarrow$ for $X_t \geq -m \Rightarrow Y_t := X_t + m \geq 0$ by linearity $\Rightarrow$ statement holds $\forall X_t \geq -m$. $\Rightarrow X_t^{(m)} := X_t \vee (-m)$. Monotone convergence gives that it is always true.

A simple bound $\mathbb{E}[X_T] \leq \mathbb{E}[X_{t_0}] < \infty$.

a) Discrete approximation.
We define

$$T_n := \frac{k+1}{2^n} \text{ if } \frac{k}{2^n} \leq T < \frac{k+1}{2^n} \text{ for a } k \geq 0. \tag{4.37}$$

Similarily define S_n. It is clear that $T \leq T_n \forall n$ and $T_n \geq T_{n+1} \geq \ldots$. Is T_n a stopping time?

$$\{T_n \leq t\} = \underbrace{\left\{T < \frac{\lceil 2^n t \rceil}{2^n}\right\}}_{\in \mathcal{F}_t} \cap \underbrace{\left\{T < \frac{\lceil 2^n t \rceil - 1}{2^n}\right\}^c}_{\in \mathcal{F}_t} \in \mathcal{F}_t \quad \checkmark \tag{4.38}$$

Also $\forall n : T_n \geq S_n$. Using that X is right-continuous it follows that

$$\lim_{n\to\infty} X_{S_n} = X_S \text{ and } \lim_{n\to\infty} X_{T_n} = X_T \tag{4.39}$$

b) Show: $X_{T_n} \leq \mathbb{E}[X_{t_0}|\mathcal{F}_{T_n}]$.
Take $K_n := \lceil t_0 2^n \rceil$.

$$\Rightarrow \mathbb{E}[X_{t_0}|\mathcal{F}_{T_n}] = \sum_{l=1}^{K_n} \mathbb{E}\left[X_{t_0}|T_n = \frac{l}{2^n}\right] \mathbb{1}_{[T_n = \frac{l}{2^n}]} \tag{4.40}$$

$$\overset{submart.}{\geq} \sum_{l=1}^{K_n} X_{\frac{l}{2^n}} \mathbb{1}_{[T_n = \frac{l}{2^n}]} = X_{T_n} \tag{4.41}$$

$\Rightarrow \{X_{T_n} : n \in \mathbb{N}\}$ is uniformely integrable, since $\{\mathbb{E}[X_{t_0}|\mathcal{F}_{T_n}] : n \in \mathbb{N}\}$ unif. integ.

$$\Rightarrow \lim_{n\to\infty} X_{T_n} = X_T \in L^1 \tag{4.42}$$

(analogue for S_n).

c) Show: $\forall A \in \mathcal{F}_{S_n}$:

$$\int_A X_{S_n} d\mathbb{P} \leq \int_A X_{T_n} d\mathbb{P} \tag{4.43}$$

Too see this: Let

$$A_j = A \cap \{S_n = \frac{j}{2^n}\} \in \mathcal{F}_{\frac{j}{2^n}} \tag{4.44}$$

$\Rightarrow \forall k \geq j : A_j \cap \{T_n > \frac{k}{2^n}\} \in \mathcal{F}_{\frac{k}{2^n}}$

$$\Rightarrow \int_{A_j \cap \{T_n \geq \frac{k}{2^n}\}} X_{\frac{k}{2^n}} d\mathbb{P} \overset{submart.}{\leq} \int_{A_j \cap \{T_n = \frac{k}{2^n}\}} X_{T_n} d\mathbb{P} + \int_{A_j \cap \{T_n \geq \frac{k+1}{2^n}\}} X_{\frac{k+1}{2^n}} d\mathbb{P} \tag{4.45}$$

Starting with $k = j$ and iterating:

$$\int_{A_j} X_{S_n} d\mathbb{P} = \int_{A_j \cap \{T_n \geq \frac{j}{2^n}\}} X_{\frac{j}{2^n}} d\mathbb{P} \leq \int_{A_j \cap \{T_n \geq \frac{j}{2^n}\}} X_{T_n} d\mathbb{P} \tag{4.46}$$

Now $\sum_j \Rightarrow$ **c)**
d) $\forall A \in \mathcal{F}_S \subset \cap_{n \geq 1} \mathcal{F}_{S_n}$

$$\Rightarrow \int_A X_{S_n} d\mathbb{P} \leq \int_A X_{T_n} d\mathbb{P} \tag{4.47}$$

Now take $\lim_{n \to \infty}$

$$\Rightarrow \forall A \in \mathcal{F}_S \int_A X_S d\mathbb{P} \leq \int_A X_T d\mathbb{P} \tag{4.48}$$

□

[30.10.2012]
[02.11.2012]

Corollary 4.12.
Let X a right-continuous adapted process and integrable. Then the following statements are equivalent:

(i) X is a martingale.

(ii) For all bounded stopping times T it holds $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Proof. "⇒" Using 2.11 with $S = 0$ we get

$$\mathbb{E}[X_T] = \mathbb{E}[\mathbb{E}[X_T|\mathcal{F}_0]] \geq \mathbb{E}[X_0] \tag{4.49}$$

But also the other inequality holds, since $-X_t$ is a submartingale, too.
"⇐" To show $\forall s < t, A \in \mathcal{F}_s$

$$\mathbb{E}[X_s \mathbb{1}_A] = \mathbb{E}[X_t \mathbb{1}_A] \tag{4.50}$$

Define two stopping times as follows: Let $T(\omega) := t$ and

$$S(\omega) := \begin{cases} s & , \omega \in A \\ t & , \text{otherwise} \end{cases} \tag{4.51}$$

Let us compute

$$\mathbb{E}[X_0] \overset{\text{hyp}}{=} \mathbb{E}[X_T] = \mathbb{E}[X_t \mathbb{1}_A] + \mathbb{E}[X_t \mathbb{1}_{A^c}] \tag{4.52}$$

but also

$$\mathbb{E}[X_0] \overset{\text{hyp}}{=} \mathbb{E}[X_S] = \mathbb{E}[X_s \mathbb{1}_A] + \mathbb{E}[X_t \mathbb{1}_{A^c}] \;\Rightarrow\; \mathbb{E}[X_s \mathbb{1}_A] = \mathbb{E}[X_t \mathbb{1}_A] \, \forall A \in \mathcal{F}_s, s < t, \tag{4.53}$$

i.e. $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$ a.s. □

Corollary 4.13.
Let X be right-continuous, adapted and integrable. Then X is a submartingale $\Leftrightarrow$ $\forall$ bounded stopping times $S \le T$ it holds

$$\mathbb{E}[X_S] \le \mathbb{E}[X_T] \tag{4.54}$$

Proof. "$\Rightarrow$"

$$\mathbb{E}[X_T] \overset{\mathcal{F}_S \subseteq \mathcal{F}_T}{=} \mathbb{E}[\mathbb{E}[X_T|\mathcal{F}_S]] \overset{4.11}{\ge} \mathbb{E}[X_S] \tag{4.55}$$

"$\Leftarrow$" Let $s < t, A \in \mathcal{F}_s$ define S and T as in the previous proof.

$$\Rightarrow \mathbb{E}[X_S] \overset{hyp}{\le} \mathbb{E}[X_T] = \mathbb{E}[X_t \mathbb{1}_A] + \mathbb{E}[X_t \mathbb{1}_{A^c}] \tag{4.56}$$

But the right side is

$$\mathbb{E}[X_S] = \mathbb{E}[X_s \mathbb{1}_A] + \mathbb{E}[X_t \mathbb{1}_{A^c}] \tag{4.57}$$

$\Rightarrow \mathbb{E}[X_s \mathbb{1}_A] \le \mathbb{E}[X_t \mathbb{1}_A], \forall s < t, A \in \mathcal{F}_s$. □

Corollary 4.14 (Optional Stopping)**.**
Let X be a (sub-)martingale and T a stopping time. Then,

$$X_t^T(\omega) \equiv X_{T(\omega)\wedge t}(\omega) \tag{4.58}$$

is also a (sub-)martingale.

Proof. Let $s < t$. Define $S = s \wedge T$ and $U = t \wedge T$. Then by definition $S \le U$. By Theorem 4.11 we get $X_S \le \mathbb{E}[X_U|\mathcal{F}_S]$. If we do the same for $-X$ we have $X_S = \mathbb{E}[X_U|\mathcal{F}_S]$ and thus $X_{s\wedge T} = \mathbb{E}[X_{t\wedge T}|\mathcal{F}_{s\wedge T}]$.

□

Next goal: Understand what is

$$\int_0^t f(B_s)dB_s = ? \tag{4.59}$$

with B a Brownian Motion. We will see

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \tag{4.60}$$

where ds will be the quadratic variation of B.

5 Continuous semimartingales and quadratic variation

5.1 Semimartingales

Definition 5.1.

a) $X \in \mathcal{A}^+$: An adapted process X is called *continuous and increasing* if for almost all $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ is continuous and increasing.

b) $X \in \mathcal{A}$: An adapted process is called *continuous with bounded variation* if for almost all $\omega \in \Omega$: $t \mapsto X_t(\omega)$ is continuous and has finite variation, i.e.

$$\forall t \geq 0, S_t(\omega) \equiv S_t(X(\omega)) := \sup_{0 \leq t_0 \leq \ldots \leq t_n \leq t, n \in \mathbb{N}} \sum_{k=1}^{n} |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)| < \infty \tag{5.1}$$

c) $X \in \mathcal{M}$: X is a continuous martingale.

d) $X \in \mathcal{M}_{loc}$: An adapted process X is a *local, continuous martingale* if $\exists$ a sequence of stopping times $T_1 \leq T_2 \leq \ldots$ with $\lim_{n\to\infty} T_n = \infty$ a.s. and X^{T_n} is a martingale $\forall n \geq 1$.

Lemma 5.2.
$X \in \mathcal{A} \Leftrightarrow X = Y - Z$ with $Y, Z \in \mathcal{A}^+$.

Proof. Take $Y = \frac{S+X}{2}$ and $Z = \frac{S-X}{2}$ where S is the variation of X. □

Lemma 5.3.

a) $X \in \mathcal{M} \Rightarrow X \in \mathcal{M}_{loc}$

b) $X \in \mathcal{M}_{loc}, X \geq 0 \Rightarrow X$ supermartingale.

c) $X \in \mathcal{M}_{loc}$ and X is bounded $\Rightarrow X \in \mathcal{M}$.

d) $X \in \mathcal{M} \Leftrightarrow X \in \mathcal{M}_{loc}$ and $\forall t \geq 0 : \{X_{T \wedge t} : T$ stopping time$\}$ is uniformely integrable.

Remark: $\exists X \in \mathcal{M}_{loc}, X$ *uniformely integrable s.t.* $X \notin \mathcal{M}$. *(ex. 3.36 in Karatzas, Shreve)*

Proof. **ad a)** Take as sequence of stopping times

$$T_n = \infty \forall n \geq 1. \tag{5.2}$$

ad b) $\forall s < t$:

$$\mathbb{E}[X_t|\mathcal{F}_s] \underset{1}{=} \mathbb{E}\left[\lim_{n\to\infty} X_{T_n \wedge t}|\mathcal{F}_s\right] \overset{Fatou}{\leq} \liminf_{n\to\infty} \mathbb{E}[X_{T_n\wedge t}|\mathcal{F}_s] \overset{X^{T_n}\in\mathcal{M}}{=} \liminf_{n\to\infty} X_{T_n \wedge s} = X_s \text{ a.s..} \tag{5.3}$$

[1]There exist $T_n \nearrow \infty$ s.t. X^{T_n} is martingale

ad c) We have $|X| \leq C < \infty$, therefore $C - X \geq 0, C + X \geq 0$. Using b) we get $C - X$ is a supermartingale and $C + X$ is a supermartingale. $\Rightarrow \pm X$ are both supermartingales $\Rightarrow X$ is a martingale.

ad d) "$\Rightarrow$": Let $X \in \mathcal{M}$. From a): $X \in \mathcal{M}_{loc}$. Let T be any stopping time and $t \in \mathbb{R}_+$ fixed. To show: $\mathbb{E}[|X_{T\wedge t}|] \leq C$ uniformely in T.

$$\mathbb{E}[|X_{T\wedge t}|] \overset{X\in\mathcal{M}}{=} \mathbb{E}[|\mathbb{E}[X_t|\mathcal{F}_{T\wedge t}]|] \overset{Jensen}{\leq} \mathbb{E}[\mathbb{E}[|X_t||\mathcal{F}_{T\wedge t}]] \leq \mathbb{E}[|X_t|] < \infty \tag{5.4}$$

The bound is uniformely in T.

"$\Leftarrow$": By assumption, $\exists$ a sequence of $T_n \nearrow \infty$ of stopping times s.t. $X^{T_n} \in \mathcal{M}$. Let T be a bounded stopping time. By Cor. 4.12 we have

$$\mathbb{E}[X_{T_n\wedge t\wedge T}] = \mathbb{E}[X_0] \tag{5.5}$$

$$\Rightarrow \mathbb{E}[X_0] = \lim_{n\to\infty} \mathbb{E}[X_{T_n\wedge T\wedge t}] \underset{\text{unif. integ.}}{=} \mathbb{E}\left[\lim_{n\to\infty} X_{T_n\wedge T\wedge t}\right] = \mathbb{E}[X_{T\wedge t}]\,\forall t \geq 0. \tag{5.6}$$

$\Rightarrow$ for all bounded T (by taking $t > T$) $\mathbb{E}[X_0] = \mathbb{E}[X_T]$. $\overset{4.12}{\Rightarrow}$ X is a martingale. □

Definition 5.4 (Semimartingale)**.**
$X \in \mathcal{S}$: A process X is called a *continuous semimartingale* if $\exists M \in \mathcal{M}_{loc}$ and $A \in \mathcal{A}$ s.t.

$$X = M + A. \tag{5.7}$$

Theorem 5.5.
Let $\mathcal{M}^0_{loc} := \{X \in \mathcal{M}_{loc} : X_0 = 0 \text{ a.s.}\}$. Then,

$$\mathcal{M}^0_{loc} \cap \mathcal{A} = \{0\} \tag{5.8}$$

and $\mathcal{S} = \mathcal{M}^0_{loc} \oplus \mathcal{A}$.

[02.11.2012]

[06.11.2012]

Remark: *Recall Doob for p=2:* $\mathbb{E}\left[\sup_{t\geq 0} X_t^2\right] \leq 4\mathbb{E}\left[X_\infty^2\right]$

Proof. Assume that we can prove that

$$\text{if } X \in \mathcal{M}^0 \cap \mathcal{A} \Rightarrow X = 0 \text{ a.s.}. \tag{5.9}$$

Then, by the definition of $\mathcal{M}_{loc}$ there exist $T_1 \leq T_2 \leq \ldots$ stopping times with $T_n \nearrow \infty$ a.s. s.t. $X^{T_n} \in \mathcal{M}$. Now let $X \in \mathcal{M}^0_{loc} \cap A \Rightarrow X^{T_n} \in \mathcal{M}^0 \cap \mathcal{A} \overset{(5.9)}{\Rightarrow} X_{T_n\wedge t} = 0$ but since $\forall t \geq 0 \lim_{n\to\infty} T_n \wedge t = t$ a.s. it holds $X_t = 0 \forall t \geq 0$.

We will now show, that (5.9) holds. So let $X \in \mathcal{M}^0 \cap \mathcal{A}$. We can also restrict ourself to processes X s.t. X is bounded and $S_\infty(X) < \infty$. Indeed, we can introduce stopping times

$$T'_n := \inf\{t > 0 : |X_t| > n \text{ or } S_t(x) > n\}. \tag{5.10}$$

Then $X^{T'_n}$ is bounded with finite variation. $\Rightarrow X^{T'_n} \in \mathcal{M}^0 \cap \mathcal{A} \forall n \overset{(5.9)}{\Rightarrow} X^{T'_n} = 0 \forall n \Rightarrow X = 0$.

Now show (5.9) for X bounded and $S_\infty(X) < \infty$. Let $\varepsilon > 0$.

$$T_0 := 0 \tag{5.11}$$

$$T_{k+1} := \inf\{t \geq T_k : |X_k - X_{T_k}| > \varepsilon\}. \tag{5.12}$$

Since X is continuous and $X \in \mathcal{A} \Rightarrow \lim_{k\to\infty} T_k = \infty$.

$$\mathbb{E}\left[X_{T_n}^2\right] = \mathbb{E}\left[\sum_{k=0}^{n-1}(X_{T_{k+1}}^2 - X_{T_k}^2)\right] \tag{5.13}$$

$$= \mathbb{E}\left[\sum_{k=0}^{n-1}(X_{T_{k+1}} - X_{T_k})^2\right] + 2\sum_{k=0}^{n-1} \underbrace{\mathbb{E}\left[X_{T_k}(X_{T_{k+1}} - X_{T_k})\right]}_{\mathbb{E}\left[X_{T_k}\mathbb{E}\left[X_{T_k}-X_{T_{k+1}}|\mathcal{F}_{T_k}\right]\right] \overset{Mart.}{=} 0} \tag{5.14}$$

$$\le \varepsilon\mathbb{E}\left[\sum_{k=0}^{n-1}|X_{T_{k+1}} - X_{T_k}|\right] \tag{5.15}$$

$$\le \varepsilon \cdot S_\infty(X) \to 0 \text{ as } \varepsilon \to 0 \tag{5.16}$$

$$\Rightarrow \mathbb{E}\left[X_{T_n}^2\right] = 0 \tag{5.17}$$

By taking $n \to \infty$ we get

$$0 \le \mathbb{E}\left[X_\infty^2\right] = \mathbb{E}\left[\liminf_{n\to\infty} X_{T_n}^2\right] \le \liminf_{n\to\infty} \mathbb{E}\left[X_{T_n}^2\right] = 0 \tag{5.18}$$

and thus $\mathbb{E}\left[X_\infty^2\right] = 0$. Using Doob Max inequality (p=2):

$$\mathbb{E}\left[\sup_{t\ge 0} X_t^2\right] \le 4\mathbb{E}\left[X_\infty^2\right] = 0 \tag{5.19}$$

Therefore $X = 0$ a.s.. □

5.2 Doob-Meyer decomposition

Theorem 5.6.
Let X be a continuous $\overset{\text{(sub-/)}}{\text{super}}$martingale, then $\exists M \in \mathcal{M}_{loc}^0$ and $A \in \mathcal{A}^+$ s.t.

$$X_t = M_t \overset{(+)}{-} A_t \tag{5.20}$$

Moreover, M and A are unique (up to indistinguishability).

Hints for the proof: **Uniqueness:** Assume $X_t = M_t - A_t = M_t' - A_t' \Rightarrow \underbrace{M_t - M_t'}_{\in\mathcal{M}_{loc}^0} = \underbrace{A_t - A_t'}_{\in\mathcal{A}} \overset{5.5}{=} 0$ a.s.

Existence in discrete time case: Let $(X_n)_{n\ge1}$ be a discrete time supermartingale $\Rightarrow Y_n := \mathbb{E}\left[X_n - X_{n+1}|\mathcal{F}_n\right] \ge 0$. Then define $A_n := \sum_{k=0}^{n-1} Y_k \Rightarrow$ is increasing in n, and it is $\mathcal{F}_{n-1}-$measurable and $M_n = X_n + A_n$ is a Martingale. Show for the case $m = n-1$:

$$\mathbb{E}\left[X_n + A_n|\mathcal{F}_{n-1}\right] = \mathbb{E}\left[X_n|\mathcal{F}_{n-1}\right] + \sum_{k=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[X_k - X_{k+1}|\mathcal{F}_k\right]|\mathcal{F}_{n-1}\right] \tag{5.21}$$

$$= \mathbb{E}\left[X_n|\mathcal{F}_{n-1}\right] + \mathbb{E}\left[X_{n-1} - X_n|\mathcal{F}_{n-1}\right] + \sum_{k=0}^{n-2}\mathbb{E}\left[X_k - X_{k+1}|\mathcal{F}_k\right] \tag{5.22}$$

$$= X_{n-1} + A_{n-1} \tag{5.23}$$

□

Corollary 5.7.
Continuous Supermartingales (and Submartingales) are continuous semi-martingales.

Proof. Let X be a continuous supermartingale. By Theorem 5.6 $X = M - A$ where $M \in \mathcal{M}_{loc}^0$ and $A \in \mathcal{A}^+$. By Lemma 5.2 we have $(-A) \in \mathcal{A}$. Therefore $X \in \mathcal{S}$. □

5.3 Quadratic Variation

Definition 5.8 (Preliminary)**.**
Let X be a stochastic processs. Then the quadratic variation of X is defined by

$$Q_t(X)(\omega) := \lim_{||\Delta||\to 0} \sum_{k=1}^{n} |X_{t_k}(\omega) - X_{t_{k-1}}(\omega)|^2 \tag{5.24}$$

where $\Delta = \{0 = t_0 \leq t_1 \leq \ldots \leq t_n = t\}$ is a partition of $[0, t]$ with ”mash-size”

$$||\Delta|| = \max_{0\leq k\leq n-1} (t_{k+1} - t_k). \tag{5.25}$$

We know that for $X = B \equiv$ Brownian Motion:

$$Q_t(B) = t \text{ in } L^2, \tag{5.26}$$

(see Lemma 2.5)

Theorem 5.9.

a) $\forall M \in \mathcal{M}_{loc}, \exists! \langle M\rangle \in \mathcal{A}_0$ s.t. $M^2 - M_0^2 - \langle M\rangle \in \mathcal{M}_{loc}^0$.

b) $\forall M, N \in \mathcal{M}_{loc}, \exists! \langle M, N\rangle \in \mathcal{A}_0$ s.t. $M \cdot N - M_0 \cdot N_0 - \langle M, N\rangle \in \mathcal{M}_{loc}^0$.

(uniqueness up to indistinguishability)

Proof. **a)** Let $M \in \mathcal{M}_{loc} \Rightarrow M^2$ is a local submartingale. By the Doob-Meyer-decomposition, $\exists A \in \mathcal{A}_0$ s.t. $M^2 = M' + A$ with $M' \in \mathcal{M}_{loc}$. We now define $\langle M\rangle := A \Rightarrow M' = M^2 - \langle M\rangle \in \mathcal{M}_{loc}$ and since $\langle M\rangle_0 = 0$ we also get $M^2 - M_0^2 - \langle M\rangle \in \mathcal{M}_{loc}^0$

b) Just use the polarisation identity

$$M \cdot N = \frac{1}{4}((M + N)^2 - (M - N)^2) \tag{5.27}$$

□

Example: *For a Brownian Motion B, we already know that*

$$B_t^2 - t \tag{5.28}$$

is a martingale and $t \mapsto t$ *is in* $\mathcal{A}_0$. $\Rightarrow$ *5.9 implies:* $\langle B\rangle_t = t$. *We also know:* $Q_t(B) = t$ *and this is* ***not*** *an accident.*

Definition 5.10 (Final version of Definition 5.8)**.**

a) $\langle M\rangle \equiv \langle M, M\rangle$ is called the quadratic variation of M.

b) $\langle M, N\rangle$ is called the covariation of M and N.

Remark: *It holds* $\langle M, N\rangle = \frac{1}{4}(\langle M + N\rangle - \langle M - N\rangle)$

Some properties:

Lemma 5.11.
$\forall M, N \in \mathcal{M}_{loc}$ it holds

a) $\langle \cdot, \cdot\rangle$ is symmetric, billinear, positive definit.

b) For all stopping times T it holds $\langle M, N\rangle^T = \langle M^T, N^T\rangle$.

c) $\langle M\rangle = \langle M - M_0\rangle$

d) $\langle M\rangle = 0 \Leftrightarrow M$ is a constant.

Proof. **ad a)** easy, use also (d).

ad b) Show $\langle M\rangle^T = \langle M^T\rangle$ and use the remark before the Lemma.

$$\underbrace{(M^2 - M_0^2 - \langle M\rangle)}_{\in\mathcal{M}_{loc}}{}^T = (M^T)^2 - M_0^2 - \langle M\rangle^T \in \mathcal{M}_{loc} \text{ (Cor 4.14)} \tag{5.29}$$

but there $\exists!\langle M^T\rangle$ s.t.

$$(M^T)^2 - M_0^2 - \langle M^T\rangle \in \mathcal{M}_{loc} \tag{5.30}$$

$\Rightarrow \langle M^T\rangle = \langle M\rangle^T$.

ad c) and d) We can assume $M - M_0$ bounded (otherwise use $T_n = \inf\{t > 0 : |M - M_0| > n^2\}$ and b)). Therefore (by 5.3 (c)) $M - M_0 \in \mathcal{M}$.

ad c) By Theorem 5.9 $\exists!\langle M - M_0\rangle \in \mathcal{A}_0$ s.t. $(M - M_0)^2 - \langle M - M_0\rangle \in \mathcal{M}^0$ but we also have

$$(M - M_0)^2 - \langle M\rangle = \underbrace{M^2 - M_0^2 - \langle M\rangle}_{\in\mathcal{M}^0} - \underbrace{2M_0(M - M_0)}_{\in\mathcal{M}^0?} \tag{5.31}$$

If $M_0(M - M_0) \in \mathcal{M}^0$, then $(M - M_0) - \langle M\rangle \in \mathcal{M}^0$. Therefore by uniqueness $\langle M\rangle = \langle M - M_0\rangle$.

Regarding $M_0(M - M_0) \in \mathcal{M}^0$, $\forall 0 \le s \le t$:

$$\mathbb{E}\left[M_0(M_t - M_0)|\mathcal{F}_s\right] = M_0\mathbb{E}\left[M_t - M_0|\mathcal{F}_s\right] \overset{M-M_0\in\mathcal{M}}{=} M_0(M_s - M) \tag{5.32}$$

Therefore $M_0(M - M_0) \in \mathcal{M}^0$.

ad d) "$\Rightarrow$": $\langle M\rangle = 0$ on $[0, t] \overset{(c)}{\Rightarrow} (M - M_0)^2 \in \mathcal{M}$ on $[0, t]$, since

$$(M - M_0)^2 - \langle M - M_0\rangle \in \mathcal{M} \tag{5.33}$$
$$\Rightarrow (M - M_0)^2 - \langle M\rangle \in \mathcal{M} \tag{5.34}$$
$$\Rightarrow (M - M_0)^2 \in \mathcal{M} \tag{5.35}$$

$$\Rightarrow \mathbb{E}\left[\sup_{0\le s\le t}(M_s - M_0)^2\right] \overset{\text{Doob}}{\le} 4\mathbb{E}\left[(M_t - M_0)^2\right] = 0 \text{ since } (M - M_0)^2 \in \mathcal{M} \tag{5.36}$$

$\Rightarrow M$ is constant on $[0, t], \forall t \ge 0 \Rightarrow M$ is constant. □

Example: *Let X be continuous, adapted process, $X_t \in L^2$ with independent and centered increments. Then,*

a) $X \in \mathcal{M}$ *and*

b) $\langle X\rangle_t = Var(X_t - X_0) \equiv \mathbb{E}\left[(X_t - X_0)^2\right]$ *a.s.*

Indeed:

a)

- adapted ✓

- $\mathbb{E}\left[|X_t| < \infty\right], \forall t \ge 0$ ✓ *since it even holds* $\mathbb{E}\left[|X_t|^2\right] < \infty \forall t \ge 0$.

- For $0 \le s \le t$*:* $\mathbb{E}\left[X_t|\mathcal{F}_s\right] = \mathbb{E}\left[X_t - X_s|\mathcal{F}_s\right] + X_s = \mathbb{E}\left[X_t - X_s\right] + X_s = X_s$

b)

- It holds

$$\mathbb{E}\left[X_t^2 - X_0^2 - \mathbb{E}\left[(X_t - X_0)^2\right]\right] = \mathbb{E}\left[X_t^2 - X_0^2 - \mathbb{E}\left[X_t^2 - X_0^2 - 2X_0(X_t - X_0)\right]\right] \tag{5.37}$$
$$= 2\mathbb{E}\left[X_0(X_t - X_0)\right] = 0 \tag{5.38}$$

since $X_0(X_t - X_0)$ *is a Martingale).* $\overset{a)}{\Rightarrow} X_t^2 - X_0^2 - \mathbb{E}\left[(X_t - X_0)^2\right] \in \mathcal{M}^0$*, i.e.* $\mathbb{E}\left[(X_t - X_0)^2\right] = \langle X\rangle_t$.

[06.11.2012]
[09.11.2012]

Definition 5.12.
For a partition $\Delta = \{t_0, t_1, ...\}$ with $t_k \to \infty$ and $0 = t_0 \leq t_1 \leq t_2..$ and a stochastic process X the *quadratic variation of X on Δ* is defined by

$$Q_t^\Delta = \sum_{k\geq 1} |X_{t\wedge t_k} - X_{t\wedge t_{k-1}}|^2 \tag{5.39}$$

The quantity

$$||\Delta|| := \sup_{k\geq 1} |t_k - t_{k-1}| \tag{5.40}$$

is the *mesh-size* of Δ.

Theorem 5.13.
Let $M \in \mathcal{M}_{loc}$ and $t \geq 0$. Then,

$$\lim_{||\Delta||\to 0} Q_t^\Delta = \langle M\rangle_t \text{ stochastically.} \tag{5.41}$$

i.e., $\forall \varepsilon > 0, \eta > 0, t \geq 0, \exists \delta > 0$ s.t.

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |Q_s^\Delta - \langle M\rangle_s| > \varepsilon\right) < \eta \tag{5.42}$$

holds $\forall \Delta$ with $||\Delta|| < \delta$.

To prove this we need one technical lemma.

Lemma 5.14.

a) Let $(A_n)_{n\geq 0}$ be an increasing process with
   - $A_0 = 0$
   - A_n is $\mathcal{F}_n$-measurable.

   Then if $\mathbb{E}[A_\infty - A_n|\mathcal{F}_n] \leq K, \forall n \geq 0, \Rightarrow \mathbb{E}\left[A_\infty^2\right] \leq 2K^2$.

b) Let $A^{(1)}$ and $A^{(2)}$ as in a) and $B := A^{(1)} - A^{(2)}$. Then, if $\exists$ a r.v. $W \geq 0$ with $\mathbb{E}\left[W^2\right] < \infty$ and $|\mathbb{E}[B_\infty - B_n|\mathcal{F}_n]| \leq \mathbb{E}[W|\mathcal{F}_n]$, there $\exists c > 0$ s.t.

$$\mathbb{E}\left[\sup_{n\geq 0} B_n^2\right] \leq c\left(\mathbb{E}\left[W^2\right] + K\sqrt{\mathbb{E}\left[W^2\right]}\right) \tag{5.43}$$

Proof. **ad a)** Define $a_n := A_{n+1} - A_n \geq 0$ since A_n is increasing.

$$\Rightarrow A_\infty^2 \overset{A_0=0}{=} \left(\sum_{n\geq 0} a_n\right)^2 = \sum_{m,n\geq 0} a_n a_m = \sum_{n\geq 0} a_n^2 + 2\sum_{n\geq 0}\left(a_n \underbrace{\sum_{m\geq n+1} a_m}_{=A_\infty - A_{n+1} = A_\infty - A_n - a_n}\right) \tag{5.44}$$

$$= 2\sum_{n\geq 0} a_n(A_\infty - A_n) \tag{5.45}$$

$$\mathbb{E}\left[A_\infty^2\right] \le 2\sum_{n\ge 0}\mathbb{E}\left[\mathbb{E}\left[a_n(A_\infty - A_n)|\mathcal{F}_n\right]\right] = 2\sum_{n\ge 0}\mathbb{E}\left[a_n \underbrace{\mathbb{E}\left[A_\infty - A_n|\mathcal{F}_n\right]}_{\le K}\right] \tag{5.46}$$

$$\le 2K\sum_{n\ge 0}\mathbb{E}\left[a_n\right] \le 2K\mathbb{E}\left[A_\infty\right] = 2K\mathbb{E}\left[A_\infty - A_0\right] = 2K\mathbb{E}\left[\mathbb{E}\left[A_\infty - A_0|F_0\right]\right] \le 2K^2 \tag{5.47}$$

ad b) Let $b_n := B_{n+1} - B_n, a_n^{(i)} := A_{n+1}^{(i)} - A_n^{(i)}$.

$$\mathbb{E}\left[B_\infty^2\right] \le 2\mathbb{E}\left[\sum_{n\ge 0}\underbrace{\mathbb{E}\left[B_\infty - B_n|\mathcal{F}_n\right]}_{|\ |\le \mathbb{E}[W|\mathcal{F}_n]} b_n\right] \tag{5.48}$$

$$\overset{|b_n|\le a_n^{(1)}+a_n^{(2)}}{\le} 2\mathbb{E}\left[\mathbb{E}\left[\sum_{n\ge 0} W(a_n^{(1)} - a_n^{(2)})|\mathcal{F}_n\right]\right] \tag{5.49}$$

$$= 2\mathbb{E}\left[W(A_\infty^{(1)} + A_\infty^{(2)})\right] \tag{5.50}$$

$$\overset{C.S.}{\le} 2\mathbb{E}\left[W^2\right]^{1/2}(\underbrace{\mathbb{E}\left[(A_\infty^{(1)})^2\right]^{1/2}}_{\le\sqrt{2}K} + \underbrace{\mathbb{E}\left[(A_\infty^{(2)})\right]^{1/2}}_{\le\sqrt{2}K}) \le 4\sqrt{2}\mathbb{E}\left[W^2\right]^{1/2}K \tag{5.51}$$

Now we introduce the martingales

$$M_n := \mathbb{E}\left[B_\infty|\mathcal{F}_n\right] \tag{5.52}$$

and

$$W_n := \mathbb{E}\left[W|\mathcal{F}_n\right] \tag{5.53}$$

and set

$$X_n := M_n - B_n \tag{5.54}$$

Since $|B_n|^2 \le 2(|X_n| + |M_n|^2)$ We have to compute/bound $|X_n|$

$$|X_n| = |\mathbb{E}\left[B_\infty - B_n|\mathcal{F}_n\right]| \tag{5.55}$$

$$\le \mathbb{E}\left[W|\mathcal{F}_n\right] \equiv W_n \tag{5.56}$$

$$\mathbb{E}\left[\sup_{n\ge 0}|B_n|^2\right] \le 2\mathbb{E}\left[\sup_{n\ge 0}|X_n|^2 + \sup_{n\ge 0}|M_n|^2\right] \tag{5.57}$$

$$\le 2\mathbb{E}\left[\sup_{n\ge 0}W_n^2\right] + 2\mathbb{E}\left[\sup_{n\ge 0}|M_n|^2\right] \tag{5.58}$$

$$\overset{Doobmaxineq.}{\le} 8(\underbrace{\mathbb{E}\left[W_\infty^2\right]}_{\mathbb{E}[W^2]} + \underbrace{= \mathbb{E}\left[B_\infty^2\right]}_{\le 2\sqrt{2}K\mathbb{E}[W^2]^{1/2}}) \tag{5.59}$$

$$\le \tilde{c}(\mathbb{E}\left[W^2\right] + K\mathbb{E}\left[W^2\right]^{1/2}) \tag{5.60}$$

□

Proof of the theorem. Let $M \in \mathcal{M}_{loc}, t \ge 0$ fixed. Let $\Delta = \{t_0, t_1, ...\}$ a partition with $||\Delta|| \le \delta$.

Case a) Let M and $\langle M\rangle$ be bounded.
Define

$$a_k^{(1)} := (M_{t_{k+1}} - M_{t_k})^2; \tag{5.61}$$

$$a_k^{(2)} := \langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k}; \tag{5.62}$$

$$b_k := a_k^{(1)} - a_k^{(2)} \tag{5.63}$$

$$\Rightarrow A_n^{(1)} := \sum_{k=0}^{n-1} a_k^{(1)} \equiv Q_{t_n}^{\Delta}(M); \tag{5.64}$$

$$A_n^{(2)} := \sum_{k=0}^{n-1} a_k^{(2)} \equiv \langle M\rangle_{t_n} \tag{5.65}$$

$$\Rightarrow B_n := A_n^{(1)} - A_n^{(2)} = \sum_{k=0}^{n-1} b_k = Q_{t_n}^{\Delta}(M) - \langle M\rangle_{t_n} \tag{5.66}$$

Define $\mathcal{F}_n := \sigma(M_{t_{k+1}}, k \leq n) \Rightarrow a_n^{(1)}, a_n^{(2)}$ are $\mathcal{F}_n$-measurable and $A_n^{(1)}, A_n^{(2)}$ are $\mathcal{F}_{n-1}$-measurable.

Since M and $\langle M\rangle$ are bounded (and M is a continuous local martingale) $\Rightarrow$ M and $\langle M\rangle$ are uniformely continuous on the interval $[0, t]$ (for any t)

$$W(\delta) := \sup_{0\leq s\leq t, 0\leq\varepsilon\leq\delta} (|M_{s+\varepsilon} - M_s|^2 + |\langle M\rangle_{s+\varepsilon} - \langle M\rangle_s|^2) \underset{\text{a.s. and also in } L^2}{\overset{\delta\to 0}{\longrightarrow}} 0 \tag{5.67}$$

We will now show: $|\mathbb{E}[B_\infty - B_n|\mathcal{F}_n]| \leq \mathbb{E}[W(\delta)|\mathcal{F}_n]$ It holds

$$B_\infty - B_n = \sum_{k\geq n} b_k \tag{5.68}$$

and

$$\mathbb{E}[b_k|\mathcal{F}_n] = 0 \forall k > n \tag{5.69}$$

since b_k is independent of $\mathcal{F}_n \forall k \geq n+1$ and $\mathbb{E}[b_k] = 0$

$$\Rightarrow |\mathbb{E}[B_\infty - B_n|\mathcal{F}_n]| = |\mathbb{E}[b_n|\mathcal{F}_n]| = |b_n| \leq a_n^{(1)} + a_n^{(2)} = \mathbb{E}\left[a_n^{(1)} + a_n^{(2)}|\mathcal{F}_n\right] \leq \mathbb{E}[W(\delta)|\mathcal{F}_n] \tag{5.70}$$

Now apply Lemma 5.14 b)

$$\Rightarrow \mathbb{E}\left[\sup_{n\geq 0} B_n^2\right] \leq c(\mathbb{E}\left[W(\delta)^2\right] + \mathbb{E}\left[W(\delta)^2\right]^{1/2}) \overset{\delta\to 0}{\longrightarrow} 0 \tag{5.71}$$

Finally

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |Q_s^{\Delta}(M) - \langle M\rangle_s|^2\right] \leq \mathbb{E}\left[(\sup_{n\in\mathbb{N}} |Q_{t_n}^{\Delta}(M) - \langle M\rangle_{t_n}| + W(\delta))^2\right] \overset{(a+b)^2\leq 2(a^2+b^2)}{\leq} 2\mathbb{E}\left[\sup_{n\geq 0} B_n^2\right] + 2\mathbb{E}\left[W(\delta)^2\right] \overset{\delta\to 0}{\longrightarrow} 0 \tag{5.72}$$

Case b) General $M, \langle M\rangle$. Let $T_n := \inf\{t \geq 0 : |M_n| \geq n \text{ or } \langle M\rangle_t \geq n\}$.

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |Q_s^{\Delta}(M) - \langle M\rangle_s| > \varepsilon\right) \leq \mathbb{P}\left(\sup_{0\leq s\leq t} |Q_s^{\Delta}(M^{T_n}) - < M^{T_n} >_s| > \varepsilon\right) + \underbrace{\mathbb{P}(T_n \leq t)}_{\leq \eta/2 \text{ for n large enough}} \tag{5.73}$$

For n large enough s.t. the right term is smaller $\eta/2$ choose δ small enough s.t. the left term is $\leq \eta/2$. □

Corollary 5.15.
Let $M, N \in \mathcal{M}_{loc}, t \geq 0$ fixed. Then,

$$\lim_{\|\Delta\| \to 0} Q_t^\Delta(M, N) = \langle M, N \rangle_t \text{ stochastically} \tag{5.74}$$

where

$$Q_t^\Delta(M, N) := \sum_{t_k \in \Delta} (M_{t_{k+1} \wedge t} - M_{t_k \wedge t})(N_{t_{k+1} \wedge t} - N_{t_k \wedge t}) \tag{5.75}$$

Lemma 5.16.
Let $M \in \mathcal{M}_{loc}$.

a) For almost all $\omega \in \Omega, \forall a < b$

$$\langle M \rangle_a(\omega) = \langle M \rangle_b(\omega) \Leftrightarrow M_t(\omega) = M_a(\omega), \forall t \in [a, b] \tag{5.76}$$

b) For almost all $\omega \in \Omega$ s.t. $\langle M \rangle_\infty(\omega) := \sup_{t \geq 0} \langle M \rangle_t(\omega) < \infty$

$$\Rightarrow \lim_{t \to \infty} M_t(\omega) \text{ exists and is finite.} \tag{5.77}$$

Remark: *For a process $A \in \mathcal{A}$ it holds $\langle A \rangle = 0$.*

$$\langle A \rangle_t = \lim_{\|\Delta\| \to 0} \sum_{k \geq 1} |A_{t_k \wedge t} - A_{t_{k+1} \wedge t}|^2 \tag{5.78}$$

$$= \lim_{\|\Delta\| \to 0} [\underbrace{\sup_{k \geq 1} |A_{t_k \wedge t} - A_{t_{k+1} \wedge t}|}_{\to 0} \underbrace{\sum_{k \geq 1} |A_{t_k \wedge t} - A_{t_{k+1} \wedge t}|}_{\leq S_t(A)}] \tag{5.79}$$

For a semimartingale $X = M + A, M \in \mathcal{M}_{loc}, A \in \mathcal{A}_0$.

Definition 5.17.
Let $X, \tilde{X} \in \mathcal{S}$ with $X = M + A$, $\tilde{X} = \tilde{M} + \tilde{A}$ where $M, \tilde{M} \in \mathcal{M}_{loc}$. We define

$$\langle X, \tilde{X} \rangle := \langle M, \tilde{M} \rangle \text{ and} \tag{5.80}$$

$$\langle X \rangle := \langle M \rangle. \tag{5.81}$$

Theorem 5.18.
Let $X, X' \in \mathcal{S}, t \geq 0$. Then

$$\lim_{\|\Delta\| \to 0} Q_t^\Delta(X, X') = \langle X, X' \rangle \text{ stochastically} \tag{5.82}$$

Proof.

$$Q_t^\Delta(X, X') = \underbrace{Q_t^\Delta(M, M')}_{\to \langle M, M' \rangle =: \langle X, X' \rangle} + Q_t^\Delta(M, A') + Q_t^\Delta(A, M') + Q_t^\Delta(A, A') \tag{5.83}$$

Now check if the last 3 summands go to 0.

$$|Q_t^\Delta(M, A')| = |\sum_{t_k \in \Delta} (M_{t_{k+1} \wedge t} - M_{t_k \wedge t})(A'_{t_{k+1} \wedge t} - A'_{t_k \wedge t}) \tag{5.84}$$

$$\leq \underbrace{\sup_{t_k \in \Delta} |M_{t_{k+1} \wedge t} - M_{t_k \wedge t}|}_{\to 0} \underbrace{\sum_{t_k \in \Delta} |A'_{t_{k+1} \wedge t} - A'_{t_k \wedge t}|}_{\leq S_t(A)} \overset{\|\Delta\| \to 0}{\longrightarrow} 0 \tag{5.85}$$

Similarly: $|Q_t^\Delta(A, M')| \overset{\|\Delta\| \to 0}{\longrightarrow} 0, |Q_t^\Delta(A, A')| \overset{\|\Delta\| \to 0}{\longrightarrow} 0.$ □

Corollary 5.19.
Let $X, X' \in \mathcal{S}, t \geq 0$.

$$\Rightarrow \langle X, X' \rangle_t \leq \sqrt{\langle X \rangle_t \langle X' \rangle_t} \leq \frac{1}{2}(\langle X \rangle_t + \langle X' \rangle_t) \tag{5.86}$$

Proof. Cauchy Schwarz and $(ab)^{1/2} \leq \frac{a+b}{2}$ for $a, b \geq 0$. □

[09.11.2012]
[13.11.2012]

5.4 L^2-bounded martingales

Definition 5.20 (L^2-bounded martingales)**.**
The space of continuous L^2*-bounded martingales* is defined by

$$H^2 := \{M \in \mathcal{M} : \sup_{t \geq 0} \mathbb{E}\left[M_t^2\right] < \infty\} \tag{5.87}$$

Example: *Let $T \in \mathbb{R}_+$ then*

$$M_t := B_{t \wedge T} \tag{5.88}$$

is in H^2, since $\mathbb{E}\left[B_{t\wedge T}^2\right] = t \wedge T \Rightarrow \sup_{t \geq 0} \mathbb{E}\left[B_{t\wedge T}^2\right] < \infty$.

Remark: *Let $M \in H^2$, then $\{M_t, t \geq 0\}$ is uniformely integrable, i.e.*

$$\sup_{t \geq 0} \mathbb{E}\left[|M_t| \mathbb{1}_{|M_t| > K}\right] \Rightarrow 0 \text{ for } K \to \infty \tag{5.89}$$

since

$$\mathbb{E}\left[|M_t| \mathbb{1}_{|M_t| > K}\right] \leq \mathbb{E}\left[\frac{|M_t|^2}{K} \mathbb{1}_{|M_t| > K}\right] \leq \frac{\sup_{t\geq 0} \mathbb{E}\left[|M_t|^2\right]}{K} \to 0 \text{ for } K \to \infty \tag{5.90}$$

From this it follows:

$$\lim_{t \to \infty} M_t = M_\infty \in L^1 \text{ exists (a.s.) and } M_t = \mathbb{E}\left[M_\infty | \mathcal{F}_t\right] \text{ a.s.} \tag{5.91}$$

Finally: $M_\infty \in L^2$.

Proposition 5.21.

a) H^2 is a Hilbert space with respect to the norm

$$\|M\|_{H^2} := \sqrt{\mathbb{E}\left[M_\infty^2\right]} = \lim_{t \to \infty} \sqrt{\mathbb{E}\left[M_t^2\right]} \tag{5.92}$$

b) Let $M_\infty^* := \sup_{t \geq 0} |M_t|$. Then an equivalent norm is

$$\|M_\infty^*\|_2 \equiv \sqrt{\mathbb{E}\left[(M_\infty^*)^2\right]} \equiv \sqrt{\mathbb{E}\left[\sup_{t \geq 0} |M_t|^2\right]} \tag{5.93}$$

c) For $M \in H_0^2 := \{X \in H^2 : X_0 = 0\}$ it holds

$$\|M\|_{H^2} = \sqrt{\mathbb{E}\left[\langle M \rangle_\infty\right]} \tag{5.94}$$

Proof. **1)** Verify that $\|\cdot\|_{H^2}$ is a norm: easy.
$\Rightarrow$ the associated scalar product is

$$(M,N)_{H^2} := \frac{1}{4}(\|M+N\|^2_{H^2} - \|M-N\|^2_{H^2}) \tag{5.95}$$

2) Check b): First inequality:

$$\|M^*_\infty\|_2^2 \equiv \mathbb{E}\left[\sup_{t\geq 0}|M_t|^2\right] \overset{\text{Doob}}{\leq} 4\sup_{t\geq 0}\mathbb{E}\left[M_t^2\right] \overset{M^2 \text{ submart.}}{=} 4\lim_{t\to\infty}\mathbb{E}\left[M_t^2\right] \equiv 4\|M\|^2_{H^2} \tag{5.96}$$

$\Rightarrow M^*_\infty$ is in L^2 ($\Rightarrow$ also in L^1).
For the second inequality: $M_t = \mathbb{E}\left[M_\infty|\mathcal{F}_t\right]$

$$\Rightarrow \|M\|^2_{H^2} = \lim_{t\to\infty}\mathbb{E}\left[M_t^2\right] \overset{submart.}{=} \sup_{t\geq 0}\mathbb{E}\left[M_t^2\right] \leq \mathbb{E}\left[\sup_{t\geq 0} M_t^2\right] \equiv \|M^*_\infty\|_2^2 \tag{5.97}$$

3) Verify the completeness of H^2.
Let $(M^n)_{n\geq 1}$ be a sequence in H^2 s.t.

$$\|M^n - M^m\|_{H^2} \overset{m,n\to\infty}{\longrightarrow} 0 \tag{5.98}$$

$\Rightarrow \exists$ sequence $M^n_\infty \in L^2$ s.t.

$$M^n_t \equiv \mathbb{E}\left[M^n_\infty|\mathcal{F}_t\right] \tag{5.99}$$

We know

$$\|M^n_\infty - M^m_\infty\|_{L^2} \overset{def}{=} \|M^n - M^m\|_{H^2} \underset{m,n\to\infty}{\overset{hyp}{\longrightarrow}} 0 \tag{5.100}$$

$\Rightarrow (M^n_\infty)_{n\geq 1}$ is Cauchy and since L^2 is complete, it converges to a limit in L^2. Let us call this limit M_∞. Define therefore the Martingale

$$M_t := \mathbb{E}\left[M_\infty|\mathcal{F}_t\right] \tag{5.101}$$

Q.: Does $M^n \to M$? Yes!

$$\mathbb{E}\left[\sup_{t\geq 0}|M^n_t - M_t|^2\right] \overset{Doob}{\leq} 4\mathbb{E}\left[(M^n_\infty - M_\infty)^2\right] = 4\|M^n - M\|^2_{H^2} \overset{n\to\infty}{\longrightarrow} 0 \tag{5.102}$$

Q.: Is M a continuous Martingale? Because of (5.102) there exists a subsequence $(n_k)_{k\geq 0}$ s.t. $\sup_{t\geq 0}|M^{n_k}_t - M_t| \overset{k\to\infty}{\longrightarrow} 0$ a.s.. We have uniformely convergence on subsequences, therefore $t\mapsto M_t$ is continuous, i.e. $M \in \mathcal{M}$.
Q.: Is $M \in H^2$?

$$\sup_{t\geq 0}\mathbb{E}\left[M_t^2\right] = \sup_{t\geq 0}\mathbb{E}\left[(\mathbb{E}\left[M_\infty|\mathcal{F}_t\right])^2\right] \leq \sup_{t\geq 0}\mathbb{E}\left[\mathbb{E}\left[M^2_\infty|\mathcal{F}_t\right]\right] = \mathbb{E}\left[M^2_\infty\right] < \infty \tag{5.103}$$

$\Rightarrow M \in H^2$.
5) Verify c): Let $M \in H^2$ with $M_0 = 0$. Let $\langle M\rangle$ be the quadratic variation of M. $\Rightarrow M^2 - \langle M\rangle$ is a (local) martingale. $\Rightarrow \mathbb{E}\left[M_t^2\right] - \mathbb{E}\left[\langle M\rangle_t\right] = \underbrace{\mathbb{E}\left[M_0^2\right]}_{=0} - \underbrace{\mathbb{E}\left[\langle M\rangle_0\right]}_{=0} \equiv 0 \forall t\geq 0$

$$\Rightarrow \|M\|^2_{H^2} = \mathbb{E}\left[M^2_\infty\right] = \lim_{t\to\infty}\mathbb{E}\left[M_t^2\right] = \lim_{t\to\infty}\mathbb{E}\left[\langle M\rangle_t\right] \underset{conv.}{\overset{monot.}{=}} \mathbb{E}\left[\langle M\rangle_\infty\right] \tag{5.104}$$

□

Example: *Let $T \in \mathbb{R}_+$ be a fixed number and B a BM.*

$$\Rightarrow M_t := B_{t\wedge T} \tag{5.105}$$

$$\|M\|_{H^2} := \begin{cases} \lim_{t\to\infty}\mathbb{E}\left[B^2_{t\wedge T}\right] = \mathbb{E}\left[B^2_T\right] = T \\ \mathbb{E}\left[\langle B_{t\wedge T}\rangle_\infty\right] = \lim_{t\to\infty} t\wedge T = T \end{cases} \tag{5.106}$$

6 Stochastic Integration

Strategy:

a) 6.1)-6.2) Define the Lebesgue-Stieltjes-Integral for functions, then extend to

$$\int_0^t X_s dAs \equiv (X \cdot A)_t - (X \cdot A)_0 \tag{6.1}$$

for X locally bounded and $A \in \mathcal{A}$.

b) 6.3)-6.5) Itô-Integral:

1) Define

$$\int_0^t X_s dMs \tag{6.2}$$

for $M \in H^2$ and X "elementary process". $\rightarrow$ Itô-isometry: $\| \underbrace{X \cdot M}_{\text{Itô-int}} \|^2_{H^2} = \| \underbrace{X^2 \cdot \langle M \rangle}_{a)} \|$

2) Extension to $X \in L^2(M)$, e.g.

$$\int_0^t B_s dB_s = ? \tag{6.3}$$

3) Extension to semi-martingales.

6.1 Lebesgue-Stieltjes Integral

Riemann case: $\Delta_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Define

$$\text{Riemann-Integral: } \lim_{\|\Delta\| \to 0} \sum_{k=0}^{n-1} f(\xi_k)(x_{k+1} - x_k) \text{ for some } \xi_k \in (x_k, x_{k+1}] \tag{6.4}$$

The limit exists e.g. when f is continuous.

$$\text{Riemann-Stieltjes: } \lim_{\|\Delta_n\| \to 0} \sum_{k=0}^{n-1} f(\xi_k)(g(x_{k+1}) - g(x_k)) \text{ for some } \xi_k \in (x_k, x_{k+1}] \tag{6.5}$$

The limit exists e.g. if g is continuous and has finite variation.

Proposition 6.1.
Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a right-continuous function. Then the following statements are equivalent.

a) g has finite variation.

b) $\exists g_1, g_2$ increasing, right-continuous s.t. $g = g_1 - g_2$.

c) $\exists$ (signed) Radon measure, μ^g, on $\mathbb{R}^+$ s.t.

$$g(t) = \mu^g([0, t]), \forall t \geq 0 \tag{6.6}$$

Proof. a $\Leftrightarrow$ b trivial.

a,b$\Leftrightarrow$ c: "$\Rightarrow$" WLOG take $g \geq 0$, rightcontinuous amd $S_t(g) < \infty$ (variation of g in $[0,t]$) and $g(0) = 0$. $\Rightarrow \mu([0,t]) := g(t) \forall t \geq 0$. $\Rightarrow \mu$ is a Radon-measure on $\mathbb{R}_+$.

"$\Leftarrow$" Given μ, define $g(t) := \mu([0,t]), \forall t \geq 0$. Therefore g is rightcontinuous and has finite variation. □

Definition 6.2 (Lebesgue-Stieltes-Integral)**.**
Let $g : \mathbb{R}_+ \to \mathbb{R}$ be right-continuous, with finite variation and let $f : \mathbb{R}_+ \to \mathbb{R}$ be a locally bounded function. Then the *Lebesgue-Stieltjes-Integral of f w.r.t. g* is defined by

$$\int_{(0,t]} f(s)\mu^g(ds) \tag{6.7}$$

where μ^g is the measure of Prop 6.1.

Notation: *We sometimes also write*

$$\int_0^t f(s)\mu^g(ds) = \int_0^t f dg = \int_0^t f(s)dg(s) = \int_0^t f(s)g(ds) \tag{6.8}$$

Remark: *(i) If $g \in C^1 \Rightarrow \int_0^t f(s)\mu^g(ds) = \int_0^t f(s)g'(s)ds$ where the last term means the usual Lebesgue-Integral.*

(ii) If g and h are continuous and of finite variation then

$$d(gh)(s) = g(s)dh(s) + h(s)dg(s) \tag{6.9}$$

Proposition 6.3.
Let g be right-continuous, increasing and let f be left-continuous and locally bounded. Then $\forall t \geq 0$

$$\lim_{\|\Delta\| \to 0} I_t^\Delta(f,g) = \int_0^t f dg \tag{6.10}$$

where

$$I_t^\Delta(f,g) := \sum_{k=0}^{n-1} f(t_k)(g(t_{k+1}) - g(t_k)) \tag{6.11}$$

and Δ is a partition of $[0,t]$, i.e. $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$.

Remark: *If f is continuous one can replace $f(t_k)$ by $f(t_{k+1})$. The BM analogue will **not** satisfy this property.*

Proof. Let $f^\Delta := \sum_{k=0}^{n-1} f(t_k)\mathbb{1}_{(t_k,t_{k+1}]}$. Since f is locally bounded $\Rightarrow \sup_{s\in[0,t]} |f^\Delta(s)| \leq C < \infty$. Also, since f is left continuous,

$$\Rightarrow \lim_{\|\Delta\| \to 0} f^\Delta(s) = f(s) \forall s \in [0,t] \tag{6.12}$$

$$I_t^\Delta(f,g) = \int_0^t f^\Delta(s)\mu^g(ds) \overset{\|\Delta\| \to 0}{\longrightarrow} \int_0^t f(s)\mu^g(ds) \overset{def}{=} \int_0^t f dg \tag{6.13}$$

□

[13.11.2012]
[16.11.2012]

6.2 Stochastic Integration w.r.t. bounded variation processes

We define "$\int_0^t X_s dA_s$" for $A \in \mathcal{A}$ and for

$$X \in \mathcal{B} := \{X : \text{ adapted, left-continuous, the trajectories are locally bounded}\}. \tag{6.14}$$

Definition 6.4.
Let $A \in \mathcal{A}, X \in \mathcal{B}$ then we define the *stochastic integral of X w.r.t. A pathwise* through

$$(X \cdot A)_t = \int_0^t XdA = \int_0^t X_s dA_s : \omega \mapsto \int_0^t X_s(\omega) dA_s(\omega) \leftarrow \text{(usual Leb.-Stieltj.-Integral)} \tag{6.15}$$

Notation: $X \cdot A \equiv ((X \cdot A)_t)_{t \geq 0}$

Properties:

Theorem 6.5.
For $A \in \mathcal{A}$ and $X, Y \in \mathcal{B}$ it holds

a) $X \cdot A \in \mathcal{A}_0$.

b) $X \cdot A$ is bilinear in X and A.

c) For any stopping time T it holds $(X \cdot A)^T = X \cdot A^T$.

d) $X \cdot (Y \cdot A) = (XY) \cdot A$.

Proof. **ad a)** $(X \cdot A)_0 = 0$ clear. (consider the partition in 6.3)
Pathwise continuous since X is locally bounded and A is continuous.
adapted:

$$\int_0^t X_s dA_s = \lim_{\|\Delta\| \to 0} \sum_{k=0}^{n-1} X_{t_k}(A_{t_{k+1}} - A_{t_k}) \text{ meas. w.r.t } \mathcal{F}_t \tag{6.16}$$

(limit of measurable functions again measurable)
Finite variation:

$$S_t((X \cdot A)(\omega)) \leq \underbrace{\sup_{0 \leq s \leq t} |X_s(\omega)|}_{<\infty} S_t(A(\omega)) \tag{6.17}$$

ad b) Trivial.
ad c)

$$(X \cdot A)^T(\omega) = \lim_{n \to \infty} \sum_{k=0}^{n-1} X_{t_k \wedge T}(\omega)[A_{t_{k+1} \wedge T}(\omega) - A_{t_k \wedge T}] \tag{6.18}$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n-1} X_{t_k}(\omega)[A_{t_{k+1} \wedge T}(\omega) - A_{t_k \wedge T}] \tag{6.19}$$

$$= (X \cdot A^T)(\omega) \tag{6.20}$$

because: if $t_k > T \Rightarrow t_{k+1} > T \Rightarrow A_{t_{k+1} \wedge T} - A_{t_k \wedge T} = 0$.
ad d)

$$(X \cdot (Y \cdot A))_t = \int_0^t X s d((Y \cdot A)_s) \tag{6.21}$$

$$= \int_0^t X_s Y_s dA_s \equiv ((XY) \cdot A)_t \tag{6.22}$$

□

6.3 Itô-Integral

We will define

$$\int_0^s X_s dB_s \tag{6.23}$$

where B is a BM. If $f, g \in C^1$ we know

$$f(g(t)) = f(g(0)) + \int_0^t f'(g(s))g'(s)ds \tag{6.24}$$

If now g is a brownian path, then g' does not exists....mmm. :(

One of the results will be for $f \in C^2$

$$f(B_t) = f(B_0) + \underbrace{\int_0^t f'(B_s)dB_s}_{\text{Itô-Integral}} + \frac{1}{2}\int_0^s f''(B_s) \underbrace{ds}_{\equiv d<B>_s} \tag{6.25}$$

If we try to define

$$I_n := \sum_{k=0}^{n-1} f(B_{t_k})(B_{t_{k+1}} - B_{t_k}), \tag{6.26}$$

then, $\lim_{n\to\infty}$ (with $\|\Delta\| \to 0$) does not exist pointwise in Ω (, i.e. pathwise). $\Rightarrow I_n$ as Lebesgue-Stieltjes-Integral can not be defined.

But one can see that the limit is fine in L^2.

Further issue: Let B be a one-dimensional standard BM. Let $t_k := \frac{k}{n}t, 0 \le k \le n$.

$$\Rightarrow \lim_{n\to\infty} \sum_{k=0}^{n-1} B_{t_k}(B_{t_{k+1}} - B_{t_k}) = \frac{B_t^2 - t}{2} \text{ in } L^2 \tag{6.27}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} B_{t_{k+1}}(B_{t_{k+1}} - B_{t_k}) = \frac{B_t^2 + t}{2} \text{ in } L^2 \tag{6.28}$$

Proof:

$$\sum_{k=0}^{n-1} B_{t_k}(B_{t_{k+1}} - B_{t_k}) = \underbrace{\sum_{k=0}^{n-1} \frac{1}{2}(B_{t_{k+1}}^2 - B_{t_k}^2)}_{=\frac{1}{2}B_t^2 (\text{since } t_n = t, B_0 = 0)} - \frac{1}{2}\underbrace{\sum_{k=0}^{n-1}(B_{t_{k+1}} - B_{t_k})^2}_{\to t \text{ in } L^2 \text{ for } n\to\infty} \tag{6.29}$$

Itô chooses (6.27) as the definition for $\int_0^t B_s dB_s$.

6.3.1 Itô-Integral for elementary processes

Definition 6.6.
Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a standard filtered probability space. $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called an *elementary process* if

a) Exists a sequence of times $0 = t_0 < t_1 < \cdots \nearrow \infty$

b) Exists a sequence of r.v. $(\xi_n)_{n\ge 0}$ uniformely bounded (i.e. $\sup_{n\ge 0} |\xi_n(\omega)| \le C \forall \omega \in \Omega$).

c) ξ_n are $\mathcal{F}_{t_n}$-measurable.

d)

$$X_t(\omega) = \xi_0(\omega)\mathbb{1}_0(t) + \sum_{n\ge 0} \xi_n(\omega)\mathbb{1}_{(t_n, t_{n+1}]}(t), 0 \le t < \infty, \omega \in \Omega \tag{6.30}$$

That means, that X is piecewise constant.

Notation: $X \in \xi \Leftrightarrow X$ *is an elementary process.*

Definition 6.7 (Itô-Integral for elementary processes)**.**
Let $X \in \xi$, $M \in H^2$. Then we define the *stochastic integral of X w.r.t. M* pathwise by

$$\int_0^t X_s dM_s \equiv (X \cdot M)_t := \sum_{k=0}^{\infty} \xi_k (M_{t_{k+1}\wedge t} - M_{t_k \wedge t}) \tag{6.31}$$

$$= \sum_{k=0}^{n-1} \xi_k (M_{t_{k+1}} - M_{t_k}) + \xi_n (M_t - M_{t_{n-1}}) \tag{6.32}$$

where n is the unique number s.t. $t \in (t_{n-1}, t_n]$.

The Itô-Isometry

Theorem 6.8.
Let $M \in H^2$ and $X \in \xi$. Then,

a) $X \cdot M \in H_0^2$

b) $\langle X \cdot M \rangle_t = \int_0^t X_s^2 d\langle M \rangle_s \equiv (X^2 \cdot \langle M \rangle)_t$

c) Isometry:

$$\|X \cdot M\|_{H^2}^2 \equiv \mathbb{E}\left[(\int_0^\infty X_s dM_s)^2 \right] = \mathbb{E}\left[\int_0^\infty X_s^2 d\langle M \rangle_s \right] \equiv \|X\|^2_{L^2(\mathbb{R}_+ \times \Omega, d\langle M \rangle \otimes \mathbb{P})} \tag{6.33}$$

Corollary 6.9.
For $M = (B_{s\wedge t})_{s \geq 0}$, then

a) $X \cdot B^t \in H_0^2$.

b) $\langle X \cdot B \rangle_t = \int_0^t X_s^2 ds$

c) $\mathbb{E}\left[(\int_0^t X_s dB_s)^2 \right] = \mathbb{E}\left[\int_0^t X_s^2 ds \right]$

Proof of the Theorem. Easy to check: $(X \cdot M)$ is adapted, $(X \cdot M)_0 = 0$, Continuity.
Martingale? Let $s < t$, say $s \in (t_k, t_{k+1}]$ and $t \in (t_n, t_{n+1}]$.

$$\mathbb{E}\left[(X \cdot M)_t | \mathcal{F}_s \right] \tag{6.34}$$

$$= \mathbb{E}\left[(X \cdot M)_s + \xi_k (M_{t_{k+1}} - M_s) + \sum_{l=k+1}^{n-1} \xi_l (M_{t_{l+1}} - M_{t_l}) + \xi_n (M_t - M_{t_n}) | \mathcal{F}_s \right] \tag{6.35}$$

$$= (X \cdot M)_s + \xi_k \underbrace{\mathbb{E}\left[M_{t_{k+1}} - M_s | \mathcal{F}_s \right]}_{=0} + \mathbb{E}\left[\xi_n \underbrace{\mathbb{E}\left[M_t - M_{t_n} | \mathcal{F}_{t_n} \right]}_{=0} | \mathcal{F}_s \right] + \mathbb{E}\left[\sum_{l=k+1}^{n-1} \xi_l \underbrace{\mathbb{E}\left[M_{t_{l+1}} - M_{t_l} | \mathcal{F}_{t_l} \right]}_{=0} | \mathcal{F}_s \right] \tag{6.36}$$

$$= (X \cdot M)_s \tag{6.37}$$

since $\mathcal{F}_s \subset \mathcal{F}_{t_n}$ and ξ_k is F_{t_n}-measurable.
L^2-boundedness follows from the uniform bound of the ξ_k.
ad b) WLOG: $s = t_k, t = t_{n+1}$ (otherwise add two points to $\{t_i\}$). To show $(X \cdot M)_t^2 - \int_0^t X_u^2 d\langle M \rangle_u$ is a martingale, i.e.

$$\mathbb{E}\left[(X \cdot M)_t^2 - \int_0^t X_u^2 d\langle M \rangle_u | \mathcal{F}_s \right] \overset{\text{if } s<t}{=} (X \cdot M)_s^2 - \int_0^s X_u^2 d\langle M \rangle_u. \tag{6.38}$$

$$\overset{5.9}{\Rightarrow} \langle X \cdot M \rangle_t = \int_0^t X_u^2 d\langle M \rangle_u \equiv (X^2 \cdot \langle M \rangle)_t \tag{6.39}$$

$$\mathbb{E}\left[(X\cdot M)_t^2-(X\cdot M)_s^2|\mathcal{F}_s\right] \tag{6.40}$$
$$=\mathbb{E}\left[((X\cdot M)_t-(X\cdot M)_s)^2|\mathcal{F}_s\right]+\underbrace{2\mathbb{E}\left[(X\cdot M)_s((X\cdot M)_t-(X\cdot M)_s)|\mathcal{F}_s\right]}_{=0\text{ by a) since }(X\cdot M)_s\ \mathcal{F}_s\text{-meas.}} \tag{6.41}$$
$$=\mathbb{E}\left[\left(\sum_{l=k}^n \xi_l(M_{t_{l+1}}-M_{t_l})\right)^2|\mathcal{F}_s\right] \tag{6.42}$$
$$=\mathbb{E}\left[\sum_{l=k}^n \xi_l^2(M_{t_{l+1}}-M_{t_l})^2|\mathcal{F}_s\right]+2\mathbb{E}\left[\sum_{k\le j<l\le n}\xi_j\xi_l(M_{t_{l+1}}-M_{t_l})(M_{t_{j+1}}-M_{t_j})\right] \tag{6.43}$$
$$=\mathbb{E}\left[\sum_{l=k}^n \xi_l^2(M_{t_{l+1}}-M_{t_l})^2|\mathcal{F}_s\right]+2\mathbb{E}\left[\sum_{k\le j<l\le n}\xi_j\xi_l\underbrace{\mathbb{E}\left[(M_{t_{l+1}}-M_{t_l})|\mathcal{F}_{t_l}\right]}_{=0}(M_{t_{j+1}}-M_{t_j})\right] \tag{6.44}$$
$$=\mathbb{E}\left[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] \tag{6.45}$$

$\Rightarrow$ (6.38) holds

c)

$$\|X\cdot M\|_{H^2}^2\equiv\mathbb{E}\left[(X\cdot M)_\infty^2\right]\overset{5.21}{=}\mathbb{E}\left[\langle X\cdot M\rangle_\infty\right]\overset{b)}{=}\mathbb{E}\left[\int_0^\infty X_u^2 d\langle M\rangle_u\right] \tag{6.46}$$

□

[16.11.2012]

[20.11.2012]

Proposition 6.10 (Kunita-Watanabe).
$M,N\in H^2$, $X,Y\in\xi$.

a) $\langle X\cdot M, Y\cdot N\rangle_t=\int_0^t X_sY_s d\langle M,N\rangle_s\equiv((XY)\cdot\langle M,N\rangle)_t$

b) $\mathbb{E}\left[\langle X\cdot M,Y\cdot N\rangle_\infty\right]\le\mathbb{E}\left[\int_0^\infty X_s^2 d\langle M\rangle_s\right]^{1/2}\mathbb{E}\left[\int_0^\infty Y_s^2 d\langle N\rangle_s\right]^{1/2}$

Proof. Claim:$(X\cdot M)_t(Y\cdot N)_t-\int_0^t X_sY_s d\langle M,N\rangle_s$ is a martingale.
We assume, that X and Y are constant on the same intervals. Otherwise one can just add the respective points.

$$(X\cdot M)_t=\sum_{l=1}^n X_{t_l}\underbrace{(M_{t_{l+1}}-M_{t_l})}_{=:\Delta M_l} \tag{6.47}$$
$$(Y\cdot N)_t=\sum_{l=1}^n Y_{t_l}\underbrace{(N_{t_{l+1}}-N_{t_l})}_{=:\Delta N_l} \tag{6.48}$$

Then

$$\mathbb{E}\left[(X\cdot M)_t(Y\cdot N)_t-(X\cdot M)_s(Y\cdot N)_s|\mathcal{F}_s\right] \tag{6.49}$$
$$=\mathbb{E}\left[\sum_{l,l'=k}^n X_{t_l}Y_{t_{l'}}\Delta M_l\Delta N_{l'}|\mathcal{F}_s\right] \tag{6.50}$$
$$\overset{k:t_k=s}{=}\mathbb{E}\left[\sum_{l=k}^n X_{t_l}Y_{t_l}\Delta M_l\Delta N_l|\mathcal{F}_s\right]+\underbrace{\mathbb{E}\left[\sum_{l\ne l'}\dots\right]}_{=0} \tag{6.51}$$
$$=\mathbb{E}\left[\int_s^t X_sY_s d\langle M,N\rangle_s|\mathcal{F}_s\right] \tag{6.52}$$

b)

$$\mathbb{E}\left[\langle X\cdot M, Y\cdot N\rangle_\infty\right] \overset{5.19}{\leq} \mathbb{E}\left[\langle X\cdot M\rangle_\infty^{1/2}\langle Y\cdot N\rangle_\infty^{1/2}\right] \tag{6.53}$$

$$\overset{C.-S.}{\leq} \mathbb{E}\left[\langle X\cdot M\rangle_\infty\right]^{1/2}\mathbb{E}\left[\langle Y\cdot N\rangle_\infty\right]^{1/2} \tag{6.54}$$

□

Goal of the week

$$\int_0^t X_s dM_s \tag{6.55}$$

$X \in \xi$ (Want a larger space! : today), $M \in H^2$(Want the space of semimartingales: friday!)

Definition 6.11 (Predictable σ-Algebra)**.**
$\mathcal{P} = \sigma(\xi)$ smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ s.t.

$$(t,\omega) \mapsto X_t(\omega) \text{ measurable } \forall X \in \xi \tag{6.56}$$

A process X is called *predictable iff* $\mathcal{P}$-measurable.

Proposition 6.12.

$$\sigma(\xi) = \sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R},\ \text{adapted, X left cont. on } (0,\infty)\}) \tag{6.57}$$

$$= \sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R},\ \text{adapted, X cont. on } (0,\infty)\}) \tag{6.58}$$

Proof. Exercise. □

Definition 6.13.
Let $M \in H^2$. We define

$$\mathcal{L}^2(M) = \{X : \mathbb{R}_+ \times \Omega \to \mathbb{R},\ \text{predictable}, \|X\|_M < \infty\} \tag{6.59}$$

with $\|\cdot\|_M$ defined as

$$\|X\|_M := \|X\|_{L^2(d\langle M\rangle \otimes dP)} := \mathbb{E}\left[\int_0^\infty X_s^2 d\langle M\rangle_s\right]^{1/2} \tag{6.60}$$

$L^2(M)$ is the space of equivalence classes

$$X \sim Y \Leftrightarrow \|X - Y\|_M = 0 \tag{6.61}$$

The Itô-Isometry is now

$$\|X\|_M \equiv \mathbb{E}\left[\int_0^\infty X_s^2 d\langle M\rangle_s\right]^{1/2} \underset{Isom}{\overset{\text{Itô}}{=}} \mathbb{E}\left[\left(\int_0^\infty X_s dM_s\right)^2\right]^{1/2} \equiv \|X\cdot M\|_{H^2} \tag{6.62}$$

Proposition 6.14.
$X \in L^2(M) \Rightarrow \exists$ a sequence of $X^n \in L^2(M) \cap \xi$ s.t.

$$\|X^n - X\|_M \overset{n\to\infty}{\longrightarrow} 0 \tag{6.63}$$

i.e.

$$\mathbb{E}\left[\int_0^\infty |X_s - X_s^n|^2 d\langle M\rangle_s\right] \overset{n\to\infty}{\longrightarrow} 0 \tag{6.64}$$

Proof. We give the proof only for the case M=B=Brownian Motion, i.e. $d\langle B\rangle_s = ds$, where ds is the Lebesgue-measure. (If $d\langle M\rangle_s \ll$ lebesgue, then the considerations are similar. If not, then the proof is tricky (see Karatzas-Shreve, Lemma 2.7))

Let B be a BM and let $T > 0$ arbitrary.

Step 1: $Z \in L^2(B)$, bounded, pathwise continuous.
Consider partitions

$$\Delta_n = \{t_0 = 0 < t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} = T\} \tag{6.65}$$

with $\|\Delta_n\| \to 0$ for $n \to \infty$. Define

$$\phi_t^n(\omega) = Z_t(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{n-1} Z_{t_k}(\omega)\mathbb{1}_{(t_k,t_{k+1}]}(t) \tag{6.66}$$

Then it holds, by continuity of $t \mapsto Z_t(\omega)$ and since $\|\Delta\|_n \to 0$:

$$\int_0^T |\phi_t^n(\omega) - Z_t(\omega)|^2 dt \overset{n\to\infty}{\longrightarrow} 0 \tag{6.67}$$

By Lebesgue (dominated convergence)

$$\mathbb{E}\left[\int_0^T |\phi_t^n - Z_t|^2 dt\right] \to 0 \tag{6.68}$$

i.e.

$$\|\phi^n - Z\|_M \to 0 \tag{6.69}$$

Step 2: $Y \in L^2(B)$, bounded.
Let K s.t. $|Y| \leq K$. We are going to introduce mollifiers ψ_n s.t.,

$$\psi_n(x) \geq 0, \psi_n \text{ continuous}, \int \psi_n dx = 1, \psi_n(x) = 0 \text{ if } x \notin [0, \frac{1}{n}] \tag{6.70}$$

For $t \leq T$ define

$$Z_t^n = \int_0^T \psi_n(t-s) Y_s ds \tag{6.71}$$

Then $t \mapsto Z_t^n$ is continuous and bounded, i.e. $|Z_t^n| \leq K$.

It holds

$$\int_0^T (Z_t^n(\omega) - Y_t(\omega))^2 dt \to 0 \; \forall \omega \in \Omega \tag{6.72}$$

and therefore by dominated convergence

$$\Rightarrow \mathbb{E}\left[\int_0^T (Z_t^n - Y_t)^2 dt\right] \overset{n\to\infty}{\longrightarrow} 0 \tag{6.73}$$

Step 3: $X \in L^2(B)$.

To make it bounded define

$$Y_t^n = \begin{cases} -n & X_t \leq -n \\ X_t & -n \leq X_t \leq n \\ n & X_t \geq n \end{cases} \quad \text{("truncation")} \tag{6.74}$$

$$\|X - Y^n\|_{L^2(B)} = \mathbb{E}\left[\int_0^T (X_t - Y_T^n)^2 dt\right] \tag{6.75}$$

$$\leq \mathbb{E}\left[\int_0^T X_t^2 \mathbb{1}_{\{|X_t| \geq n\}} dt\right] \overset{n\to\infty}{\longrightarrow} 0 \tag{6.76}$$

again by dominated convergence. Note that we could use that X was bounded in the previous steps. Here we have to use the hypothesis that $X \in L^2(B)$. □

Theorem 6.15.
Let $X \in L^2(M)$. Then $\exists!(X \cdot M) \in H_0^2$ s.t., if $X^n \in \xi$ is a sequence with

$$\|X - X^n\|_M \overset{n\to\infty}{\longrightarrow} 0 \tag{6.77}$$

then also

$$\|X \cdot M - X^n \cdot M\|_{H^2} \overset{n\to\infty}{\longrightarrow} 0 \tag{6.78}$$

Thus

$$L^2 - \lim_{n\to\infty}(X^n \cdot M)_t = X \cdot M_t \tag{6.79}$$

uniformely in t. The map $L^2(M) \to H_0^2, X \mapsto X \cdot M$ is an isometry, i.e.

$$\|X\|_M = \|X \cdot M\|_{H^2} \tag{6.80}$$

Proof. Let $X \in L^2(M)$.
Step 1: Definition of $(X \cdot M)$.
By Prop. 6.14: $\exists X^n \in \xi : \|X - X^n\|_M \to 0$. Therefore

$$\|X^n \cdot M - X^m \cdot M\|_{H^2} \overset{\text{Isometry}}{=} \|X^n - X^m\|_M \overset{m,n\to\infty}{\longrightarrow} 0, \tag{6.81}$$

i.e. $(X^n \cdot M)$ is a cauchy sequence in H^2 which is a Hilbert space. $\Rightarrow \lim_{n\to\infty} X^n \cdot M$ exists and is in H^2. So we can define $X \cdot M := \lim_{n\to\infty} X^n \cdot M$.
Step 2: Show that $X \cdot M$ is independent of X^n.
Let Y^n be a second approximating sequences, i.e.

$$\|Y^n - X\|_M \to 0 \tag{6.82}$$

Then

$$\|X^n \cdot M - Y^n \cdot M\|_{H^2} = \|X^n - Y^n\|_M \overset{n\to\infty}{\longrightarrow} 0 \tag{6.83}$$

Thus we have

$$\lim_{n\to\infty} X^n \cdot M = \lim_{n\to\infty} Y^n \cdot M \tag{6.84}$$

Lastly we have to check, whether $\|X \cdot M - X^n \cdot M\|_{H^2} \to 0$.

$$\|X \cdot M - X^n \cdot M\|_{H^2} \overset{Doob}{\leq} 4 \sup_t \mathbb{E}\left[((X^n \cdot M)_t - (X \cdot M)_t)^2\right] \tag{6.85}$$

$$= 4\|X^n - X\|_M \to 0 \tag{6.86}$$

□

Definition 6.16.
We define

$$\int_0^t X_s dM_s := (X \cdot M)_t \tag{6.87}$$

as *Itôs Integral*, where $X \cdot M$ is the unique process from the previous Theorem.

6.4 Properties of Itôs Integral.

Kunita-Watanabe holds exactly as in the previous setting.

Corollary 6.17.
Let $M, N \in H^2, X \in L^2(M), Y \in L^2(N)$. Then

a) $\langle X \cdot M \rangle_t = \int_0^t X_s^2 d\langle M \rangle_s = (X^2 \cdot \langle M \rangle)_t$

b) $\langle X \cdot M, Y \cdot N \rangle_t = \int_0^t X_s Y_s d\langle M, N \rangle_s = ((XY) \cdot \langle M, N \rangle)_t$

c) $|\mathbb{E}[\langle X \cdot M, Y \cdot N \rangle_t]| \le \mathbb{E}\left[\int_0^t |X_s||Y_s||d\langle M, N \rangle|\right] \le \sqrt{\mathbb{E}\left[\int_0^t X_s^2 d\langle M \rangle_s\right]} \sqrt{\mathbb{E}\left[\int_0^t Y_s^2 d\langle N \rangle_s\right]}$

Lemma 6.18.
Let $X \in L^2(M)$ and $Y \in L^2(X \cdot M)$. Then

$$XY \in L^2(M) \tag{6.88}$$

and the associative property holds, i.e.

$$Y \cdot (X \cdot M) = (YX) \cdot M. \tag{6.89}$$

Proof. **Step 1:** $XY \in L^2(M)$
It holds

$$\langle X \cdot M \rangle = X^2 \cdot \langle M \rangle \tag{6.90}$$

and thus

$$\infty \overset{Y \in L^2(X \cdot M)}{>} \mathbb{E}\left[\int_0^\infty Y_t^2 d\langle X \cdot M \rangle_t\right] = \mathbb{E}\left[\int_0^\infty Y_t^2 d(X^2 \cdot \langle M \rangle)_t\right] \overset{\text{Assoc. Stieltj.}}{=} \mathbb{E}\left[\int_0^\infty Y_t^2 X_t^2 d\langle M \rangle_t\right] \tag{6.91}$$

Step 2: Associativity.
Let $N \in H^2$ arbitrary. Then

$$\langle (YX) \cdot M, N \rangle \overset{6.17}{=} (YX) \cdot \langle M, N \rangle \underset{\text{Stieltj.}}{\overset{\text{Assoc.}}{=}} Y \cdot (X \cdot \langle M, N \rangle) \overset{6.17}{=} Y \cdot \langle X \cdot M, N \rangle \overset{6.17}{=} \langle Y \cdot (X \cdot M), N \rangle \tag{6.92}$$

Hence we have

$$\langle [(YX) \cdot M] - [Y \cdot (X \cdot M)], N \rangle = 0 \; \forall N \in H^2 \tag{6.93}$$

and thus $(YX) \cdot M = Y \cdot (X \cdot M)$. □

Proposition 6.19.
Let $X \in L^2(M), T$ a stopping time. Then

$$(X \cdot M)^T = X \cdot M^T = (X \mathbb{1}_{[0,T]}) \cdot M \tag{6.94}$$

Proof. Follows from the Lemma above since

$$M^T = \mathbb{1}_{[0,T]} M \tag{6.95}$$

□

Lemma 6.20.
Let $X, Y \in L^2(M), 0 \leq s \leq u < t$. Then the following properties hold

a) $\int_s^t X_v dM_v = \int_s^u X_v dM_v + \int_u^t X_v dM_v$

b) $\int_s^t (\alpha X_v + \beta Y_v) dM_v = \alpha \int_s^t X_v dM_v + \beta \int_s^t Y_v dM_v$

c) $s < t \Rightarrow \mathbb{E}\left[\int_s^t X_v dM_v\right] = 0$

d) $\mathbb{E}\left[\int_0^t X_v dM_v | \mathcal{F}_s\right] = \int_0^s X_v dM_v$

Proof. **a)** and **b)** are obvious. **c)** and **d)** hold since

$$N_t := \int_0^t X_v dM_v \tag{6.96}$$

is a Martingale. □

[20.11.2012]
[23.11.2012]

6.5 The Itô-Integral for continuous local semimartingales

Let V be a semimartingale. Therefore we can write $V = M + A$ with $M \in \mathcal{M}_{loc}$ and $A \in \mathcal{A}$. We already defined

$$(X \cdot A)_t = \int_0^t X_s dA_s \tag{6.97}$$

where $X \in \mathcal{B} := \{X :$ adapted, left-continuous, the trajectories are locally bounded$\}$.
By definition $M \in \mathcal{M}_{loc}$ *iff* $\exists (T_n)$ stopping times $T_n \nearrow \infty$ s.t. M^{T_n} a Martingale. We also know for a Martingale M

$$(X \cdot M)^T = X \cdot M^T \tag{6.98}$$

Therefore for a local martingale M the following definition makes sense

$$X \cdot M = \lim_{n \to \infty} X \cdot M^{T_n} \tag{6.99}$$

and so for a Seminartingale $V = M + A$

$$X \cdot V = (X \cdot M) + (X \cdot A) \tag{6.100}$$

We are now doing this calculation step by step.

Definition 6.21.
For $M \in \mathcal{M}_{loc}$ we define

$$\mathcal{L}^2_{loc}(M) = \{X : X \text{ is measurable, predictable and } \forall t \in [0, \infty) : \mathbb{P}\left(\int_0^t X_s^2 d\langle M \rangle_s < \infty\right) = 1\} \tag{6.101}$$

$$L^2_{loc}(M) = \text{space of equivalence classes.} \tag{6.102}$$

Lemma 6.22.
Let $M \in \mathcal{M}_{loc}$. It holds $X \in \mathcal{L}^2_{loc}(M) \Leftrightarrow X$ is predictable, $\exists$ stopping times $(T_n)_{n \in \mathbb{N}} \nearrow \infty$ s.t.

$$\mathbb{E}\left[\int_0^{T_n} X_s^2 d\langle M \rangle_s\right] < \infty \quad \forall n \in \mathbb{N}. \tag{6.103}$$

$$(\equiv X \in \mathcal{L}^2(M^{T_n})) \tag{6.104}$$

Proof. "⇒": Construct T_n:

$$T_n = \inf\{t : \int_0^t X_s^2 d\langle M\rangle_s \geq n\} \nearrow \infty \tag{6.105}$$

By definition $\int_0^{T_n} X_s^2 d\langle M\rangle_s \leq n$ and therefore

$$\mathbb{E}\left[\int_0^{T_n} X_s^2 d\langle M\rangle_s\right] \leq n \tag{6.106}$$

"⇐": Assume $\exists (T_n)$ s.t. $\mathbb{E}\left[\int_0^{T_n} X_s^2 d\langle M\rangle_s\right] < \infty$. Then

$$\mathbb{E}\left[\int_0^{T_n\wedge t} X_s^2 d\langle M\rangle_s\right] < \infty \tag{6.107}$$

$$\Rightarrow \mathbb{P}\left(\int_0^{T_n\wedge t} X_s^2 d\langle M\rangle_s < \infty\right) = 1 \tag{6.108}$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}\left(\int_0^{T_n\wedge t} X_s^2 d\langle M\rangle_s < \infty\right) = 1 \tag{6.109}$$

$$\underset{1}{\Rightarrow} \mathbb{P}\left(\int_0^{t} X_s^2 d\langle M\rangle_s < \infty\right) = 1 \tag{6.110}$$

□

Definition 6.23.
Let $M \in \mathcal{M}_{loc}$ and $X \in L^2_{loc}(M)$. We define the stochastic integral as

$$X \cdot M := \lim_{n\to\infty} (X \cdot M^{T_n}) \tag{6.111}$$

Remark: *Does the limit exist?$m \geq n, t \leq T_n$*

$$(X \cdot M^{T_m})_t = (X \cdot M^{T_m})_t^{T_n} = (X \cdot M^{T_m\wedge T_n}) = (X \cdot M^{T_n})_t \tag{6.112}$$

Therefore the sequence 'stabilizes' at a certain point ⇒ Convergence.

Definition 6.24.
Let $V \in \mathcal{S}$ be a semimartingale with $V = M + A$ where $M \in \mathcal{M}_{loc}, A \in \mathcal{A}$. Let $X \in \mathcal{B}$. We define

$$(X \cdot V) := (X \cdot M) + (X \cdot A) \tag{6.113}$$

Proposition 6.25.
Let $V, W \in \mathcal{S}$ and $X, Y \in B$.

a) $(X, V) \mapsto X \cdot V$ is bilinear.

b) $V \in \mathcal{M}_{loc} \Rightarrow X \cdot V \in \mathcal{M}^0_{loc}$
$V \in \mathcal{A}_0 \Rightarrow X \cdot V \in \mathcal{A}_0$

c) Associativity $(XY) \cdot V = X \cdot (Y \cdot V)$

d) $\langle X \cdot V, Y \cdot W\rangle = (XY) \cdot \langle V, W\rangle (\equiv 0$ if V or $W \in \mathcal{A}.)$

e) $(X \cdot V)^T = (X\mathbb{1}_{[0,t]} \cdot V) = (X \cdot V^T)$

f) Let $a, b \in \mathbb{R} \Rightarrow \mathbb{P}(X_t = 0$ on $[a, b]$ or V_t is const. on $[a, b) \Rightarrow X \cdot V$ is const. on $[a, b]) = 1$

[1] Limes reinziehen, da Folge von absteigenden Mengen, vergl. Ana III Satz 2.10

Proof. **a)** Obvious.

b) Let $V \in \mathcal{M}_{loc}$. Then $\exists S_n \nearrow \infty$ s.t. $V^{S_n} \in \mathcal{M}$. Thus $(X \cdot V^{S_n}) \in \mathcal{M}$. But since $(X \cdot V^{S_n}) = (X \cdot V)^{S_n}$ it follows that $(X \cdot V) \in \mathcal{M}_{loc}$.
For $V \in \mathcal{A}$ see Theorem 6.5.

c) Theorem 6.5 and Lemma 6.18.

d) Corollary 6.17.

e) Theorem 6.5 and Proposition 6.19.

f) Clear for $V \in \mathcal{A}$ by the definition of $(X \cdot V)$ (Lebesgue-Stieltjes).
Now let $V \in \mathcal{M}_{loc}$. By the assumption it holds either

$$X_0(\omega) = 0 \text{ on } [a, b] \tag{6.114}$$

or

$$\langle V\rangle(\omega) \text{ constant on } [a, b]. \tag{6.115}$$

Hence

$$t \mapsto (X^2 \cdot \langle V\rangle)_t = \int_0^t X_s^2 d\langle V\rangle_s \tag{6.116}$$

is constant on $[a, b]$. Since $(X^2 \cdot \langle V\rangle)_t = \langle X \cdot V\rangle_t$ we get that $X \cdot V$ is constant on $[a, b]$. □

Theorem 6.26 (Convergence of Stochastic Integrals)**.**
Let $V \in \mathcal{S}$, and $X^n, Y \in B$ s.t. $|X^n| \le Y \ \forall n$. If

$$X_t^n \overset{n\to\infty}{\longrightarrow} 0 \text{ a.s., } \forall t \ge 0, \tag{6.117}$$

then

$$X^n \cdot V \to 0 \ \mathbb{P}\text{-stochastically, uniformly on compacts.} \tag{6.118}$$

i.e.

$$\forall t \ge 0, \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}\left(\sup_{0\le s\le t} |X^n \cdot V|_s \ge \varepsilon\right) = 0. \tag{6.119}$$

Proof. If $V \in \mathcal{A}_0$ then the statement follow from dominated convergence. So now let $V \in \mathcal{M}_{loc}$ and let T be a stopping time s.t. $V^T \in H^2$ and X^T bounded. Since $(X^n)^T \to 0$, we get by dominated convergence

$$\|(X^n)^T\|_{V^T} = \mathbb{E}\left[\int_0^\infty ((X_s^n)^T)^2 d\langle V^T\rangle_s\right] \to 0 \tag{6.120}$$

Hence

$$(X^n)^T \to 0 \text{ in } L^2(V^T) \tag{6.121}$$

and the L^2-isometry (Theorem 6.15) gives

$$(X^n \cdot V)^T \to 0 \text{ in } H^2 \tag{6.122}$$

and thus

$$(X^n \cdot V)^T \to 0 \text{ uniformly on } \mathbb{R}_+ \ \mathbb{P}\text{-stochastic} \tag{6.123}$$

$$\Rightarrow (X^n \cdot V) \to 0 \text{ locally uniformly } \mathbb{P}\text{-stochastic} \tag{6.124}$$

□

Theorem 6.27 (Approximation by Riemann-sums).
Let $V \in S, X \in B, t > 0$. $\Delta_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ partitions of $[0, t]$, s.t. $\|\Delta_n\| \overset{n\to\infty}{\longrightarrow} 0$. Then for

$$I_s^{\Delta_n}(X, V) := \sum_{t_k \in \Delta_n} X_{t_k}(V_{s\wedge t_{k+1}} - V_{s\wedge t_k}), \tag{6.125}$$

$I^{\Delta_n}(X, V)$ converges stochastically uniformly on $[0, t)$ towards $\int_0^s X_u dV_u$.

Proof. WLOG assume $X_0 = 0$ and X bounded (otherwise there exist $T_n \nearrow \infty$ s.t. X^{T_n} bounded). Consider $X_t^{\Delta_n} = \sum_{t_k\in\Delta_n} X_{t_k} \mathbb{1}_{(t_k,t_{k+1}]}$. Since X is left-continuous $X_t^{\Delta_n} \overset{n\to\infty}{\longrightarrow} X_t$ pointwise. Thus

$$I_s^{\Delta_n}(X, V) = \int_0^s X_u^{\Delta_n} dV_u \tag{6.126}$$

$$= \underbrace{\int_0^s (X_u^{\Delta_n} - X_u) dV_u}_{\to 0 \text{by Theorem 6.26}} + \int_0^s X_u dV_u \tag{6.127}$$

□

Theorem 6.28 (Integration by parts).
Let $X, Y \in \mathcal{S}$. Then it holds

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_y + \langle X, Y\rangle_t \tag{6.128}$$

and in particular

$$X_t^2 = X_0^2 + 2\int_0^t X_s dX_s + \langle X\rangle_t. \tag{6.129}$$

Proof. We show the second statement. The general case follows from polarisation.
Let Δ_n be a partition of $[0, t]$.

$$\langle X\rangle_t \leftarrow \sum_{t_k\in\Delta_n} (X_{t_{k+1}} - X_{t_k})^2 = \sum_{t_k\in\Delta_n} (X_{t_{k+1}} - X_{t_k})(X_{t_{k+1}} - X_{t_k}) \tag{6.130}$$

$$= \sum_{t_k\in\Delta_n} X_{t_{k+1}}(X_{t_{k+1}} - X_{t_k}) - \underbrace{\sum_{t_k\in\Delta_n} X_{t_k}(X_{t_{k+1}} - X_{t_k})}_{=I_t^{\Delta_n}(X,X)} \tag{6.131}$$

$$= \sum_{t_k\in\Delta_n} X_{t_{k+1}}^2 - \sum_{t_k\in\Delta_n} (X_{t_{k+1}} - X_{t_k})X_{t_k} - \sum_{t_k\in\Delta_n} X_{t_k}^2 - I_t^{\Delta_n}(X, X) \tag{6.132}$$

$$\to X_t^2 - X_0^t - 2I_t(X, X) \tag{6.133}$$

for $\|\Delta_n\| \to \infty$. □

Corollary 6.29.
Let X=B=BM.

$$B_t^2 = 2\int_0^t B_s dB_s + \langle B\rangle_t = 2\int_0^t B_s dB_s + t \tag{6.134}$$

$$\int_0^t B_s dB_s = \frac{B_t^2 - t}{2} \tag{6.135}$$

If we write this in differential notation this is

$$d(XY)_t = X_t dY_t + Y_t dX_t + \langle X, Y\rangle_t \tag{6.136}$$
$$= X_t dY_t + Y_t dX_t + dX_t dY_t \tag{6.137}$$

if we define $dX_t dY_t = d\langle X, Y\rangle_t$. Hence

$$(dX_t)^2 = dX_t dX_t = d\langle X\rangle_t \tag{6.138}$$

If $X \in \mathcal{A}_0$ or $Y \in \mathcal{A}_0$ we have

$$dX_t dY_t = 0 \tag{6.139}$$

Thus $\forall X, Y, Z \in \mathcal{S}$:

$$(dX_t dY_t)dZ_t = dX_t(dY_t dZ_t) = 0 \tag{6.140}$$

since $(dX_t dY_t)dZ_t = (d\underbrace{\langle X, Y\rangle}_{\in\mathcal{A}})dZ_t$.

Now consider a BM B. Then we have

$$B_t^2 = B_0^2 + 2\int_0^t B_s dB_s + t \tag{6.141}$$
$$\Rightarrow dB_t^2 = 2B_t dB_t + dt \tag{6.142}$$

Rules for calculation:

$$(dB_t)^2 = dt \tag{6.143}$$
$$dB_t dt = dt dB_t = 0 \tag{6.144}$$
$$(dt)^2 = 0 \tag{6.145}$$

For $d \geq 2$ one gets

$$dB_t^i dB_t^j = \delta_{ij} dt \tag{6.146}$$
$$dB_t^i dt = dt dB_t^i = 0 \tag{6.147}$$
$$(dt)^2 = 0 \tag{6.148}$$

Back to $d = 1$. When we write dV_t we should interpret it as a map from $\{(a, b) \in \mathbb{R}^2, a < b\} \to \mathbb{R}^\Omega$.

$$dV_t : [a, b] \mapsto \int_a^b dV_t = V_b - V_a \tag{6.149}$$

$$d(X \cdot V)_t \equiv X_t dV_t : [a, b] \mapsto \int_a^b X_t dV_t \equiv (X \cdot V)_b - (X \cdot V)_a \tag{6.150}$$

Now recall the associative property, i.e.

$$Y \cdot (X \cdot V) = (YX) \cdot V. \tag{6.151}$$

In the new notation this is

$$d(Y \cdot (X \cdot V)) = Y_t d(X \cdot V)_t = (Y_t X_t) dV_t. \tag{6.152}$$

Kunita-Watabe

$$\langle X \cdot V, Y \cdot W\rangle = (XY) \cdot \langle V, W\rangle \tag{6.153}$$
$$\langle X \cdot V\rangle = X^2 \cdot \langle V\rangle \tag{6.154}$$

becomes

$$X_t dV_t Y_t dW_t = d(X \cdot V)_t d(Y \cdot W)_t = X_t Y_t dV_t dW_t \tag{6.155}$$
$$(d(X \cdot V)_t)^2 = X_t^2 (dV_t)^2. \tag{6.156}$$

Example: *Let $X_t = B_t^2$. We want to get $\langle X \rangle_t$.*

$$d\langle X \rangle_t = (dX_t)^2 \tag{6.157}$$
$$= (dB_t^2)^2 \tag{6.158}$$
$$\overset{6.29}{=} (2B_t dB_t + dt)^2 \tag{6.159}$$
$$= 4B_t^2 \underbrace{(dB_t)^2}_{=dt} + \underbrace{4B_t dB_t dt}_{=0} + \underbrace{(dt)^2}_{=0} \tag{6.160}$$
$$= 4B_t^2 dt \tag{6.161}$$

and hence

$$\langle X \rangle_t = \langle B^2 \rangle = 4 \int_0^t B_s^2 ds \tag{6.162}$$

Now consider the case

$$f \in C^\infty, X_t \text{ "regular function" (finite variation)} \tag{6.163}$$

Then

$$d(f(X))_t = f'(X_t)dX_t + \frac{1}{2} f''(X_t)(dX_t)^2 + \underbrace{\frac{1}{3} f'''(X_t)(dX_t)^3 + \dots}_{=0} \tag{6.164}$$

since $(dX_t)^n = 0$ for $n \geq 3$ (see (6.140)). In the case of a BM we get as a result

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2} f''(B_t)(dB_t)^2 \tag{6.165}$$

This is Itô's-Formula!

[23.11.2012]

[27.11.2012]

7 The Itô-Formula and applications

7.1 The Itô-Formula

Theorem 7.1 (Itô-Formula)**.**
Let $F \in C^2(\mathbb{R}^d, \mathbb{R})$ and $X = (X^1, ..., X^d)$ with $X_i \in \mathcal{S}$. Then $F(X) \in \mathcal{S}$ and

$$F(X_t) = F(X_0) + \sum_{k=1}^{d} \int_0^t \partial_k F(X_s) dX_s^k + \sum_{k,l=1}^{n} \frac{1}{2} \int_0^t \partial_{k,l}^2 F(X_s) d\langle X^k, X^l \rangle_s, \tag{7.1}$$

Remark: *Itô-Formula in differentialform is*

$$dF(X_t) = \sum_{k=1}^{d} \partial_k F(X_t) dX_t^k + \frac{1}{2} \sum_{k,l=1}^{n} \partial_{k,l}^2 F(X_t) d\langle X^k, X^l \rangle_t \tag{7.2}$$

Corollary 7.2.
Let $F \in C^2(\mathbb{R}^d, \mathbb{R})$, $(B_t)_{t \geq 0}$ a d-dimensional BM. Then,

$$F(B_t) = F(B_0) + \int_0^t \nabla F(B_s) dB_s + \frac{1}{2} \int_0^t \Delta F(B_s) ds \tag{7.3}$$

Proof. We use $\langle B^k, B^l \rangle_t = \delta_{k,l} dt$ to see this. □

Corollary 7.3.
Let $F \in C^2(\mathbb{R}^{d+1}, \mathbb{R})$, $(B_t)_{t \geq 0}$ a d-dimensional BM. Then,

$$F(t, B_t) = F(0, B_0) + \int_0^t \nabla F(s, B_s) dB_s + \int_0^t \dot{F}(s, B_s) ds + \frac{1}{2} \int_0^t \Delta F(s, B_s) ds \tag{7.4}$$

where ∇F is the gradient and Δ is the Laplace-operator of F with differentials w.r.t. the space-variables and $\dot{F}$ is the time-derivative.

Remark: *Corollary 7.2 in differential form:*

$$dF(B_s) = \nabla F(B_s) dB_s + \frac{1}{2} \Delta F(B_s) ds \tag{7.5}$$

Corollary 7.3 in differential form:

$$dF(t, B_t) = \nabla F(t, B_t) dB_t + \frac{1}{2} \Delta F(t, B_t) dt + \dot{F}(t, B_t) dt \tag{7.6}$$

Proof of Theorem 7.1. **Step 1)** Prove (7.1) for F being a polynomial.
Let's see first, that (7.1) holds true for $F \equiv 1$. Now assume that (7.1) holds for a polynomial F. We have to show that (7.1) holds for $G(x_1, ..., x_d) = x_m F(x_1, ..., x_d)$. Then Step 1 holds by induction

and linearity.

$$G(X_t) - G(X_0) = X_t^m F(X_t) - X_0^m F(X_0) \tag{7.7}$$

$$\overset{\text{integr.}}{\underset{\text{by parts}}{=}} \int_0^t X_s^m dF(X_s) + \int_0^t F(X_s) dX_s^m + \langle X^m, F(X)\rangle_s \tag{7.8}$$

$$\overset{\text{Itô Form.}}{\underset{\text{for F}}{=}} \sum_{l=1}^d \int_0^t X_s^m \partial_s F(X_s) dX_s^l + \sum_{l,k=1}^d \frac{1}{2} \int_0^t X_s^m \partial_{k,l}^2 F(X_s) d\langle X^k, X^l\rangle_s \tag{7.9}$$

$$+ \int_0^t F(X_s) dX_s^m \tag{7.10}$$

$$+ \sum_{l=1}^d \int_0^t \partial_l F(X_s) d\langle X^m, X^l\rangle_s \tag{7.11}$$

Where we used in the last step that

$$\langle X^m, F(X)\rangle_s = dX_s^m dF(X)_s \tag{7.12}$$

$$= dX_s^m (\sum_{l=1}^d \partial_l F(X_s) dX_s^l + \sum_{k,l=1}^d \frac{1}{2} \partial_{k,l} F(X_s) d\langle X^k, X^l\rangle_s) \tag{7.13}$$

$$= \sum_{l=1}^d \partial_l F(X_s) dX_s^m dX_s^l + \sum_{k,l=1}^d \frac{1}{2} \partial_{k,l} F(X_s) \underbrace{dX_s^m dX_s^k dX_s^l}_{=0} \tag{7.14}$$

Thus we have

$$G(X_t) - G(X_0) = \sum_{k=1}^d \int_0^t (F(X_s)\delta_{k,m} + X_s^m \partial_k F(X_s)) dX_s^k \tag{7.15}$$

$$+ \frac{1}{2} \int_0^t \sum_{k,l=1}^d \partial_{k,l}^2 F(X_s) X_s^m + \partial_k F(X_s) \delta_{l,m} d\langle X^l, X^k\rangle_s \tag{7.16}$$

$$= \sum_{k=1}^d \int_0^t \partial_k G(X_s) dX_s^k + \frac{1}{2} \sum_{k,l=1}^d \int_0^t \partial_{k,l}^2 G(X_s) d\langle X^k, X^l\rangle_s \tag{7.17}$$

Step 2) Extension to $F \in C_0^2(\mathbb{R}^d, \mathbb{R})$ (with bounded support). By the Weierstrass-Approximation theorem we can get F as the limit of polynomials F_n, i.e.

$$F_n \to F \tag{7.18}$$

$$\partial_k F_n \to \partial_k F \tag{7.19}$$

$$\partial_k \partial_l F_n \to \partial_k \partial_l F \tag{7.20}$$

$\Rightarrow$ Itô-Formula holds for $F_n \Rightarrow$ also for $F \in C_0^2(\mathbb{R}^d, \mathbb{R})$.

Step 3) Extension to $F \in C^2(\mathbb{R}^d, \mathbb{R})$.
Let $K_n = [-n, n]^d$ and

$$T_n = \inf\{t > 0 : X_t \notin K_n\} \tag{7.21}$$

Then $T_n \nearrow \infty$ as $n \to \infty$. Now consider $F_n = F \mathbb{1}_{K_n} \in C_0^2(\mathbb{R}^d, \mathbb{R})$. We know that the formula holds for F_n. Therefore it holds for all $\{\omega \in \Omega : T_n(\omega) > t\}$. But as $n \to \infty$ $T_n(\omega) > t \forall \omega \in \Omega \forall t \geq 0$. Therefore the formula holds for all Ω. □

Corollary 7.4.
Let $X = X_0 + M + A, M \in \mathcal{M}^0_{loc}, A \in \mathcal{A}_0$ and $F \in C^2(\mathbb{R}, \mathbb{R})$. Then

$$F(X_t) = F(X_0) + \tilde{M}_t + \tilde{A}_t \tag{7.22}$$

with

$$\tilde{M} \in \mathcal{M}^0_{loc} \text{ and } \tilde{A} \in \mathcal{A}_0 \tag{7.23}$$

where

$$\tilde{M}_t = \int_0^t F'(X_s)dM_s \tag{7.24}$$

$$\tilde{A}_t = \int_0^t F'(X_s)dA_s + \frac{1}{2}\int_0^t F''(X_s)d\langle M\rangle_s \tag{7.25}$$

Let us compute e.g. the quadratic variation of $F(X_t)$.

Corollary 7.5.
Let $X \in \mathcal{S}^d, F \in C^2(\mathbb{R}^d, \mathbb{R})$. Then

$$\langle F(X)\rangle_t = \sum_{k,l=1}^d \int_0^t \partial_k F(X_s)\partial_l F(X_s)d\langle X^k, X^l\rangle_s \tag{7.26}$$

In particular, if $X = B$ is a BM

$$\langle F(B)\rangle_t = \sum_{k=1}^d \int_0^t (\partial_k F(B_s))^2 ds = \int_0^t (\nabla F(B_s))^2 ds \tag{7.27}$$

Proof. The differential form to be proven is

$$d\langle F(X)\rangle_t = \sum_{k,l=1}^d \partial_k F(X_t)\partial_l F(X_t)d\langle X^k, X^l\rangle_t \tag{7.28}$$

Remember: $d\langle X, Y\rangle_t \equiv dX_t dY_t$. Therefore

$$d\langle F(X)\rangle_t \equiv (dF(X_t))^2 \overset{\text{Itô}}{=} (\sum_{k=1}^d \partial_k F(X_t)dX_t^k + \frac{1}{2}\sum_{k,l=1}^d \partial_k\partial_l F(X_t)d\langle X^k, X^l\rangle_t)^2 \tag{7.29}$$

$$\overset{dX_t^k dX_t^l dX_t^m=0}{=} \sum_{k,l=1}^d \partial_k F(X_t)\partial_l F(X_t)\underbrace{dX_t^k dX_t^l}_{=d\langle X^k,X^l\rangle} \tag{7.30}$$

The statement for the BM follows from

$$d\langle B^k, B^l\rangle_s = \delta_{k,l}ds. \tag{7.31}$$

□

Remember one exercise: If $M_t := \exp(\alpha B_t - \frac{1}{2}\alpha^2 t) \in \mathcal{M}$ and B_t is a continuous process with $B_0 = 0$. Then B is a BM. M_t is an example for a so called 'exponential martingale' and will later be the 'Levy characterization'.

Proposition 7.6.

a) Let B be a d-dimensional BM, $f \in C^2(\mathbb{R}^{d+1}, \mathbb{R})$ and

$$Af := \frac{1}{2}\Delta f + \frac{\partial f}{\partial t} \tag{7.32}$$

Then,

$$M_t := f(t, B_t) - f(0, B_0) - \int_0^t Af(s, B_s)ds \in \mathcal{M}^0_{loc} \tag{7.33}$$

In particular, if $Af = 0$, then

$$(f(t, B_t))_{t\geq 0} \in \mathcal{M}^0_{loc} \tag{7.34}$$

b) If $f \in C^2(\mathbb{R}^d)$, then

$$M_t := f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \Delta f(B_s)ds \in \mathcal{M}^0_{loc} \tag{7.35}$$

In particular if f is harmonic on $\mathbb{R}^d$, i.e. $\Delta f = 0$, then $(f(B_t))_{t\geq 0} \in \mathcal{M}_{loc}$ (is a local martingale).

c) Let $D \subset \mathbb{R}^d$ and $T = \inf\{t \geq 0 : B_t \notin D\}$. Then, if f is harmonic on D,

$$f(B^T) - f(B_0) \in \mathcal{M}^0_{loc}. \tag{7.36}$$

Proof. **ad a)** Follows from Cor. 7.3:

$$M_t = f(t, B_t) - f(0, B_0) - \int_0^t (Af)(s, B_s)ds = \int_0^t (\nabla f)(s, B_s)dB_s \in \mathcal{M}_{loc} \tag{7.37}$$

ad b) Follows similarly from Cor. 7.2.

ad c) Take B^T in b). Then one will get M_t^T is $\mathcal{M}^0_{loc}$. Important: We need at least $f \in C^2(D')$ for an D' s.t. $\bar{D} \subset D'$. □

Lemma 7.7.
Let M_t as in Prop. 7.6 a). Then

$$\langle M\rangle_t = \int_0^t |\nabla f(s, B_s)|^2 ds \tag{7.38}$$

Proof.

$$dM_t = (\nabla f)(s, B_s)dB_s \tag{7.39}$$

$$\Rightarrow d\langle M\rangle_t = (dM_t)^2 = (\nabla f(t, B_t))^2 dt \tag{7.40}$$

□

A generalisation:

Proposition 7.8.
Let B be a d-dimensional BM. $\sigma(x) := (\sigma_{i,j}(x))_{1\le i,j\le d}$ a Matrix with continuous coefficients and let X be a continuous, adapted d-dimensional process with

$$X_t^k = X_0^k + \sum_{l=1}^{d} \int_0^t \sigma_{ij}(X_s) dB_s^l \tag{7.41}$$

Then,

a) X^k is a local martingale.

b) For all $f \in C^2(\mathbb{R}_+ \times \mathbb{R}^d)$, let

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t Af(s, X_s) ds \tag{7.42}$$

with

$$Af(t,x) = \frac{\partial}{\partial t} f(t,x) + \frac{1}{2} \sum_{k,l=1}^{d} a_{kl}(x) \partial_{k,l}^2 f(t,x) \tag{7.43}$$

and $a_{kl} = \sum_{m=1}^d \sigma_{km}\sigma_{lm}$ $(\equiv (\sigma\sigma^T)_{kl})$. Then M_t^f is a local martingale.

[27.11.2012]
[30.11.2012]

Proof. **a)** Follows since B is a martingale.
b) We compute first:

$$d\langle X^k, X^l \rangle_t \equiv dX_t^k dX_t^l \overset{\text{hyp}}{=} \sum_{i,j=1}^{d} \sigma_{k,j}(X_t)\sigma_{l,i}(X_t) \underbrace{dB_t^j dB_t^i}_{=d\langle B^i, B^j\rangle_t = \delta_{ij} dt} \tag{7.44}$$

$$= \sum_{i=1}^{d} \sigma_{ki}\sigma_{li} dt = a_{kl} dt \tag{7.45}$$

Thus

$$f(t, X_t) \underset{\text{Form.}}{\overset{\text{Itô}}{=}} f(0, X_0) + \int_0^t \partial_s f(s, X_s) ds + \sum_{k=1}^{d} \int_0^t \partial_k f(s, X_s) dX_s^k + \frac{1}{2} \sum_{k,l=1}^{d} \int_0^t \partial_{k,l} f(s, X_s) \underbrace{d\langle X^k, X^l \rangle_s}_{=a_{k,l}(X_s) ds} \tag{7.46}$$

And therefore

$$M_t^f = \sum_{k=1}^{d} \int_0^t \partial_k f(s, X_s) dX_s^k \in \mathcal{M}_{loc} \tag{7.47}$$

□

7.2 Exponential Martingales

Lemma 7.9.
Let $F \in C^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, s.t. $\partial_t F + \frac{1}{2}\partial_{xx}^2 F = 0$ and $M \in \mathcal{M}_{loc}$.

$$\Rightarrow \tilde{M}_t := F(\langle M \rangle_t, M_t) \in \mathcal{M}_{loc} \tag{7.48}$$

Proof.

$$d\tilde{M}_t = \frac{\partial F}{\partial t} d\langle M\rangle_t + \frac{\partial F}{\partial x} dM_t + \frac{1}{2}\partial^2_{xx} F \cdot d\langle M\rangle_t + \underbrace{\frac{1}{2}\partial^2_{tt}(d\langle M\rangle)^2}_{=0} \tag{7.49}$$

$$\overset{\text{Hyp}}{=} \frac{\partial F}{\partial x}(\langle M\rangle_t, M_t) dM_t \in \mathcal{M}_{loc} \tag{7.50}$$

□

Definition 7.10.
Let $\lambda \in \mathbb{C}, M \in \mathcal{M}$, then

$$\mathcal{E}_\lambda(M)_t := e^{\lambda M_t - \frac{1}{2}\lambda^2 \langle M\rangle_t} \tag{7.51}$$

is called *exponential local martingale*.

Lemma 7.11.
$\lambda \in \mathbb{C}, M \in \mathcal{M}_{loc}$.

$$\Rightarrow \mathcal{E}_\lambda(M) \in \mathcal{M}_{loc} + i\mathcal{M}_{loc} \equiv \mathbb{C}\mathcal{M}_{loc} \tag{7.52}$$

Proof. Take $F(t, x) := e^{\lambda x - \frac{1}{2}\lambda^2 t}$ and apply Lemma 7.9. □

Example: *Choose* $\lambda = i$.

$$\Rightarrow \cos(M_t)^{\frac{1}{2}\langle M\rangle_t} \in \mathcal{M}_{loc} \tag{7.53}$$

$$\sin(M_t) e^{\frac{1}{2}\langle M\rangle_t} \in \mathcal{M}_{loc} \tag{7.54}$$

$$\tag{7.55}$$

Example for a BM. $X_t = F(t, B_t) = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}, \lambda \in \mathbb{R}$.

$$dX_t = d(F(X)) = \partial_x F(B_t) dB_t + \underbrace{\frac{1}{2}\Delta_x F(t, B_t) dt + \partial_t F(t, B_t) dt}_{=0} = \lambda X_t dB_t \tag{7.56}$$

Hence $dX_t = \lambda X_t dB_t$. *Therefore*

$$X_t - \underset{=1}{X_0} = \int_0^t dX_s = \lambda \int_0^t X_s dB_s \tag{7.57}$$

$$\Rightarrow X_t = 1 + \lambda \int_0^t X_s dB_s \tag{7.58}$$

Q.: Is $\mathcal{E}_\lambda(M) \in \mathcal{M}$, i.e. a real, not just a local martingale?
A.: In general no!

Theorem 7.12.
$\mathcal{E}_\lambda(M) \in \mathbb{C}\mathcal{M}$ if at least one of the following conditions are satisfied:

a) M is bounded and $\lambda \in \mathbb{R}$.

b) $\langle M\rangle$ is bounded and $\lambda \in i\mathbb{R}$.

c) $M_0 = 0, \mathbb{E}[\mathcal{E}_\lambda(M)_t] = 1, \forall t \geq 0$, and $\lambda \in \mathbb{R}$.

Proof. **a)**

$$|\mathcal{E}(M)| \leq |\exp(\lambda M_t)\underbrace{\exp(-\frac{\lambda^2}{2}\underbrace{\langle M\rangle_t}_{\geq 0})}_{\leq 1}| \leq \underbrace{|\exp(\lambda M_t)|}_{\text{bounded}} \tag{7.59}$$

Thus $\mathcal{E}(M)$ is bounded hence a martingale.

b)

$$|\mathcal{E}(M)| \leq |\underbrace{\exp(i|\lambda|M_t)}_{\leq 1}\exp(\frac{|\lambda|^2}{2}\langle M\rangle_t)| \tag{7.60}$$

$$\leq \underbrace{|\exp(\frac{|\lambda|^2}{2}\langle M\rangle_t)|}_{\text{bounded}} \tag{7.61}$$

Thus $\mathcal{E}(M)$ is bounded hence a martingale.

ad c) $\mathcal{E}_\lambda(M)_t = e^{\lambda M_t - \frac{1}{2}\lambda^2\langle M\rangle_t} \geq 0$. By Lemma 5.3 we know that $\mathcal{E}_\lambda(M)$ is a supermartingale.

$$\Rightarrow 1 \overset{\text{hyp.}}{=} \mathbb{E}\left[\mathcal{E}_\lambda(M)_t\right] \geq \mathbb{E}\left[\mathcal{E}_\lambda(M)_0\right] \equiv 1 \tag{7.62}$$

$\Rightarrow \mathcal{E}_\lambda(M) \in \mathcal{M}$. (see Remark below.) □

Remark: *Let M_t be a super-martingale s.t. $\mathbb{E}[M_t] = c$ for all t. Claim: M_t is a martingale!*

$$\mathbb{E}[X_t|\mathcal{F}_s] - X_s \leq 0 \tag{7.63}$$

but

$$\mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_s] - X_s] = \mathbb{E}[X_t] - \mathbb{E}[X_s] = 0 \tag{7.64}$$

hence

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \text{ a.e.} \tag{7.65}$$

Let B be a 2-dimensional BM.

$$\Rightarrow f(B_t) = f(B_0) + \int_0^t \nabla f(B_s)dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)ds \tag{7.66}$$

Q.: If f is harmonic on $\mathbb{R}^2$, does it follow that

$$f(B) \in \mathcal{M}? \tag{7.67}$$

Is $\nabla f \in L^2(B)$?

Answer: In general not. Counterexample: Take $f(x,y) = e^{x^2-y^2}\cos(2xy)$.

$$\frac{\partial f(x,y)}{\partial x} = 2xe^{x^2-y^2}\cos(2xy) - e^{x^2-y^2}\sin(2xy)2y \tag{7.68}$$

$$\frac{\partial f(x,y)}{\partial y} = -2yxe^{x^2-y^2}\cos(2xy) - e^{x^2-y^2}\sin(2xy)2x \tag{7.69}$$

$$\Rightarrow \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0 \tag{7.70}$$

$\Rightarrow$ f is harmonic, but $f(B)$ is not a martingale for all t. The problem is that e.g. $\nabla F \notin L^2(B)$ or $f(B_t) \notin L^1$ for t large enough, because:

$$\mathbb{E}[f(B_t)] = \int_{\mathbb{R}^2} f(x,y)\frac{1}{2\pi t}e^{-\frac{x^2+y^2}{2t}}dxdy \tag{7.71}$$

which is not good for $t > 1/2$.

7.3 Levy characterization of the BM

Theorem 7.13 (Levy)**.**
Let X be a d-dimensional, adapted and continuous stochastic process with $X_0 = 0$. Then the following statements are equivalent.

a) X is a d-dimensional BM w.r.t. $\mathcal{F}_t$.

b) $X \in \mathcal{M}^0_{loc}$ and $\langle X^k, X^l \rangle_t = \delta_{k,l} \cdot t, \forall 1 \leq k, l \leq d$.

c) $X \in \mathcal{M}^0_{loc}$ and for all $f = (f_1, ..., f_d)$ with $f_k \in L^2(\mathbb{R}_+, \mathbb{R})$,

$$M_t := \exp\left[i \sum_{k=1}^{d} \int_0^t f_k(s) dX^k_s + \frac{1}{2} \sum_{k=1}^{d} \int_0^t f_k^2(s) ds \right] \in \mathcal{M} + i\mathcal{M} (\equiv \mathbb{C}\mathcal{M}) \tag{7.72}$$

Proof. **"a⇒b":** is already known.
"b⇒c":

$$d(f \bullet X)_t = \sum_{k=1}^{d} f_k(s) dX^k_s \tag{7.73}$$

$$\Rightarrow (f \bullet X)_t = \underbrace{(f \bullet X)_0}_{=0} + \sum_{k=1}^{d} \int_0^t f_k(S) dX^k_s \text{ and} \tag{7.74}$$

$$\langle f \bullet X \rangle_t = \sum_{k,l=1}^{d} \int_0^t f_k(s) f_l(s) \underbrace{d\langle X^k, X^l \rangle_s}_{=\delta_{k,l} ds \text{ by hyp.}} \tag{7.75}$$

$$= \sum_{k=1}^{d} \int_0^t f_k^2(s) ds \tag{7.76}$$

Since $f_k \in L^2(\mathbb{R}_+, \mathbb{R})$

$$\langle f \bullet X \rangle_t = \int_0^t \sum_{k=1}^{d} f_k(s)^2 ds < \infty \tag{7.77}$$

Now $\lambda = i, N_t = \sum_{k=1}^{d} \int_0^t f_k(s) dX^k_s$. $\Rightarrow$ $M_t = \mathcal{E}_{\lambda=i}(N)_t$ and since $\lambda \in i\mathbb{R}$ and $\langle N \rangle_t$ bounded we have $M_t \in \mathbb{C}\mathcal{M}$ by Theorem 7.12 .
"c⇒a": Let $z \in \mathbb{R}^d, T > 0$. Define

$$f_k(s) = z_k \mathbb{1}_{[0,T)}(s) \tag{7.78}$$

Then,

$$\sum_{k=1}^{d} \int_0^t f_k(s) dX^k_s = \sum_{k=1}^{d} z_k X^k_{t \wedge T} \equiv (z, X_{t \wedge T}), \tag{7.79}$$

$$\sum_{k=1}^{d} \int_0^t f_k^2(s) ds = \sum_{k=1}^{d} z_k^2 (t \wedge T) \equiv \|z\|^2 \cdot (t \wedge T) \tag{7.80}$$

The assumption implies that

$$M_t = \exp[i(z, X_{t \wedge T}) + \frac{1}{2}\|z\|^2 (t \wedge T)] \in \mathbb{C}\mathcal{M} \tag{7.81}$$

$\Rightarrow$ For $0 < s < t < T : \forall A \in \mathcal{F}_s$

$$\mathbb{E}\left[\mathbb{1}_A e^{i(z,X_t) + \frac{1}{2}\|z\|^2 t} | \mathcal{F}_s \right] = \mathbb{1}_A e^{i(z,X_s) + \frac{1}{2}\|z\|^2 s} \tag{7.82}$$

Therefore

$$\mathbb{E}\left[\mathbb{1}_A e^{i(z,X_t-X_s)}|\mathcal{F}_s\right] = \underbrace{\mathbb{E}\left[\mathbb{1}_A e^{i(z,X_t-X_s)} e^{\frac{1}{2}\|z\|^2(t-s)}|\mathcal{F}_s\right]}_{=\mathbb{1}_A \text{by } (7.82)} e^{-\frac{1}{2}\|z\|^2(t-s)} \tag{7.83}$$

$$\Rightarrow \mathbb{E}\left[\mathbb{1}_A e^{i(Z,X_t-X_s)}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_A e^{i(Z,X_t-X_s)}|\mathcal{F}_s\right]\right] = \mathbb{E}\left[\mathbb{1}_A e^{-\frac{1}{2}\|z\|^2(t-s)}\right] = \mathbb{P}(A)\, e^{-\frac{1}{2}\|z\|^2(t-s)} \tag{7.84}$$

$\Rightarrow \forall A \in \mathcal{F}_s : \mathbb{E}\left[\mathbb{1}_A e^{i(z,X_t-X_s)}\right] = \mathbb{E}[\mathbb{1}_A]\, e^{-\frac{1}{2}\|z\|^2(t-s)} \Rightarrow \mathbb{E}\left[e^{i(X_t-X_s)}\right] = e^{-\frac{1}{2}\|z\|^2(t-s)}$ and $X_t - X_s$ is independent of $\mathcal{F}_s$($\Rightarrow$ of X_s). $\Rightarrow X$ is a BM. □

We get some corollaries for $d = 1$.

Corollary 7.14.
Let $X \in \mathcal{M}^0_{loc}$ with $\langle X\rangle_t = t$. Then X is a BM.

Corollary 7.15.
Let $X \in \mathcal{M}^0_{loc}$ with

$$t \mapsto X_t^2 - t \in \mathcal{M}^0_{loc} \tag{7.85}$$

Then X is a BM.

Remark: *Continuity is needed! Otherwise, let N_t a Poisson Process with intensity 1, then*

$$\{M_t := N_t - t\}_{t\geq 0} \tag{7.86}$$

is a martingale in continuous time with cadlag trajectories. Also $\langle M\rangle_t = t$, but M_t is not a BM!

[30.11.2012]
[03.12.2012]

7.4 Applications of Ito's Calculus

7.4.1 Brownian Bridge (BB)

A Brownian Bridge for $t \in [0, 1]$ is a BM with $X_0 = 0$ conditioned on $X_1 = 0$.

Definition 7.16 (Brownian Bridge)**.**
A *Brownian Bridge* is a continuous Gaussian Process $(X_t, 0 \leq t \leq 1)$ (where $0 \leq t \leq 1$ is the lifespan) s.t.

(i) $\mathbb{E}[X_t] = 0 \forall t \in [0, 1]$.

(ii) $Cov(X_s, X_t) = s(1-t) \forall 0 \leq s \leq t \leq 1$

We can see $X_t \sim \mathcal{N}(0, t(1-t))$. Therefore $X_1 \sim \mathcal{N}(0, 0), X_0 \sim \mathcal{N}(0, 0)$. So the processes starts and ends at 0.

We know $|X_t| \approx \sqrt{\mathbb{E}\left[X_t^2\right]} = \sqrt{t(1-t)}$. So for t well inside $[0, 1]$ we have $\approx \sqrt{t}$.

Construction

a) Let $B = (B_t)$ be a standard BM. Then

$$X_t = B_t - tB_1 \tag{7.87}$$

is a BB. Check:

- $X_0 = 0 = X_1$ ✓,
- $\mathbb{E}[X_t] = \mathbb{E}[B_t] - t\mathbb{E}[B_1] = 0$ ✓,
- Gaussian Process ✓,
- continuous ✓,
- Now let $0 \le s \le t \le 1$.

$$Cov(X_s, X_t) = \mathbb{E}[(B_s - sB_1)(B_t - tB_1)] \tag{7.88}$$
$$= \mathbb{E}[B_sB_t] - s\mathbb{E}[B_1B_t] - t\mathbb{E}[B_sB_1] + st\mathbb{E}\left[B_1^2\right] \tag{7.89}$$
$$= s \wedge t - st - ts + st = s(1-t)\checkmark \tag{7.90}$$

b) BB is a BM conditioned on $\{B_1 = 0\}$. Problem: $\mathbb{P}(B_1 = 0) = 0$. So for the law

$$\mathcal{L}(X_t, 0 \le t \le 1) = \lim_{\varepsilon \to 0} \mathbb{P}(BM \| B_1| < \varepsilon) \tag{7.91}$$
$$\Rightarrow \mathbb{P}(X_{t_1} \in \cdot, \dots, X_{t_k} \in \cdot) = \lim_{\varepsilon \to 0} \mathbb{P}(B_{t_1} \in \cdot, \dots, B_{t_k} \in \cdot \| B_1| < \varepsilon) \tag{7.92}$$

c) Let B be a BM. Then

$$X_t = \begin{cases} (1-t)B_{\frac{t}{1-t}} & 0 \le t < 1 \\ 0 & t = 1 \end{cases} \tag{7.93}$$

is a BB. Well defined? For $t \nearrow 1 : W_{\frac{t}{1-t}} \sim \frac{1}{\sqrt{1-t}} \Rightarrow X_t \sim \sqrt{1-t} \xrightarrow{t \to 1} 0$. Also $t \mapsto \frac{t}{1-t}$ is monoton, goes to ∞ for $t \to 1$. Check the other conditions:

$$\mathbb{E}[X_t] = (1-t)\mathbb{E}\left[B_{\frac{t}{1-t}}\right] = 0\checkmark \tag{7.94}$$
$$(s \le t)\ Cov(X_s, X_t) = (1-t)(1-s)\mathbb{E}\left[B_{\frac{t}{1-t}} B_{\frac{s}{1-s}}\right] = s(1-t)\checkmark \tag{7.95}$$

Lemma 7.17.
For a BB it holds $(X_t, 0 \le t \le 1) \in \mathcal{S}$. Furthermore $\langle X \rangle_t = t$, but it's not a BM, since it is not a martingale.

Proof. Use $X_t = (1-t)B_{\frac{t}{1-t}}$. Define $B'_t = B_{\frac{t}{1-t}}$. Then B'_t is a martingale w.r.t. $\mathcal{F}'_t = \mathcal{F}_{\frac{t}{1-t}}$. Choose $F(t,x) = (1-t)x$.

$$X_t = (1-t)B'_t = F(t, B'_t) \tag{7.96}$$
$$\Rightarrow F(t, B'_t) = \int_0^t \partial_s F(s, B'_s)ds + \int_0^t \partial_x F(s, B'_s)dB'_s + \frac{1}{2}\int_0^t \underbrace{\partial_x^2 F(s, B'_s)}_{=0} d\langle B' \rangle_s \tag{7.97}$$
$$= \underbrace{-\int_0^t B'_s ds}_{\text{finite variation}} + \underbrace{\int_0^t (1-s)dB'_s}_{\text{martingale term}} \tag{7.98}$$

Thus X_t is a semimartingale. Now for the variation:

$$\langle \int_0^\cdot (1-s)dB'_s \rangle_t = \int_0^t (1-s)^2 d\langle B' \rangle_s \overset{1}{=} \int_0^t (1-s)^2 d\frac{s}{1-s} = \int_0^t (1-s)^2 \frac{(1-s)+s}{(1-s)^2} ds = t \tag{7.99}$$

Therefore by Levy $W_t := \int_0^t (1-s)dB'_s$ is a BM! For the finite variation term we can write

$$-\int_0^t B'_s ds = -\int_0^t \frac{X_s}{1-s} ds \tag{7.100}$$

[1] $\langle B' \rangle_t = \frac{t}{1-t}$ since it's a time change of a BM.

Thus we get:

$$X_t = -\int_0^t \frac{X_s}{1-s} ds + W_t \tag{7.101}$$

where W_t is a BM. And in differential form

$$dX_t = -\frac{X_t}{1-t} dt + dW_t \tag{7.102}$$

□

Remark: *Brownian Bridge* $(X_t, 0 \leq t \leq 1)$*:*

(i) Gaussian process with $\mathbb{E}[X_t] = 0, Cov(X_s, X_t) = s(1-t)$.

(ii) $X_t = B_t - tB_1$ *for B a BM.*

(iii) $X_t = (1-t)B_{\frac{t}{1-t}}$ *for B a BM.*

(iv) Solution of the SDE: $dX_t = -\frac{X_t}{1-t}dt + dW_t$ *where W is a BM.*

7.4.2 Ornstein-Uhlenbeck Process (OU)

Definition 7.18.
Let $B = (B_t)_{t \geq 0}$ be a standard BM. Let $\lambda > 0$, then

$$Y_t = \frac{e^{-\lambda t}}{\sqrt{2\lambda}} B_{e^{2\lambda t}} (t \geq 0) \tag{7.103}$$

is a Ornstein-Uhlenbeck Process.

The process does not necessarily start in 0. $Y'_t = Y_t - Y_0$ is an OU issued at 0. We can see:

$$\mathbb{E}[Y_t] = \frac{e^{-\lambda t}}{\sqrt{2\lambda}} \mathbb{E}[B_{e^{2\lambda t}}] = 0 \tag{7.104}$$

$$\mathbb{E}\left[Y_t^2\right] = \frac{e^{-2\lambda t}}{2\lambda} \mathbb{E}\left[B_{e^{2\lambda t}}^2\right] = \frac{1}{2\lambda} \tag{7.105}$$

Lemma 7.19.
Let Y be an OU-Process. Then it holds $(Y_t) \in \mathcal{S}$ and $\langle Y \rangle_t = t$, but Y is not a martingale.

Proof. We set $B'_t = B_{e^{2\lambda t}}$, then

$$Y_t = \frac{e^{-\lambda t}}{\sqrt{2\lambda}} B'_t \tag{7.106}$$

B'_t is a martingale wrt.t. $\mathcal{F}'_t = \mathcal{F}_{e^{2\lambda t}}$. ($t \mapsto e^{2\lambda t}$ is increasing.) Now choose $F(t,x) = \frac{e^{-\lambda t}}{\sqrt{2\lambda}} x$. Then $Y_t = F(t, B'_t)$.

$$Y_t = F(t, B'_t) = \int_0^s \partial_s F(s, B'_s) ds + \int_0^t \partial_x F(s, B'_s) dB'_s \tag{7.107}$$

$$= \underbrace{-\lambda \int_0^t \frac{e^{-\lambda s}}{\sqrt{2\lambda}} B'_s ds}_{\text{finite variation process}} + \underbrace{\int_0^t \frac{e^{-\lambda s}}{\sqrt{2\lambda}} dB'_s}_{\text{martingale part}} \tag{7.108}$$

Hence Y_t is a semimartingale. For the variation, see that

$$\langle \int_0^{\cdot} \frac{e^{-\lambda s}}{\sqrt{2\lambda}} dB'_s \rangle_t = \int_0^t \frac{e^{-2\lambda s}}{2\lambda} d\langle B' \rangle_s \tag{7.109}$$

$$= \int_0^t \frac{e^{-2\lambda s}}{2\lambda} d(e^{2\lambda s}) \tag{7.110}$$

$$= \int_0^t \frac{e^{-2\lambda s}}{2\lambda} 2\lambda e^{2\lambda s} ds = t \tag{7.111}$$

$$\Rightarrow dY_t = -\lambda \frac{e^{-\lambda t}}{\sqrt{2\lambda}} B'_t dt + dW_t \tag{7.112}$$

where W_t is a BM.

$$dY_t = -\lambda Y_t dt + dW_t \tag{7.113}$$

□

So the OU is the solution of the 'easiest' linear stochastic differential equation.

Remark: *"A particle in a Brownian Potential".*

Newton: $F = m \cdot a$. (m=1). $F = ma = a = \dot{v} = -\xi v + W$ where W is a random force action of the particle.

7.4.3 Bessel Processes (BP)

Let $(B_t)_{t\geq 0}$ be a d-dimensional BM, issued at $x \neq 0$ on some probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^x)$. We define $R_t := \|B_t\| = \sqrt{(B_t^1)^2 + (B_t^2)^2 + \cdots + (B_t^d)^2}$

Remark: $y \in \mathbb{R}^d, \|y\| = \|x\|$. *Then there exists a rotation matrix s.t.* $y = Ox$ *and* $OO^T = \mathbb{1}$.

Since the distribution of a standard BM is symmetric around 0, the distribution of R_t solely depends on $\|x\| = r$. Hence from now on we will write

$$\hat{\mathbb{P}}^r = \mathbb{P}^{(r,0,\dots,0)} \tag{7.114}$$

where $\mathbb{P}^{(r,0,\dots,0)}$ is the mass of a BM issued at $(r, 0, ..., 0)$.

Definition 7.20.
Let $r \geq 0, d \geq 2$. Then $R_t = \|B_t\|$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \hat{\mathbb{P}}^r)$ is a Bessel Process of dimension d.

Consider $F : \mathbb{R}^d \to \mathbb{R}, x = (x_1, ..., x_n) \mapsto \sqrt{x_1^2 + ... + x_n^2} \Rightarrow R_t = F(B_t)$ and $\nabla F = \frac{x}{\|x\|}$.

Theorem 7.21.
$B = (B_t)$ a d-dim BM, $d \geq 2$, $B_0 = x$. $R_t = \|B_t\|$.

a) $X_t := \sum_{k=1}^d X_t^k$ where $X_t^k := \int_0^t \frac{B_s^k}{R_s} dB_s$. Then $(X_t)_{t\geq 0}$ is a 1-dim BM.

a) $dR_t = \frac{d-1}{2R_t} dt + dW_t$ where W_t is a BM but $\neq B$.

[03.12.2012]
[07.12.2012]

Proof. **a)** $Leb(0 \le s \le t : R_s = 0) \le Leb(0 \le s \le t : B_s = 0) = 0$.

$$\langle X^k, X^l\rangle_t = \int_0^t \frac{B_s^k B_s^l}{R_s^2} \underbrace{d\langle B^k, B^l\rangle_s}_{\delta_{kl} ds} = \begin{cases} 0 & k \neq l \\ \int_0^t \frac{(B_s^k)^2}{R_s^2} ds & k = l \end{cases} \tag{7.115}$$

$$\Rightarrow \langle X\rangle_t = \sum_{k,l} \langle X^k, X^l\rangle_t = \sum_k \int_0^t \frac{(B_s^k)^2}{R_s^2} ds = \int_0^t \frac{\sum_k (B_s^k)^2}{R_s^2} ds \overset{\sum_k (B_s^k)^2 = R_s^2}{=} t \tag{7.116}$$

By Levy: X is a BM.

b) $R_t = \|B_t\| = F(B_t), F : \mathbb{R}^d \to \mathbb{R}_+, x = (x_1, ..., x_d) \mapsto \sqrt{(x_1)^2 + ... + (x_d)^2}$. Ito's Formula. Caution: singularity of $\nabla F, \nabla^2 F$ at $x = 0$! Way out: $\forall \varepsilon > 0 : \|B_\varepsilon\| > 0$. $K \in \mathbb{N}, F_K \equiv F$ on $B^c_{1/k}(0)$.

Define $T_{K,l} = \inf\{t \ge \frac{1}{l} : \|B_t\| \le 1/K\} \underset{K\to\infty}{\nearrow} \inf\{t \ge 1/l : \|B_t\| = 0\} = +\infty$. But on $\{(t,\omega) : T_{K,l}(\omega) \ge t \ge 1/l\}$ Ito's formula is valid for F_K and $F_K \equiv F$.

$$F(B_t) = F(B_{1/l}) + \int_{1/l}^t \sum_{i=1}^d \partial_i F(B_s) dB_s^i + 1/2 \int_{1/l}^t \sum_{i,j} \partial_{i,j} F(B_s) d\langle B^i, B^j\rangle_s = \Delta \tag{7.117}$$

Note: $\partial_i F(x) = \frac{x_i}{\|x\|}, \partial_{i,j} F(x) = \frac{\delta_{ij}}{\|x\|} - \frac{x^i x^j}{\|B_s\|^2}$

$$\Delta = ... = R_{1/l} + X_t - X_{1/l} + \frac{1}{2} \int_{1/l}^t \frac{d-1}{R_s} ds \tag{7.118}$$

Let K, l to infinity, by continuity

$$R_t = R_0 + X_t + \frac{1}{2} \int_0^t \frac{d-1}{R_s} ds \tag{7.119}$$

□

Remark:

$$dR_t = \underbrace{\frac{d-1}{R_t} dt}_{\text{blows up for } R_t \text{ small}} + dX_t \tag{7.120}$$

$\Rightarrow$ *pushed away from 0.*

Proposition 7.22.
Let $d = 1, \alpha \ge 0$.

a) $\mathbb{P}(\|B_t\| = \alpha$ for some $t) = 1 (d = 1)$

b) $d = 2, \alpha > 0, \mathbb{P}^x(\|B_t\| = \alpha$ for some $t) = 1 (x \neq 0)$

c) $d \ge 3, \mathbb{P}^x(\|B_t\| = \alpha$ for some $t) = \min\{1, \frac{\alpha}{\|x\|}\}^{d-2}$

d) $d \ge 2, \mathbb{P}^x(\|B_t\| = 0$ for some $t > 0) = 0$

e) $d \ge 3, \mathbb{P}^x(\lim_{t\to\infty} \|B_t\| = +\infty) = 1$ BM in $d \ge 3$ is transient

8 Stochastic differential equations

Problem/Setting: X is a d-dimensional stochastic process, we know its evolution, i.e.

$$\text{(EQ1)} \begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = \xi \end{cases} \tag{8.1}$$

where W is a BM on $\mathbb{R}^n$, ξ can be a random variable or a constant.

Definition 8.1.
We define

$$b(t, x) = [b_i(t, x)]_{1 \le i \le d} \text{ the } \textit{drift vector}. \tag{8.2}$$
$$\sigma(t, x) = [\sigma_{i,j}(t, x)]_{1 \le i \le d, 1 \le j \le n} \text{ the } \textit{dispersion matrix}. \tag{8.3}$$

From now on tacitly assume that W is a standard n-dimensional BM and that ξ is a random vector and that the two are independent.

Assumptions: $\forall i, j$:

$$b_i : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \tag{8.4}$$
$$\sigma_{i,j} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \tag{8.5}$$
$$a_{ij} = (\sigma\sigma^T)_{ij} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \tag{8.6}$$

are measurable.

Notation:

$$(a_{ij})_{1 \le i,j \le d} \text{ with } a_{ij} = \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} \tag{8.7}$$

is called *Diffusion Matrix*.

Definition 8.2.
We define the following norms

$$\|b(t, x)\| := \sqrt{\sum_{i=1}^{d} b_i(t, x)^2} \tag{8.8}$$
$$\|\sigma(t, x)\| := \sqrt{\sum_{i=1}^{d} \sum_{j=1}^{n} \sigma_{i,j}^2(t, x)} \tag{8.9}$$

Q.: What do we understand under a solution of EQ1?

8.1 Strong solutions to SDE

Given:

- Standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$.
- W, ξ both given

- $\mathcal{F}_t^W = \sigma(W_s, s \leq t), \mathcal{F}_t = \mathcal{F}_t^W \vee \sigma(\xi) = \sigma(W_s, 0 \leq s \leq t, \xi)$

Definition 8.3 (Strong solution)**.**
A *strong solution* to EQ1 is a $\mathbb{R}^d$-process (X_t) (on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$) s.t.

a) $X_0 = \xi$ a.s.

b) X is $\mathcal{F}_t$-adapted.

c) X is a continuous semimartingale s.t. $\forall t < \infty$

$$\int_0^t \|b(s, X_s)\| + \|\sigma(s, X_s)\|^2 ds < \infty \text{ } \mathbb{P}\text{-a.s.} \tag{8.10}$$

d) $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$ $\mathbb{P}$-a.s. (the Ito Integral)

Definition 8.4 (Strong uniqueness)**.**
For (EQ1) holds *strong uniqueness* if the following holds: If X and $\tilde{X}$ are strong solutions to (EQ1) then X and $\tilde{X}$ are indistinguishable, i.e.

$$\mathbb{P}\left(X_t = \tilde{X}_t \forall t\right) = 1 \tag{8.11}$$

Check lecture notes for a deterministic example where uniqueness does not hold.

Definition 8.5.
A function f is called *locally lipschitz continuous iff*

$$\forall n \geq 1 \exists 0 < K_n < \infty \text{ s.t.} \forall x, y : \|x\| \leq n, \|y\| \leq n, \|f(x) - f(y)\| \leq K_n \|x - y\| \tag{8.12}$$

Theorem 8.6.
Assume b, σ are locally lipschitz. Then strong uniqueness for (EQ1) holds.

Remark: *The exact condition is*

$$\forall n \in \mathbb{N} \exists K_n < \infty \forall t \geq 0 \forall x, y \in \mathbb{R}^d : \|x\| \leq n, \|y\| \leq n : \tag{8.13}$$

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K_n \|x - y\| \tag{8.14}$$

Lemma 8.7 (Gronwall's Lemma)**.**
Let $g : [0, t] \to \mathbb{R}$ continous, $h : [0, T] \to \mathbb{R}$ integrable, $\beta \geq 0$. Then if

$$0 \leq g(t) \leq h(t) + \beta \int_0^t g(s) ds \forall t \in [0, T] \tag{8.15}$$

then

$$g(t) \leq h(t) + \beta \int_0^t h(s) e^{\beta(t-s)} ds \forall t \in [0, T] \tag{8.16}$$

Remark: *If* $h \equiv 0 \Rightarrow g(t) = 0 \forall t \in [0, T]$*. Therefore if* $0 \leq g(t) \leq \beta \int_0^t g(s)ds \Rightarrow g = 0$*!*

Proof.

$$\frac{d}{dt}(e^{-\beta t} \int_0^t g(s) ds) = ... \tag{8.17}$$

□

Proof of the Thm. Let $X, \tilde{X}$ be strong solutions. Define

$$\tau_m = \inf\{t \geq 0 : \|X_t\| \geq m\}, \tag{8.18}$$
$$\tilde{\tau}_m = \inf\{t \geq 0 : \|\tilde{X}_t\| \geq m\}. \tag{8.19}$$

Easy: $\tilde{\tau}_m, \tau_m \nearrow \infty$ as $m \to \infty$. Define $S_m = \tau_m \wedge \tilde{\tau}_m$.

$$g(t) := \mathbb{E}\left[\|X_t^{S_m} - \tilde{X}_t^{S_m}\|^2\right] \tag{8.20}$$
$$= \mathbb{E}\left[\| \int_0^{t\wedge S_m} (b(s,X_s) - b(s,\tilde{X}_s)) + \int_0^{t\wedge S_m} (\sigma(s,X_s) - \sigma(s,\tilde{X}_s)) dW_s\|^2\right] \tag{8.21}$$
$$= \sum_{i=1}^d \mathbb{E}\left[\left(\int_0^{t\wedge S_m} \underbrace{b_i(s,X_s) - b_i(s,\tilde{X}_s)}_{=a} ds + \underbrace{\sum_{j=1}^n \int_0^{t\wedge S_m} \sigma_{ij}(s,X_s) - \sigma_{ij}(s,\tilde{X}_s) dW_s^j}_{b+c+d...}\right)^2\right] \tag{8.22}$$
$$\overset{(a+b)^2\leq 2a^2+2b^2}{\leq} C(d,n) \sum_{i=1}^d \mathbb{E}\left[(\int_0^{t\wedge S_m} b_i(s,X_s) - b_i(s,\tilde{X}_s) ds)^2\right] + C\sum_{i,j} \mathbb{E}\left[(\int_0^{t\wedge S_m} \sigma_{ij}(s,X_s) - \sigma_{ij}(s,\tilde{X}_s) dW_s^j)^2\right] \tag{8.23}$$
$$= \Delta \tag{8.24}$$

use: $(a+b+c+...)^2 \leq 2a^2 + 2b^2 + 2c^2 + ...$ By Cauchy Schwarz $(\int f \cdot 1 dy)^2 \leq \int f^2 ds \int 1 dx$ for the first integral, and Ito isometry for the second.

$$\Delta \leq Ct \sum_{i=1}^d \mathbb{E}\left[\int_0^{t\wedge S_m} (b_i(s,X_s) - b_i(s,\tilde{X}_s))^2 ds\right] + C\sum_{i,j} \mathbb{E}\left[\int_0^{t\wedge S_m} (\sigma_{ij}(s,X_s) - \sigma_{ij}(s,\tilde{X}_s)^2 ds\right] \tag{8.25}$$
$$\leq Ct\mathbb{E}\left[\int_0^{t\wedge S_m} \sum_{i=1}^d (b_i(s,X_s) - b_i(s,\tilde{X}_s))^2 ds\right] + C\mathbb{E}\left[\int_0^{t\wedge S_m} \sum_{ij} (\sigma_{ij}(s,X_s) - \sigma_{ij}(s,\tilde{X}_s))^2 ds\right] \tag{8.26}$$
$$= Ct\mathbb{E}\left[\int_0^{t\wedge S_m} \underbrace{\|b(s,X_s) - b(s,\tilde{X}_s)\|^2}_{\leq K_m^2 \|X_s - \tilde{X}_s\|^2} ds\right] + C\mathbb{E}\left[\int_0^{t\wedge S_m} \underbrace{\|\sigma(s,X_s) - \sigma(s,\tilde{X}_s)\|^2}_{\leq ...} ds\right] \tag{8.27}$$
$$\leq CtK_m^2 \int_0^t \underbrace{\mathbb{E}\left[\|X_s^{S_m} - \tilde{X}_s^{S_m}\|^2 ds\right]}_{g(s)} + CK_m^2 \int_0^t \underbrace{\mathbb{E}\left[\|X_s^{S_m} - \tilde{X}_s^{S_m}\|^2\right]}_{g(s)} ds \tag{8.28}$$
$$\leq CK_m^2(1+t) \int_0^t g(s) ds \tag{8.29}$$

Now fix $T > 0$, then $cK_m^2(1+t) \leq cK_m^2(1+T) =: \beta$. Then by Gronwall $g \equiv 0$. But $g(t) = \mathbb{E}\left[\|X_t^{S_m} - \tilde{X}_t^{S_m}\|^2\right] = 0 \forall t \in [0,T]$. Therefore for all such t, $X_t^{S_m} = \tilde{X}_t^{S_m}$ a.s.. Let $m \to \infty, S_m \to \infty$. Then $X_t = \tilde{X}_t$ a.s. $\forall t \in [0,T]$. (by continuity and boundedness statement of theorem) □

[07.12.2012]
[11.12.2012]

Theorem 8.8 (Global existence)**.**
Assume $\mathbb{E}\left[\|\xi\|^2\right] < \infty$ and $\exists K > 0$ s.t.

$$\forall t \geq 0 x, y, \in \mathbb{R}^d, \tag{8.30}$$
$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\| \text{ (globally lipschitz)} \tag{8.31}$$

and

$$\forall t \geq 0, x \in \mathbb{R}^d \tag{8.32}$$
$$\|b(t,x)\| + \|\sigma(t,x)\| \leq K(1 + \|x\|) \text{ (linear growth)} \tag{8.33}$$

Then

a) $\exists!$ strong solution of (EQ1)

b) $\forall T \geq 0, \exists C > 0$ s.t. $\forall 0 \leq t \leq T$

$$\mathbb{E}\left[\|X_t\|^2\right] \leq C(T)(1 + \mathbb{E}\left[\|\xi\|^2\right]) \tag{8.34}$$

Remark: *The theorem also holds without the condition* $\mathbb{E}\left[\|\xi\|^2\right] < \infty$

Proof. **Idea:** Picard-Lindelöf-Iteration. Let

$$f(X_t) := \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \tag{8.35}$$

and we define

$$X_t^0 := \xi \tag{8.36}$$
$$X_t^{k+1} := f(X_t^k). \tag{8.37}$$

Hence, X_t^k is an adapted and continuous semimartingale. We want to show that $X_t^k \stackrel{k\to\infty}{\longrightarrow} X_t$ with $f(X_t) = X_t$ (fixpoint), i.e. X_t is the solution of (EQ1). But first we need the following lemma. □

Lemma 8.9.
For all $T > 0, \exists C > 0$ (which depends on K and T) s.t. $\forall k \geq 0$

$$\mathbb{E}\left[\|X_t^k\|^2\right] \leq C(1 + \mathbb{E}\left[\|\xi\|^2\right]) \; \forall 0 \leq t \leq T. \tag{8.38}$$

Proof. $k = 0$:

$$\mathbb{E}\left[\|X_t^0\|^2\right] = \mathbb{E}\left[\|\xi\|^2\right] \leq 1 + \mathbb{E}\left[\|\xi\|^2\right] \checkmark \tag{8.39}$$

For any k:

$$\mathbb{E}\left[\|X_t^{k+1}\|^2\right] = \sum_{i=1}^d \mathbb{E}\left[(X_t^{k+1,i})^2\right] \tag{8.40}$$

$$\underset{(\sum_{i=1}^m \alpha_i)^2 \leq m \sum_{i=1}^m \alpha_i^2}{\overset{X^{k+1}=f(X^k)}{\leq}} 3\sum_{i=1}^d \mathbb{E}\left[(\xi^i)^2 + (\int_0^t b_i(s, X_s^k)ds)^2 + (\sum_{j=1}^n \int_0^t \sigma_{ij}(s, X_s^k)dW_s^j)^2\right] \tag{8.41}$$

$$\underset{\text{Itô for } \sigma}{\overset{\text{Hölder for } b_i}{\leq}} 3\mathbb{E}\left[\|\xi\|^2\right] + 3t\mathbb{E}\left[\underbrace{\int_0^t \|b(s, X_s^k)\|^2 ds}_{\leq K^2 2\int_0^t (1+\|X_s^k\|)^2 ds}\right] + 3\mathbb{E}\left[\underbrace{\int_0^t \|\sigma(s, X_s^k)\|^2 ds}_{\leq K^2 2\int_0^t (1+\|X_s^k\|^2)ds}\right] \tag{8.42}$$

$$\overset{0\leq t\leq T}{\leq} 3\mathbb{E}\left[\|\xi\|^2\right] + 6K^2(T+1)\int_0^t (1 + \mathbb{E}\left[\|X_s^k\|^2\right])ds \tag{8.43}$$

Thus

$$\Rightarrow \underbrace{\mathbb{E}\left[\|X_t^{k+1}\|^2\right]}_{=:g^{k+1}(t)} \leq 3\mathbb{E}\left[\|\xi\|^2\right] + 6K^2(T+1)\int_0^t (1+\mathbb{E}\left[\|X_s^k\|^2\right])ds \tag{8.44}$$

Then

$$g^{k+1}(t) \leq C_1 + C_2 \int_0^t (1+g_s^k)ds \tag{8.45}$$

$$\leq C_1 + C_2\int_0^t 1ds + C_2\int_0^t ds_1(C_1 + C_2\int_0^{s_1} ds_2 1 + g_{s_2}^k) \tag{8.46}$$

$$\leq ... \tag{8.47}$$

Recursively and

$$\int_0^t ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{k-1}} ds_k 1 = \frac{t^k}{k!} \tag{8.48}$$

$$\Rightarrow \mathbb{E}\left[\|X_t^{k+1}\|^2\right] \leq C(T,K)(1+\mathbb{E}\left[\|\xi\|^2\right] \forall 0 \leq t \leq T \tag{8.49}$$

□

Continuation of the proof of the theorem.
Step 1) For X^k continuous, adapted and well-defined, then also X^{k+1} is continuous, adapted and well-defined.
Indeed: - Continuity and adaptedness from the definition of the integral.
- Condition c) of Def 8.2 holds:

$$\int_0^t (\|b(s,X_s^k)\| + \|\sigma(s,X_s^k)\|^2)ds \overset{\text{C.S. on } b}{\leq} t\int_0^t \|b(s,X_s^k)\|^2 ds + \int_0^t \|\sigma(s,X_s^k)\|^2 ds \tag{8.50}$$

$$\leq (1+t)2K^2 \int_0^t (1+\|X_s^k\|^2)ds < \infty \forall t < \infty \tag{8.51}$$

Step 2: Estimate $X^{k+1} - X^k$
For fixed k it holds

$$X^{k+1} - X^k = B + M \tag{8.52}$$

with

$$B_t = \int_0^t b(s,X_s^k) - b(s,X_s^{k-1})ds, \tag{8.53}$$

$$M_t = \int_0^t \sigma(s,X_s^k) - \sigma(s,X_s^{k-1})dW_s. \tag{8.54}$$

Claim: We have

$$\mathbb{E}\left[\sup_{0\leq s\leq t} \|X_s^{k+1} - X_s^k\|^2\right] \leq 2\mathbb{E}\left[\sup_{0\leq s\leq t}\|M_s\|^2\right] + 2\mathbb{E}\left[\sup_{0\leq s\leq t}\|B_t\|^2\right] \tag{8.55}$$

Proof:

$$\|B_t\|^2 = \sum_{i=1}^{d}(B_t^i)^2 \tag{8.56}$$

$$= \sum_{i=1}^{d}(\int_0^t b_i(s,X_s^k) - b_i(s,X_s^{k-1})ds)^2 \tag{8.57}$$

$$\overset{CS and 0\leq t\leq T}{\leq} T\sum_{i=1}^{d}\int_0^t (b_i(s,X_s^k) - b_i(s,X_s^{k-1}))^2 ds \tag{8.58}$$

$$= T\int_0^t \underbrace{\|b(s,X_s^k) - b(s,X_s^{k-1})\|^2}_{\leq K^2\|X_s^k - X_s^{k-1}\|^2 by Lipschitz} ds \tag{8.59}$$

Hence

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\|B_s\|^2\right] \overset{1}{\leq} K^2 T\int_0^t ds\, \underbrace{\mathbb{E}\left[\|X_s^k - X_s^{k-1}\|^2\right]}_{=\mathbb{E}\left[\sup_{0\leq s\leq t}\|X_s^k - X_s^{k-1}\right]} \tag{8.60}$$

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\|M_s\|^2\right] = \mathbb{E}\left[\sup_{0\leq s\leq t}\sum_{i=1}^{d}(M_s^i)^2\right] \tag{8.61}$$

$$\leq \sum_{i=1}^{d}\mathbb{E}\left[\sup_{0\leq s\leq t}(M_s^i)^2\right] \tag{8.62}$$

$$\overset{Doob}{\leq} 4\sum_{i=1}^{d}\mathbb{E}\left[(M_t^i)^2\right] \tag{8.63}$$

$$\leq 4\sum_{i=1}^{d}\mathbb{E}\left[(\sum_{j=1}^{n}\int_0^t(\sigma_{ij}(s,X_s^k) - \sigma_{ij}(s,X_s^{k-1}))dW_s^j)^2\right] \tag{8.64}$$

$$\overset{ItoIsom}{=} 4\sum_{i=1}^{d}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t(\sigma_{ij}(s,X_s^k) - \sigma_{ij}(s,X_s^{k-1}))^2 ds\right] \tag{8.65}$$

$$= 4\mathbb{E}\left[\int_0^t ds\, \underbrace{\|\sigma(s,X_s^k) - \sigma(s,X_s^{k-1})\|^2}_{\leq K^2\|X_s^k - X_s^{k-1}\|^2}\right] \tag{8.66}$$

Thus

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\|M_s\|^2\right] \leq 4K^2\int_0^t \mathbb{E}\left[\sup_{0\leq s\leq t}\|X_u^k - X_u^{k-1}\|^2\right] \tag{8.67}$$

$$\Rightarrow \mathbb{E}\left[\sup_{0\leq s\leq t}\|X_s^{k+1} - X_s^k\|^2\right] \leq 2K^2(4+T)\int_0^t ds\mathbb{E}\left[\sup_{0\leq u\leq s}\|X_u^k - X_u^{k-1}\|^2\right] \tag{8.68}$$

Iterations as in Lemma 8.9 give

$$\leq \frac{(c_1 t)^k}{k!}c_s with c_1 = 2K^2(T+4) and \tag{8.69}$$

$$c_2 = T\sup_{0\leq s\leq T}\mathbb{E}\left[\|X_s^1 - \xi\|^2\right] < \infty \tag{8.70}$$

[1]Supremum wird ganz rechts bei t angenommen da integral über was positives

last $< \infty$ since

$$\mathbb{E}\left[\|X_s^1 - \xi\|^2\right] \le 2\mathbb{E}\left[\|X_s^1\|^2\right] + 2\mathbb{E}\left[\|\xi\|^2\right] \overset{lemma}{\le} 2(c+1)\mathbb{E}\left[\|\xi\|^2\right] \tag{8.71}$$

We have

$$\mathbb{E}\left[\sup_{0\le s\le t} \|X_s^{k+1} - X_s^k\|^2\right] \le C_2 \frac{(C_1 t)^k}{k!} \tag{8.72}$$

Step 3: uniform convergence on $[0, T]$ for all fixed $T > 0$.

$$\mathbb{P}\left(\sup_{0\le s\le T} \|X_s^{k+1} - X_s^k\| \ge \frac{1}{2^{k+1}}\right) \overset{Cebicevand(8.72)}{\le} 4c_2 \frac{(4c_1 T)^k}{k!} \tag{8.73}$$

Since $\sum_k \sup_{0\le s\le T} \|X_s^{k+1} - X_s^k\| \ge \frac{1}{2^{k+1}} < \infty$ we can use Borel Cantelli which implies

$$\exists \Omega^* : \mathbb{P}(\Omega^*) = 1 s.t. \forall \omega \in \Omega^* \exists N = N(\omega) s.t. \tag{8.74}$$

$$\forall k \ge N(\omega) \sup_{0\le s\le T} \|X_s^{k+1} - X_s^k\| \le \frac{1}{2^{k+1}} \tag{8.75}$$

$$\Rightarrow \forall k \ge N(\omega), m \ge 1 \sup_{0\le s\le T} \|X_s^{m+k} - X_s^k\| \le \frac{1}{2^k} \tag{8.76}$$

Hence the sequence $\{X_t^k, 0 \le t \le T\}_{k\ge1}$ converges in the sup-norm to a continuous process $\{X_t, 0 \le t \le T\} \forall \omega \in \Omega^*$. $\Rightarrow$ But T is any positive time.

$$\Rightarrow X^k \overset{unif}{\to} X for any bounded time interval. \tag{8.77}$$

Step 4: Verify b)

$$\mathbb{E}\left[\|X_t\|^2\right] = \mathbb{E}\left[\lim_{k\to\infty} \|X_t^k\|^2\right] \tag{8.78}$$

$$\le \liminf_{k\to\infty} \mathbb{E}\left[\|X_t^k\|^2\right] \tag{8.79}$$

$$\overset{Lemma}{\le} C(1 + \mathbb{E}\left[\|\xi\|^2\right]) \tag{8.80}$$

Step 5: Check that $X_t = \lim_{k\to\infty} X_t^k$ satisfies (EQ1)

$$\underbrace{X_t^{k+1}}_{\to X_t} = \underbrace{\xi}_{\to X_0} + \underbrace{\int_0^t b(s, X_s^k) ds}_{\to \int_0^t b(s,X_s)ds??} + \underbrace{\int_0^t \sigma(s, X_s^k) dW_s}_{\to \int_0^t \sigma(x,X_s)dW_s??} \tag{8.81}$$

□

[11.12.2012]
[14.12.2012]

Recap:

$$X_t = \frac{e^{-\lambda t}}{\sqrt{2\lambda}} B_{e^{2\lambda t}} \rightsquigarrow dX_t = -\lambda X_t dt + d\tilde{B}_t \text{ (SDE)} \tag{8.82}$$

Are there unique solutions? Yes under the right conditions.

8.2 Examples

8.2.1 Brownian Motion with drift

Let $v \in \mathbb{R}^d$ (drift vector) and $\sigma > 0$ a constant and W a BM. Then, the SDE

$$dX_t = vdt + \sigma dW_t \tag{8.83}$$

has a unique strong solution

$$X_t = X_0 + \int_0^t vds + \int_0^t \sigma dW_s = X_0 + vt + \sigma W_t \tag{8.84}$$

It holds

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] = vt \tag{8.85}$$

$$Cov(X_t^i, X_t^j) = \sigma^2 Cov(W_t^i, W_t^j) = \sigma^2 \delta_{ij} t \tag{8.86}$$

8.2.2 Ornstein-Uhlenbeck

Let $\lambda > 0$ a constant, consider the SDE

$$dX_t = -\lambda X_t dt + dW_t \tag{8.87}$$

$\exists!$ strong solution given by

$$X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dW_s \tag{8.88}$$

How does one get this formula? Let us set $\frac{d\ln(X_t)}{dt} = -\lambda \Rightarrow X_t = e^{-\lambda t} X_0$. Then

$$\Rightarrow Y_t := e^{\lambda t} X_t \tag{8.89}$$

$$\Rightarrow dY_t = e^{\lambda t} dX_t + \lambda e^{\lambda t} X_t dt \tag{8.90}$$

$$= e^{\lambda t}[-\lambda X_t dt + dW_t + \lambda X_t dt] = e^{\lambda t} dW_t \tag{8.91}$$

Hence

$$e^{\lambda t} X_t = Y_t = \int_0^t e^{\lambda s} dW_s + Y_0 \tag{8.92}$$

$$\Rightarrow X_t = e^{-\lambda t} \underbrace{X_0}_{=Y_0} + \int_0^t e^{-\lambda(t-s)} dW_s \tag{8.93}$$

Let's check if this is really a solution.

$$X_t = e^{-\lambda t} X_0 + e^{-\lambda t} \int_0^t e^{\lambda s} dW_s \tag{8.94}$$

$$\Rightarrow dX_t = -\lambda e^{-\lambda t} X_0 dt - \lambda e^{-\lambda t} dt \int_0^t e^{\lambda s} dW_s + e^{-\lambda t} e^{\lambda t} dW_s \tag{8.95}$$

$$= -\lambda(\underbrace{e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dW_s}_{=X_t})dt + dW_s \checkmark \tag{8.96}$$

The stationary distribution of the O.U. process is given by the initial condition

$$X_0 \sim \mathcal{N}(0, \frac{1}{2\lambda}) \tag{8.97}$$

Then $X_t \sim \mathcal{N}(0, \frac{1}{2\lambda})$ and $\mathrm{Cov}(X_s, X_t) = \frac{1}{2\lambda} e^{-\lambda|t-s|}$. The OU Process is a Gaussian process. Indeed:

Lemma 8.10.
Let

$$M_t = \int_0^t h(s)dW_s \tag{8.98}$$

with $h \in L^2(\mathbb{R}_+)$. Then it holds $M_t = \mathcal{N}(0, \langle M\rangle_t)$.

Proof. Let's calculate $\langle M\rangle_t$ first.

$$dM_t = h(t)dW_t \tag{8.99}$$

$$d\langle M\rangle_t = (h(t))^2(dW_t)^2 = (h(t))^2 dt \tag{8.100}$$

$$\Rightarrow \langle M\rangle_t = \underbrace{\int_0^t (h(s))^2 ds}_{\text{deterministic}} < \infty \text{ by hypothesis.} \tag{8.101}$$

$\overset{7.12}{\Rightarrow}$ We know that for $\xi \in \mathbb{R}$

$$e^{i\xi M_t + \frac{\xi^2}{2}\langle M\rangle_t} \tag{8.102}$$

is a martingale. Thus

$$\mathbb{E}\left[e^{i\xi M_t}\right] e^{\frac{\xi^2}{2}\langle M\rangle_t} = \mathbb{E}\left[e^{i\xi M_0}\right] e^{\frac{\xi^2}{2}\langle M\rangle_0} = 1 \tag{8.103}$$

$$\Rightarrow \mathbb{E}\left[e^{i\xi M_t}\right] = e^{-\frac{\xi^2}{2}\langle M\rangle_t} \tag{8.104}$$

□

In our case $h(s) = e^{-\lambda(t-s)}$. Thus

$$\int_0^t e^{-\lambda(t-s)} dW_s \sim \mathcal{N}(0, \underbrace{\int_0^t e^{-2\lambda(t-s)} ds}_{=\frac{1-e^{-2\lambda t}}{2\lambda}}) \tag{8.105}$$

Now assume that X_0 is independent of W. Then

$$e^{-\lambda t} X_0 \sim \mathcal{N}(0, \frac{e^{-2\lambda t}}{2\lambda}) \tag{8.106}$$

$$\Rightarrow X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dW_s \overset{\text{indep.}}{\sim} \mathcal{N}\left(0, \frac{e^{-2\lambda t}}{2\lambda} + \frac{1-e^{-2\lambda t}}{2\lambda}\right) = \mathcal{N}\left(0, \frac{1}{2\lambda}\right) \checkmark \tag{8.107}$$

Now calculate for $s \leq t$

$$Cov(X_s, X_t) = ? \tag{8.108}$$

Recall that $X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-u)} dW_u$. Hence (with independence of X_0 and W)

$$Cov(X_s, X_t) = e^{-\lambda t} e^{-\lambda s} \underbrace{Var(X_0)}_{Cov(X_0, X_0)} + e^{-\lambda(t+s)} Cov(\underbrace{\int_0^s e^{\lambda u} dW_u}_{=:M_s}, \int_0^t e^{\lambda v} dW_v) \tag{8.109}$$

Need to get

$$Cov(M_s, M_t) = Cov(M_s, M_s) - \underbrace{Cov(M_s, M_t - M_s)}_{=0} = Var(M_s) \tag{8.110}$$

$$\Rightarrow Cov(X_s, X_t) = e^{-2\lambda(t+s)}\frac{1}{2\lambda} + e^{-\lambda(t+s)}\mathbb{E}\left[(\int_0^s e^{\lambda u}dW_u)^2\right] \tag{8.111}$$

$$\overset{\text{Itô Isom.}}{=} e^{-2\lambda(t+s)}\frac{1}{2\lambda} + e^{-\lambda(t+s)}\mathbb{E}\left[\int_0^s e^{2\lambda u}du\right] \tag{8.112}$$

$$= e^{-2\lambda(t+s)}\frac{1}{2\lambda} + e^{-\lambda(t+s)}\frac{e^{2\lambda s}-1}{2\lambda} \tag{8.113}$$

$$= \frac{e^{-\lambda(t-s)}}{2\lambda} ☺ \tag{8.114}$$

Remark: *Intuition: The drift* $b(t,x) = -\lambda x$ *towards* $0 \in \mathbb{R}^d$ *leads to* X *being stationary, i.e.*

$$\mathbb{E}[X_t] \to 0 \tag{8.115}$$

$$\mathbb{E}\left[X_t^2\right] \to \frac{1}{2\lambda} \tag{8.116}$$

8.2.3 Geometric Brownian Motion

Let $\sigma \neq 0$ and $\mu \in \mathbb{R}$. Consider the SDE

$$\begin{cases} dX_t = \mu X_t dt + \sigma X_t dW_t \\ X_0 = x > 0 \end{cases} \tag{8.117}$$

Then there exists a unique strong solution given by

$$X_t = xe^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}, \ t \geq 0 \tag{8.118}$$

To get (8.118) we set

$$Y_t = \ln(X_t) \tag{8.119}$$

$$\Rightarrow dY_t \overset{\text{Itô-Isom.}}{=} \frac{dX_t}{X_t} - \frac{1}{2}\frac{(dX_t)^2}{X_t^2} = \frac{\mu X_t dt + \sigma X_t dW_t}{X_t} - \frac{1}{2}\frac{\sigma^2 X_t^2 dt}{X_t^2} = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t \tag{8.120}$$

$\Rightarrow$ Y_t is a BM with drift $\mu - \frac{\sigma^2}{2}$.

$$\ln(X_t) = Y_t = Y_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t \tag{8.121}$$

$$\Rightarrow X_t = e^{Y_0}e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t} \tag{8.122}$$

But since $X_0 = x \Rightarrow e^{Y_0} = x$ ☺.

8.2.4 Brownian Bridge

Let $a, b \in \mathbb{R}, T > 0$. Then the Brownian Bridge from a at time $t = 0$ to b at time $t = T$ is the solution of

$$\begin{cases} dX_t = \frac{b-X_t}{T-t}dt + dW_t & ,0 \leq t \leq T \\ X_0 = a \end{cases} \tag{8.123}$$

The solution is

$$X_t = \begin{cases} a(1-\frac{t}{T}) + \frac{bt}{T} + (T-t)\int_0^t \frac{1}{T-s}dW_s & ,0 \leq t < T \\ b & ,t = T \end{cases} \tag{8.124}$$

Does $X_t \to b$ for $t \nearrow T$? Consider the case $T = 1, a = 0 = b$. Then:

$$X_t = (1-t)W_{\frac{t}{1-t}} \tag{8.125}$$

For $t \nearrow 1 : W_{\frac{t}{1-t}} \sim \frac{1}{\sqrt{1-t}} \Rightarrow X_t \sim \sqrt{1-t} \overset{t\to 1}{\longrightarrow} 0$

8.2.5 Linear system (d=1)

Let us consider the case where the drift is given by

$$a(t,x) = a_1(t)x + a_2(t) \tag{8.126}$$

and the dispersion is given by

$$\sigma(t,x) = \sigma_1(t)x + \sigma_2(t) \tag{8.127}$$

with $a_1, a_2, \sigma_1, \sigma_2$ bounded in time. Then our SDE is given by

$$dX_t = a(t,X_t)dt + \sigma(t,X_t)dW_t = X_t dY_t + dZ_t \tag{8.128}$$

$$X_0 = \xi \tag{8.129}$$

with

$$Y_t = \int_0^t a_1(s)ds + \int_0^t \sigma_1(s)dW_s \tag{8.130}$$

$$Z_t = \int_0^t a_s(2)ds + \int_0^t \sigma_2(s)dW_s \tag{8.131}$$

We know that ∃! strong solution: Let

$$\mathcal{E}_t^Y := \exp(Y_t - \frac{1}{2}\langle Y\rangle_t) \tag{8.132}$$

$\Rightarrow X_t = \mathcal{E}_t^Y(\xi + \int_0^t (\mathcal{E}_s^Y)^{-1}(dZ_s - \sigma_1(s)\sigma_2(s)ds))$. How does one get that? We have

$$\langle Y\rangle_t = \int_0^t \sigma_1(s)^2 ds \tag{8.133}$$

$$\Rightarrow \mathcal{E}_t^Y = \exp\left[\int_0^t \left(\sigma_1(s) - \frac{\sigma_1(s)^2}{2}\right)ds + \int_0^t \sigma_1(s)dW_s\right] \tag{8.134}$$

Consider

$$Q_t := \frac{X_t}{\mathcal{E}_t^Y} = X_t[(\mathcal{E}_t^Y)^{-1}] \tag{8.135}$$

$$\Rightarrow dQ_t \underset{\text{by parts}}{\overset{\text{Integr.}}{=}} \frac{dX_t}{\mathcal{E}_t^Y} + X_t d[(\mathcal{E}_t^Y)^{-1}] + dX_t d[(\mathcal{E}_t^Y)^{-1}] \tag{8.136}$$

But

$$d[(\mathcal{E}_t^Y)^{-1}] = d(e^{-Y_t + \frac{1}{2}\langle Y\rangle_t}) \tag{8.137}$$

$$\overset{\text{Itô Form.}}{=} (\mathcal{E}_t^Y)^{-1}(-dY_t + \frac{1}{2}d\langle Y\rangle_t + \frac{1}{2}d\langle Y\rangle_t) \tag{8.138}$$

$$= (\mathcal{E}_t^Y)^{-1}(-\underbrace{dY_t}_{a_1(t)dt} + \underbrace{d\langle Y\rangle_t}_{\sigma_1(t)^2 dt}) \tag{8.139}$$

$$\Rightarrow dQ_t = \frac{dX_t}{\mathcal{E}_t^Y} + \frac{X_t}{\mathcal{E}_t^Y}(-dY_t + d\langle Y\rangle_t) + \frac{dX_t}{\mathcal{E}_t^Y}(-dY_t + d\langle Y\rangle_t) \tag{8.140}$$

$$= (\mathcal{E}_t^Y)^{-1}\left(dX_t + X_t(-dY_t + d\langle Y\rangle_t) + dX_t(-dY_t + d\langle Y\rangle_t)\right) \tag{8.141}$$

$$= (\mathcal{E}_t^Y)^{-1}(\underbrace{X_t dY_t + dZ_t}_{\overset{(8.128)}{=} dX_t} - X_t dY_t + X_t d\langle Y\rangle_t + (X_t dY_t + dZ_t)(d\langle Y\rangle_t - dY_t) \tag{8.142}$$

$$= (\mathcal{E}_t^Y)^{-1}(dZ_t + X_t d\langle Y\rangle_t - X_t d\langle Y\rangle_t - dZ_t dY_t) \tag{8.143}$$

$$= (\mathcal{E}_t^Y)^{-1}(dZ_t + \underbrace{dZ_t}_{=\sigma_2(t)dW_t} \underbrace{dY_t}_{=\sigma_1(t)W_t}) \tag{8.144}$$

$$= (\mathcal{E}_t^Y)^{-1}(dZ_t - \sigma_1(t)\sigma_2(t)dt) \tag{8.145}$$

And hence with $Q_t = \frac{X_t}{\xi_t}$

$$\frac{X_t}{\mathcal{E}_t^Y} = \underbrace{\frac{X_0}{\mathcal{E}_0^Y}}_{=\frac{\xi}{1}} + \int_0^t (\mathcal{E}_s^Y)^{-1}(dZ_s - \sigma_1(s)\sigma_2(s)ds) \tag{8.146}$$

$$\Rightarrow X_t = \mathcal{E}_t^Y(\xi + \int_0^t (\mathcal{E}_s^Y)^{-1}(dZ_s - \sigma_1(s)\sigma_2(s)ds)) \tag{8.147}$$

[14.12.2012]

[18.12.2012]

9 Connection to PDE: The Feynman-Kac Formula

Discrete time:

$$\begin{cases} \nabla u = g & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{9.1}$$

$\leftrightarrow$ had a probability formula written as $\mathbb{E}\,[]$ with some stopping time $\tau_{\partial\Omega}$.

Today we consider the heat equation.

9.1 Heat equation

Let $u(t, x)$ be the temperature in an isotropic material without dispersion at time t and position $x \in \mathbb{R}^d$. Let D be the diffusion constant. Then it holds

$$\frac{\partial u}{\partial t} = \frac{D}{2}\Delta u \tag{9.2}$$

This is the Heat-equation. Now we add an initial condition, and hence have

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{D}{2}\Delta u \\ u(x,0) = f(x) \end{cases} \tag{EQ1}$$

More generically we have:

$$\partial_t u + div\vec{\gamma} = \sigma \text{ (loss/source of energy)} \tag{9.3}$$

$$\vec{\gamma} = -\frac{1}{2}D(x)\vec{\nabla}u \text{ (current)} \tag{9.4}$$

1) By scaling in space and time we can assume Wlog D=1. One can see that

$$p_t(x,y) := \frac{e^{-\frac{(x-y)^2}{2t}}}{(\sqrt{2\pi t})^d} \tag{9.5}$$

solves (1) with $u(x, 0) = \delta_y(x)$. For general f

$$u(x,t) := \int_{\mathbb{R}^d} p_t(x,y)f(y)dy \equiv \mathbb{E}^x[f(W_t)] \tag{9.6}$$

solves (1). Here W is a BM starting from x.

We now consider a generalisation, with an external cooling:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - K(x)u \\ u(x,0) = f(x) \end{cases} \tag{EQ2}$$

Here $K(x)$ is the cooling rate at the position x.

Solution (Kac '49)

$$u(x,t) = \mathbb{E}^x\left[f(W_t)e^{-\int_0^t K(W_s)ds}\right] \tag{EQ3}$$

(EQ3) is called the Feynman-Kac formula.

Parenthesis: Consider a particle with mass m in a (conservativ) potential field $V(x)$. In Quantum-Mechanics the state of the system is given by a complex function $\psi_t(x) \in L^2(\mathbb{R}^3)$. Evolution: (Schrödinger eq.)

$$i\hbar\partial_t\psi = \frac{\hbar^2}{2^m}\Delta\psi + V(x)\psi \tag{9.7}$$

where $\hbar = \frac{h}{2\pi}$ is the Planck constant.

Feynman Idea (1948):

$$\psi_t(x)'' ='' \text{ average over all possible trajectories of } e^{i\frac{S(y)}{\hbar}} \text{ with } S \text{ the 'action' of } y. \tag{9.8}$$

$\Rightarrow$ He wrote

$$\psi_t(x) = Const \int_A e^{i\frac{S(y_s)}{\hbar}}\psi_0(y(t)) \underbrace{Dy}_{''\infty\text{-dim. leb. meas.}''} \tag{9.9}$$

with $A = \{\text{Continuous functions } y \text{ mit } y(0) = x\}$ and

$$S(y) = \int_0^t \underbrace{\frac{m}{2}(\dot{y}(s)^2)}_{kineticenergy} - V(Y(s))ds \tag{9.10}$$

This is mathematically ill-defined. Kac noticed that if you consider "purely imaginary" times ($t \to it$) $\Rightarrow$ the Schrödinger equation becomes (EQ2). Using the idea of Feynman he got the representation of (EQ2) above.

Definition 9.1.
Let $f : \mathbb{R}^d \to \mathbb{R}$, $K : \mathbb{R}^d \to \mathbb{R}_+$ be continuous functions. Assume, v is a continuous real function on $\mathbb{R}^d \times [0, T]$, $v \in C^{2,1}(\mathbb{R}^d \times [0, T))$ s.t.

$$\begin{cases} -\frac{\partial v}{\partial t} + Kv = \frac{1}{2}\Delta v & \text{on } \mathbb{R}^d \times [0, T) \\ v(x, T) = f(x) & , x \in \mathbb{R}^d \end{cases} \tag{EQ4}$$

Then v is called a solution of the Cauchy problem for the backwards heat equation (EQ4) with potential K and final condition f.

Theorem 9.2.
Let v as in Def 9.1. Assume that

$$\max_{0\le t\le T} |v(t, x)| \le Ce^{a\|x\|^2}, \forall x \in \mathbb{R}^d \tag{9.11}$$

for a constant $C > 0$ and $0 < a < \frac{1}{2Td}$. Then v has the stochastic representation

$$(5) \quad v(x, t) = E^x(f(W_{T-t})e^{-\int_0^{T-t} K(W_s)ds}),\ 0 \le t \le T, x \in \mathbb{R}^d \tag{9.12}$$

Moreover, v is unique.

Corollary 9.3.
By taking $t \mapsto T - t$ one gets the stochastic representation of (2) given by

$$u(x, t) = \mathbb{E}^x\left[f(W_t)e^{-\int_0^t K(W_s)ds}\right] \tag{9.13}$$

a

Proof of the Theorem. Let $g(\vartheta) := v(W_\vartheta, t+\vartheta)e^{-\int_0^\vartheta K(W_s)ds}$. What is $dg(\vartheta)$?

$$d(e^{-\int_0^\vartheta K(W_s)ds}) = e^{-\int_0^\vartheta K(W_s)ds}(-K(W_\vartheta))d\vartheta \tag{9.14}$$

$$d(v(W_\vartheta, t+\vartheta)) = \dot{v}(W_\vartheta, t+\vartheta)d\vartheta + \nabla v(W_\vartheta, t+\vartheta)dW_\vartheta + \underbrace{\frac{1}{2}\Delta v(W_\vartheta, t+\vartheta)d\vartheta}_{\overset{(EQ4)}{=} -\dot{v}(W_\vartheta,t+\vartheta)d\vartheta + Kv(W_\vartheta,t+\vartheta)d\vartheta} \tag{9.15}$$

$$= \nabla v(W_\vartheta, t+\vartheta)dW_\vartheta + Kv(W_\vartheta, t+\vartheta)d\vartheta \tag{9.16}$$

And thus

$$\Rightarrow dg \underset{integ.}{\overset{part.}{=}} -vKe^{-\int_0^\vartheta Kds}d\vartheta + e^{-\int_0^\vartheta Kds}(Kvd\vartheta + \nabla v dW_\vartheta) \tag{9.17}$$

$$= e^{-\int_0^\vartheta K(W_s, t+s)ds}\nabla v(W_\vartheta, t+\vartheta)dW_\vartheta \tag{9.18}$$

Hence we have

$$g(\vartheta) = g(0) + \int_0^\vartheta e^{-\int_0^u K(W_s,t+s)ds}\nabla v(W_u, t+u)dW_s \tag{9.19}$$

□

$\Rightarrow g$ is a local martingale with $g(0) = v(W_0, t) = v(x, t)$. Let us introduce the stopping time

$$S_n := \inf\{t \geq 0 : \|W_t\| \geq n\sqrt{d}\}, n \geq 1. \tag{9.20}$$

Let $r \in (0, T-t)$. Then

$$v(x,t) = \mathbb{E}^x[v(W_0, t)] = \mathbb{E}^x[g(0)] = \mathbb{E}^x[g(S_n \wedge t)] \tag{9.21}$$

$$= \underbrace{\mathbb{E}^x\left[v(W_{S_n}, t+S_n)e^{-\int_0^{S_n} K(W_s)ds}\mathbb{1}_{\{S_n \leq r\}}\right]}_{(A)} + \underbrace{\mathbb{E}^x\left[v(t+r, W_r)e^{-\int_0^r K(W_s)ds}\mathbb{1}_{\{S_n > r\}}\right]}_{(B)} \tag{9.22}$$

ad (B) As $n \nearrow \infty$ and $r \nearrow T-t$, by dominated convergence

$$(B) \Rightarrow \mathbb{E}^x\left[v(T, W_{T-t})e^{-\int_0^{T-t} K(W_s)ds}\right] \checkmark \tag{9.23}$$

Remains to show: As $n \nearrow \infty$ (A)$\searrow 0$.

$$|A| \underset{r\in(0,T-t)}{\overset{K\geq 0}{\leq}} \mathbb{E}^x\left[|v(W_{S_n}, \underbrace{t+S_n}_{\in(0,T)})|\mathbb{1}_{\{S_n \leq r\}}\right] \tag{9.24}$$

$$\leq C\mathbb{E}^x\left[e^{a\|W_{S_n}\|^2}\mathbb{1}_{S_n \leq r}\right] \tag{9.25}$$

$$\underset{S_n}{\overset{\text{Def of}}{\leq}} Ce^{adn^2}\mathbb{E}^x[\mathbb{1}_{S_n \leq T}] \tag{9.26}$$

$$\underset{1}{\leq} Ce^{adn^2}\sum_{l=1}^d \mathbb{P}^x\left(\max_{0\leq t\leq T}|W_t^{(l)}| \geq n\right) \tag{9.27}$$

$$\leq Ce^{adn^2}\sum_{l=1}^d \mathbb{P}^x\left(\max_{0\leq t\leq T} W_t^{(l)} \geq n\right) + \mathbb{P}^x\left(\max_{0\leq t\leq T} -W_t^{(l)} \geq n\right) \tag{9.28}$$

$$\underset{\text{princ.}}{\overset{\text{refl.}}{=}} 2Ce^{adn^2}\sum_{l=1}^d \mathbb{P}^x\left(W_T^{(l)} \geq n\right) + \mathbb{P}^x\left(-W_T^{(l)} \geq n\right) \tag{9.29}$$

[1] $\mathbb{P}(S_n \leq T) \leq \mathbb{P}\left(\max_{0\leq t\leq T}\sum(W_t^{(l)})^2 \geq n^2 d\right) \leq \mathbb{P}\left(\exists l : (W_t^{(l)})^2 \geq n^2\right)$

We know

$$P^x(\pm W_T^{(l)} \geq n) \leq \sqrt{\frac{T}{2\pi}} \frac{e^{-\frac{(n \mp x^{(l)})^2}{2T}}}{n \mp x^{(l)}} \overset{n \gg 1}{\approx} e^{-\frac{n^2}{2T}} \tag{9.30}$$

$\Rightarrow |A| \leq \tilde{C} e^{adn^2} e^{-\frac{n^2}{2T}} \to 0$ since we assumed $a < \frac{1}{2dT}$.

[18.12.2012]
[08.01.2013]

10 Brownian Martingale

10.1 Time changes

Goal: Show the following: Let $X \in \mathcal{M}^0_{loc}$ with $\langle X\rangle_\infty = \infty$, then if we set

$$\tau_t = \inf\{s > 0 : \langle X\rangle_s > t\} \tag{10.1}$$

it holds that

$$B_t := X_{\tau_t} \tag{10.2}$$

is a BM (w.r.t. $\mathcal{F}_{\tau_t}$) and $X_t = B_{\langle X\rangle_t}$.

Definition 10.1.
Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ a monotone increasing, right-continuous function with $f_\infty := \lim_{t\to\infty} f(t) \in \bar{\mathbb{R}}_+$. Then the right-inverse of f, denoted by $f^{[-1]}$, is defined by

$$f^{[-1]}(t) := \inf\{s \geq 0 : f(s) > t\} \tag{10.3}$$
$$\equiv \sup\{s \geq 0 : f(s) \leq t\} \tag{10.4}$$
$$\equiv Leb(\mathbb{1}_{f\leq t}) \tag{10.5}$$

with $\inf\{\emptyset\} = \infty$.

Lemma 10.2.

a) $f^{[-1]} : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone increasing and right-continuous.

b) $\left(f^{[-1]}\right)^{[-1]} = f$.

c) $f\left(f^{[-1]}\right) \geq s \wedge f_\infty$. If f is continuous (in t) and $f_\infty = \infty$, then $f\left(f^{[-1]}\right) = s$.

d) $f^{[-1]}$ is constant on $[f(t_-), f(t))$, $\forall t \geq 0$.

Proof. **ad a)** It's easy to see that $f^{[-1]}$ is increasing. Now verify that $f^{[-1]}$ is right-continuous. Since $f^{[-1]}$ is increasing we have $f^{[-1]}(t) \leq \lim_{\vartheta\searrow t} f^{[-1]}(\vartheta)$. To show: $\lim_{\vartheta\searrow t} f^{[-1]}(\vartheta) \leq f^{[-1]}(t)$.

Let $s := f^{[-1]}(t) \Rightarrow \forall\varepsilon > 0$ it holds $f(s+\varepsilon) > t$ and for all $\vartheta \in (t, f(s+\varepsilon))$ we have $f^{[-1]}(\vartheta) \leq s+\varepsilon$ since $f^{[-1]}(\vartheta) = \sup\{u : \underbrace{f(u) \leq \vartheta < f(s+\varepsilon)}_{\Rightarrow u<s+\varepsilon}\}$.

Thus we now have $\lim_{\vartheta\searrow t} f^{[-1]}(\vartheta) \leq \lim_{\varepsilon\searrow 0} s+\varepsilon = s = f^{[-1]}(t)$. □

Definition 10.3.
A time change $(T_t)_{t\geq 0}$ is an increasing, right-continuous process $T : \Omega \times \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ with T_t is a stopping time $\forall t$.

Example:
- $T_t = e^{2\lambda t}, \lambda > 0$
- *$T_t = t \wedge \tau$ with τ stopping time.*
- *$T_t = t + \tau$ with τ stopping time.*

- $T_t = \inf\{s \geq 0 : A_s > t\}$ *where A is an adapted, right-continuous, increasing process.* $(*)$

$\Rightarrow$ *From Def 10.1:* $T_t = A_t^{[-1]}$ *and we know that:* T_t *is a stopping time* $\Leftrightarrow A_s := \mathbb{1}_{[0,T_t)}(s)$ *is adapted. Thus all time changes are of the form* $(*)$ *with* $A_t = \inf\{s \geq 0 : T_s > t\}$.

Definition 10.4.
Let $g : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ be an increasing, right-continuous function. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is called g-continuous if

$$f\Big|_{[g(t_-),g(t)]} \tag{10.6}$$

is constant $\forall t$ (with $g(t) < \infty$)

Example: *Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *continuous, increasing, then* f *is* $f^{[-1]}$*-continuous. Indeed:* $\forall s \in [f^{[-1]}(t_-), f^{[-1]}(t)] < \infty \Rightarrow f(s) = f(f^{[-1]}(t))$.

Definition 10.5.
Let $(X_t)_{t\geq 0}$ an adapted process with $X_t \in \bar{\mathbb{R}}$. If either $(T_t)_{t\geq 0}$ is a finite time change (i.e. $T_t < \infty$ a.s.) or $X_\infty = \lim_{t\to\infty} X_t \in \bar{\mathbb{R}}$ exists a.s., then we define the time changed process by

$$\hat{X} : \mathbb{R}_+ \times \Omega \to \bar{\mathbb{R}} \tag{10.7}$$

$$(t,\omega) \mapsto \hat{X}_t(\omega) := X_{T_t(\omega)}(\omega) \tag{10.8}$$

This process is adapted to $\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$.

Remark: *If* $X \in \mathcal{M} \Rightarrow \hat{X}$ *is not always a Martingale. For example:* $X = BM, T_t = \inf\{s > 0 : \max_{0\leq u\leq s} X_u > t\}$. *By the continuity of the BM we have* $\hat{X}_t = t \notin \mathcal{M}$.

Definition 10.6.
Let $(T_t)_{t\geq 0}$ a time change. A process $(X_t)_{t\geq 0}$ is called $(T_t)_{t\geq 0}$-continuous if for a.e. $\omega : X(\omega)$ is $T(\omega)$-continuous, i.e. $t \mapsto X_t(\omega)$ is constant on all intervals $[T_{t_-}(\omega), T_t(\omega)]$.

This ensures the continuity of $\hat{X}$!

Lemma 10.7.
Let $X \in \mathcal{M}_{loc}$ and $T_t := \inf\{s \geq 0 : \langle X\rangle_s > t\} \equiv \langle X\rangle_t^{[-1]}$. Then, X is $(T_t)_{t\geq 0}$-continuous.

Proof. For given ω in a set of measure 1, and $s \in \mathbb{R}_+$ s.t. $(T_t)_{t\geq 0}$ has a jump at s,

$$[T_{s_-}(\omega), T_s(\omega)] = [a,b](b > a) \tag{10.9}$$

$$\Leftrightarrow \langle X\rangle(\omega) \text{ is constant on } [a,b] \tag{10.10}$$

$$\Leftrightarrow X_s(\omega) \text{ is constant on } [a,b] \tag{10.11}$$

□

Theorem 10.8.
Let $(T_t)_{t\geq 0}$ be a time change and $X \in H^2$ with X is T-continuous.

$$\Rightarrow \hat{X} \in \hat{H}^2 := \{\text{continuous } L^2 - \text{bounded Mart. w.r.t } (\hat{\mathcal{F}}_t)_{t\geq 0}\} \tag{10.12}$$

Moreover:

$$\langle \hat{X}\rangle_t \equiv \langle X_{T_t}\rangle \stackrel{!}{=} \widehat{\langle X\rangle}_t - \widehat{\langle X\rangle}_0 \equiv \langle X\rangle_{T_t} - \langle X\rangle_{T_0} \tag{10.13}$$

Proof (Sketch). X T-continuous $\overset{\text{proof of 10.7}}{\Rightarrow}$ $\langle X\rangle$ T-continuous. $\Rightarrow \hat{X}_t := X_{T_t}$ and $\widehat{\langle X\rangle}_t = \langle X\rangle_{T_t}$ are continuous, since X and $\langle X\rangle$ are constant on jumping points of T. Now since $X \in H^2$ it holds

$$X_t = \mathbb{E}\left[X_\infty|\mathcal{F}_t\right] \tag{10.14}$$

and furthermore

$$X_{T_t} = \mathbb{E}\left[X_\infty|\hat{\mathcal{F}}_t\right]. \tag{10.15}$$

Thus $(\hat{X}_t)_{t\geq 0}$ is a $(\hat{\mathcal{F}}_t)_{t\geq 0}$-Martingal and is L^2-bounded. For the latter see

$$\mathbb{E}\left[\sup_{t\geq 0} X^2_{T_t}\right] \leq \mathbb{E}\left[\sup_{t\geq 0} X^2_t\right] < \infty \tag{10.16}$$

Now let's show the formula. First one can see, that

$$|X^2_{T_t} - \langle X\rangle_{T_t}| \leq \sup_{t\geq 0} X^2_t + \langle X\rangle_\infty \tag{10.17}$$

The right part is in L^1 since

$$X \in H^2 \Rightarrow \sup_{t\geq 0} X_t \in L^2 \tag{10.18}$$

and

$$X^2_\infty - \langle X\rangle_\infty \in \mathcal{M}_{loc},\ X^2_\infty \in L^1 \Rightarrow \langle X\rangle_\infty \in L^1 \tag{10.19}$$

i.e. unif. integrable. Now one can stop and see

$$\rightarrow X^2_{T_t} - \langle X\rangle_{T_t} = \mathbb{E}\left[X^2_\infty - \langle X\rangle_\infty|\mathcal{F}_{T_t}\right] \tag{10.20}$$

i.e. $\hat{X}^2 - \widehat{\langle X\rangle}$ is $\hat{\mathcal{F}}$-Martingale. $\Rightarrow \langle\hat{X}\rangle = \widehat{\langle X\rangle}_t - \widehat{\langle X\rangle}_0$ □

[08.01.2013]
[11.01.2013]

Remark: *We need the term $\widehat{\langle X\rangle}_0$. For example if we consider a timechange $T_t = t + c, c > 0$.*

Corollary 10.9.
Let $X \in \mathcal{M}_{loc}, T \equiv (T_t)_{t\geq 0}$ a finite time change, and assume that X is T-continuous. Then,

$$\hat{X} \in \hat{M}_{loc} := \{\text{continuous local martingales w.r.t. } \hat{\mathcal{F}}_t\} \tag{10.21}$$

and

$$\langle\hat{X}\rangle = \widehat{\langle X\rangle} - \widehat{\langle X\rangle}_0 \tag{10.22}$$

Proof. WLOG $X_0 = 0$ and let σ be a stopping time s.t. $X^\sigma \in H^2$. Define the stopping time

$$\hat{\sigma} := \inf\{s \geq 0 : T_s \geq \sigma\} \tag{10.23}$$

$$\Rightarrow \hat{X}^{\hat{\sigma}}_t \equiv \hat{X}_{\hat{\sigma}\wedge t} = X_{T_{\hat{\sigma}\wedge t}} = \begin{cases} X_{\sigma\wedge T_t} & \sigma \geq T_0 \\ X_{T_0} & \sigma < T_0 \end{cases}.$$

Thus

$$\hat{X}^{\hat{\sigma}} - X_{T_0} = \widehat{X^\sigma} - X^\sigma_{T_0} \tag{10.24}$$

Similarly one gets

$$\widehat{\langle X\rangle}^{\hat{\sigma}} - \langle X\rangle_{T_0} = \widehat{\langle X^{\sigma}\rangle} - \langle X^{\sigma}\rangle_{T_0} \tag{10.25}$$

Now consider a sequence of stopping times $(\sigma_n)_{n\geq 1}$ s.t. $\sigma_n \nearrow \infty$ and $X^{\sigma_n} \in H^2$(e.g. $\sigma_n = \inf\{t : |X_t| > n\}$). Then it also holds that $\hat{\sigma}_n \nearrow \infty$, since $\{\hat{\sigma}_n \leq t\} = \{\sigma_n \leq T_t\}$. $\overset{10.8}{\Rightarrow} \widehat{X^{\sigma_n}} \in \hat{H}^2$ $\overset{(10.24)}{\Rightarrow} \hat{X}^{\hat{\sigma}_n} \in \hat{H}^2$ and thus we have that $\hat{X}$ is a local martingale. For the formula, one can calculate

$$\underbrace{\langle \hat{X}^{\hat{\sigma}_n}\rangle}_{=\langle\hat{X}\rangle^{\hat{\sigma}_n}} \overset{(10.24)}{=} \langle\widehat{X^{\sigma_n}}\rangle_t \overset{Thm 10.8}{=} \widehat{\langle X^{\sigma_n}\rangle} - \widehat{\langle X^{\sigma_n}\rangle}_0 \overset{(10.25)}{=} \widehat{\langle X\rangle}^{\hat{\sigma}_n} - \langle X\rangle_{T_0} \tag{10.26}$$

Taking $n \nearrow \infty$, since $\hat{\sigma}_n \nearrow \infty$ a.s. we get the result

$$\langle \hat{X}\rangle_t = \widehat{\langle X\rangle}_t - \underbrace{\widehat{\langle X\rangle}_0}_{=\langle X\rangle_{T_0}} \tag{10.27}$$

□

10.2 Applications

Theorem 10.12.
Let $(X_t)_{t\geq 0}$ be a d-dimensional BM w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ and τ a finite stopping time. Then,

$$B_t := X_{t+\tau} - X_\tau \tag{10.28}$$

is a d-dimensional BM w.r.t $(\mathcal{F}_{\tau+t})_{t\geq 0}$.

Proof. Let $T_t := t + \tau$. Then $\hat{X}_t = X_{t+\tau}$. $\overset{10.9\ \&\ 10.8}{\Rightarrow}$ B_t is a Martingale w.r.t $(\mathcal{F}_{\tau+t})_{t\geq 0} = (\hat{\mathcal{F}}_t)_{t\geq 0}$. Moreover:

$$\langle B^i, B^j\rangle_t \underset{Polarisation}{\overset{10.8}{=}} \langle X^i, X^j\rangle_{t+\tau} - \langle X^i, X^j\rangle_\tau = \delta_{ij}(t+\tau) - \delta_{ij}\tau = t\delta_{ij}. \tag{10.29}$$

By the Levy-characterization, B is a d-dimensional BM w.r.t. $(\mathcal{F}_{t+\tau})_{t\geq 0}$. □

Theorem 10.13 (Dubins-Schwarz)**.**
Let $X \in \mathcal{M}^0_{loc}$ with $\langle X\rangle_\infty = \infty$ a.s.. Then

$$B_t := X_{T_t} \tag{10.30}$$

with

$$T_t := \inf\{s \geq 0 : \langle X\rangle_s > t\} \equiv \langle X\rangle_t^{[-1]} \tag{10.31}$$

is a standard 1-dimensional BM w.r.t $(\mathcal{F}_{T_t})_{t\geq 0}$ and

$$X_t = B_{\langle X\rangle_t} \tag{10.32}$$

Proof. T_t is a finite time change, because $\langle X\rangle_\infty = \infty$ a.s.. By Lemma 10.7, we know that X is T-continuous. By Cor 10.9: $(\underbrace{B_t}_{\equiv\hat{X}_t})_{t\geq 0} \in \mathcal{M}^0_{loc}$. It starts from 0 since $X_0 = 0, T_0 = 0$. Also

$$\langle B\rangle_t = \widehat{\langle X\rangle}_t - \widehat{\langle X\rangle}_0 = \langle X\rangle_{T_t} - \underbrace{\langle X\rangle_{T_0}}_{=0} = \langle X\rangle_{\langle X\rangle_t^{[-1]}} \underset{t\mapsto\langle X\rangle_t \text{ cont., incr., } \langle X\rangle_\infty=\infty}{\overset{10.2c)}{=}} t \tag{10.33}$$

Thus B is a local martingal with $\langle B\rangle_t = t$. By Levy we get that B is a BM. Furthermore

$$B_{\langle X\rangle_t} = X_{T_{\langle X\rangle_t}} = X_t \tag{10.34}$$

where we use in the last "=" that

$$T_u = \inf\{s \geq 0 : \langle X\rangle_s > u\} \tag{10.35}$$

$$T_{\langle X\rangle_t} = \inf\{s \geq 0 : \langle X\rangle_s > \langle X\rangle_t\} \overset{\langle X\rangle_t cont.}{=} t \tag{10.36}$$

□

Definition 10.14.
Let τ be a stopping time. A process $(B_t)_{t\geq 0}$ is called BM stopped by τ if

$$\bullet B \in \mathcal{M}^0_{loc} \tag{10.37}$$

$$\bullet \langle B\rangle_t = t \wedge \tau \tag{10.38}$$

Theorem 10.15.
Let $X \in \mathcal{M}^0_{loc}$ with $X_\infty(\omega) := \lim_{t\to\infty} X_t(\omega)$ exists and $\langle X\rangle_\infty < \infty$ a.s.. Define

$$B_t := \begin{cases} X_{T_t} & \text{if } t < \langle X\rangle_\infty \\ X_\infty & \text{if } t \geq \langle X\rangle_\infty \end{cases} \tag{10.39}$$

with

$$T_t = \inf\{s \geq 0 : \langle X\rangle_s > t\}. \tag{10.40}$$

Then $(B_t)_{t\geq 0}$ is a BM stopped by $\langle X\rangle_\infty$

Proof. For given n, consider

$$T_t^{(n)} := T_t \wedge n. \tag{10.41}$$

Then $T_t^{(n)}$ is a finite time change. Now define

$$B_t^{(n)} := X_{T_n^{(n)}}. \tag{10.42}$$

By Cor 10.9:

$$\langle B^{(n)}\rangle_t = \langle X\rangle_{T_t^{(n)}} - \underbrace{\langle X\rangle_{T_0^{(n)}}}_{=0} \tag{10.43}$$

$$= \langle X\rangle_{T_t\wedge n} \tag{10.44}$$

$$= t \wedge \langle X\rangle_n \tag{10.45}$$

Taking $n \to \infty$ finishes the proof. □

[11.01.2013]
[15.01.2013]

11 Girsanov's theorem

11.1 An example

Let $Z = (Z_1, ..., Z_n)$ be $\mathcal{N}(0, \mathbb{1})$-distributed on a space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mu = (\mu_1, ..., \mu_n) \in \mathbb{R}^n$ be a fixed vector. Define a new measure by

$$\mathbb{Q}(d\omega) = e^{\sum_{k=1}^n \mu_k Z_k(\omega) - \frac{1}{2}\sum_{k=1}^n \mu_k^2} \mathbb{P}(d\omega). \tag{11.1}$$

One can compare this to the moment generating function to see, that this is still a probability measure. We now have

$$\mathbb{P}(Z_1 \in dz_1, ..., Z_n \in dz_n) = \frac{1}{(2\pi)^{n/2}} \prod_{k=1}^n e^{-\frac{z_k^2}{2}} dz_k \tag{11.2}$$

and

$$\mathbb{Q}(Z_1 \in dz_1, ..., Z_n \in dz_n) = \frac{1}{(2\pi)^{n/2}} \prod_{k=1}^n e^{-\frac{(Z_k - \mu_k)^2}{2}} dz_k, \tag{11.3}$$

i.e. $Z \sim \mathcal{N}(\mu, \mathbb{1})$ with respect to $\mathbb{Q}$. Thus $\{\tilde{Z}_k := Z_k - \mu_k, k = 1, ..., n\}$ are iid. $\mathcal{N}(0, 1)$-distributed r.v. with respect to $\mathbb{Q}$.

"The Girsanov Theorem extends this idea of *invariance of Gaussian finite-dimensional distributions* under appropriate translations and changes of the underlying probability measure, from the discrete to the continuous setting. Rather than beginning with an n-dimensional vector $(Z_1, ..., Z_n)$ of independent, standard normal random variables, we begin with a d-dimensional Brownian motion under $\mathbb{P}$, and then construct a new measure $\mathbb{Q}$ under which a "translated" process is a d-dimensional Brownian motion." - [KS91, p. 190]

11.2 Change of measure

Consider a filtered standard probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let $T \in \mathbb{R}_+$ and for all $t \in [0, T]$ let $\mathbb{Q}_t$ a probability measure with $\mathbb{Q}_t \ll \mathbb{P}$. If we take $Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}}$ as the Radon-Nikodym-derivative, we have

- $Z_t \geq 0$ on Ω.
- $Q_t = Z_t \mathbb{P}$, i.e. $\int_A d\mathbb{Q}_t = \int_A Z_t d\mathbb{P}, \forall A \in \mathcal{F}_t$.
- $\mathbb{E}_{\mathbb{P}}[Z_t] = 1$

Definition 11.1.
$(\mathbb{Q}_t)_{t\in[0,T]}$ is consistent, if

$$\mathbb{Q}_s = \mathbb{Q}_t \text{ on } (\Omega, \mathcal{F}_s)\ \forall 0 \leq s \leq t \tag{11.4}$$

If $\mathbb{Q}$ is consistent, then $\forall A \in \mathcal{F}_s$ $(s < t)$

$$\int_A Z_s d\mathbb{P} \overset{def}{=} \int_A d\mathbb{Q}_s \overset{consistent}{=} \int_A d\mathbb{Q}_t \overset{def}{=} \int_A Z_t d\mathbb{P} \tag{11.5}$$

Thus we have $Z_s = \mathbb{E}[Z_t|\mathcal{F}_s]$. So Z is a martingale on $[0, T]$.

Viceversa: For all Martingales $(Z_t)_{t\in[0,T]}$, with

- $Z_t \geq 0$
- $\mathbb{E}[Z_t] = 1, \forall t \in [0, T]$

$\mathbb{Q}_t := Z_t \mathbb{P}$ is a family of consistent probability measures.

Lemma 11.2.
For all $Z > 0, Z \in \mathcal{M}_{loc}, \exists! L \in \mathcal{M}_{loc}$ s.t. $Z = \mathcal{E}^L = \exp(L - \frac{1}{2}\langle L \rangle)$. It is given by

$$L_t = \ln(Z_0) + \int_0^t \frac{1}{Z_s} dZ_s. \tag{11.6}$$

Proof. Ito-Formula:

$$\ln(Z_t) = \underbrace{\ln(Z_0) + \int_0^t \frac{1}{Z_s} dZ_s}_{=L_t} - \frac{1}{2} \underbrace{\int_0^t \frac{1}{Z_s^2} d\langle Z_s \rangle}_{\overset{(\Delta)}{=} \langle L \rangle_t} \tag{11.7}$$

$$= L_t - \frac{1}{2}\langle L \rangle_t \tag{11.8}$$

Regarding (Δ): $\langle L \rangle_t = \langle \frac{1}{Z} \bullet Z \rangle_t = (\frac{1}{Z^2} \bullet \langle Z \rangle)_t$.
Uniqueness follows from

$$\tilde{L}_t - \frac{1}{2}\langle \tilde{L} \rangle_t = \ln(Z_t) = L_t - \frac{1}{2}\langle L \rangle_t \tag{11.9}$$

$$\Rightarrow \underbrace{L_t - \tilde{L}_t}_{\in \mathcal{M}_{loc}} = \underbrace{\frac{1}{2}(\langle \tilde{L} \rangle_t - \langle L \rangle_t)}_{\in \mathcal{A}} \tag{11.10}$$

Thus $L_t = \tilde{L}_t$. □

Remark: $Z = \exp(L - \frac{1}{2}\langle L \rangle)$. *If* $Z_0 = 1 \Rightarrow L_0 = 0$ *and from Theorem 7.12 we know that* Z *is a martingale (not just local!)* $\Leftrightarrow \mathbb{E}[Z_t] = 1 \forall t$.

Q.: Is

$$M \in \mathcal{M} \text{ w.r.t } \mathbb{P} \Leftrightarrow M \in \mathcal{M} \text{ w.r.t. } \mathbb{Q}? \tag{11.11}$$

No! But it holds

$$S \in \mathcal{S} \text{ w.r.t } \mathbb{P} \Leftrightarrow S \in \mathcal{S} \text{ w.r.t. } \mathbb{Q} \tag{11.12}$$

$$S = M_1 + A_1 \qquad S = M_2 + A_2 \tag{11.13}$$

where M_1 is the martingale part w.r.t. $\mathbb{P}$, M_2 is the martingale part w.r.t. $\mathbb{Q}$.
Q.: How does one determine M_2, A_2?
Consider $Z \in \mathcal{M}$ (not only local) and $T \in \mathbb{R}_+$ fixed. Set $\mathbb{Q}_T := Z_T \mathbb{P}$.

Lemma 11.3.
Let $0 \leq s \leq t \leq T$ and let Y be $\mathcal{F}_t$-measurable with $\mathbb{E}_{\mathbb{Q}_T}(|Y|) < \infty$. Then,

$$\mathbb{E}_{\mathbb{Q}_T}(Y | \mathcal{F}_s) = \frac{1}{Z_s} \mathbb{E}_{\mathbb{P}}(Y Z_t | \mathcal{F}_s) \text{ a.s. w.r.t. } \mathbb{Q}_T \text{ and } \mathbb{P}. \tag{11.14}$$

Proof. Let $A \in \mathcal{F}_s$.

$$\int_A \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}[YZ_t|\mathcal{F}_s] \underbrace{d\mathbb{Q}_T}_{\overset{cons.}{=} d\mathbb{Q}_s = Z_s d\mathbb{P}} = \int_A \mathbb{E}_{\mathbb{P}}[YZ_t|\mathcal{F}_s]d\mathbb{P} \tag{11.15}$$

$$= \mathbb{E}_{\mathbb{P}}[\underbrace{\mathbb{1}_A}_{\mathcal{F}_s\text{-meas.}} \mathbb{E}_{\mathbb{P}}[YZ_t|\mathcal{F}_s]] \tag{11.16}$$

$$= \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\mathbb{1}_A YZ_t|\mathcal{F}_s]] \tag{11.17}$$

$$= \int_A Y \underbrace{Z_t d\mathbb{P}}_{dQ_t} \tag{11.18}$$

$$\overset{cons.}{=} \int_A Y d\mathbb{Q}_T \tag{11.19}$$

□

Notation: *We write*

$$\mathcal{M}^0_{loc,T} = \{\text{cont. local martingales } (M_t)_{t\in[0,T]} \text{ w.r.t } (\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}) : M_0 = 0\} \tag{11.20}$$

$$\tilde{\mathcal{M}}^0_{loc,T} = \{\text{cont. local martingales } (M_t)_{t\in[0,T]} \text{ w.r.t } (\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q}) : M_0 = 0\} \tag{11.21}$$

Theorem 11.4.
Let $M \in \mathcal{M}^0_{loc,T}$ and $Z \in \mathcal{M}, Z_t > 0, \mathbb{E}[Z_t] = 1 \forall t$ and $\mathbb{Q}_t = Z_t\mathbb{P}$, then

$$\tilde{M}_t := M_t - \langle M, L\rangle_t \in \tilde{M}^0_{loc,T} \tag{11.22}$$

with

$$L_t := \ln(Z_0) + \int_0^t \frac{1}{Z_s} dZ_s \tag{11.23}$$

and it holds

$$\langle \tilde{M}\rangle_t = \langle M\rangle_t \tag{11.24}$$

on $[0, T] \times \Omega$ a.s. w.r.t. $\mathbb{P}$ and $\mathbb{Q}_T$.

Proof. WLOG $M, \langle M\rangle, \langle L\rangle$ bounded in t and ω. Then $\tilde{M}$ is bounded because

$$\langle M, L\rangle \leq \sqrt{\langle M\rangle_t \langle L\rangle_t} \tag{11.25}$$

Now, since $L_t := \ln(Z_0) + \int_0^t \frac{1}{Z_s} dZ_s$

$$\langle M, L\rangle_t = \langle M, \frac{1}{Z} \cdot Z\rangle_t \tag{11.26}$$

$$\underset{Watanabe}{\overset{Kunita}{=}} \frac{1}{Z} \cdot \langle M, Z\rangle_t \tag{11.27}$$

Using integration by parts we can now see

$$Z_t\tilde{M}_t = Z_0 \underbrace{\tilde{M}_0}_{=0} + \int_0^t Z_s d\tilde{M}_s + \int_0^t \tilde{M}_s dZ_s + \langle Z, \tilde{M}\rangle_t \tag{11.28}$$

$$= \int_0^t Z_s dM_s - \int_0^t Z_s \underbrace{d\langle M, L\rangle_s}_{\frac{1}{Z_s} d\langle M,Z\rangle_s} + \int_0^t \tilde{M}_s dZ_s + \underbrace{\langle Z, \tilde{M}\rangle_t}_{\langle Z,M\rangle_t} \tag{11.29}$$

$$= \int_0^t Z_s dM_s + \int_0^t \tilde{M}_s dZ_s \tag{11.30}$$

Thus $Z_t\tilde{M}_t \in \mathcal{M}^0_{loc,T}$ (∗).
But $\forall 0 \leq s \leq t \leq T$:

$$\mathbb{E}_{\mathbb{Q}_T}(\tilde{M}_t|\mathcal{F}_s) \overset{11.3}{=} \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}(\tilde{M}_t Z_t|\mathcal{F}_s) \tag{11.31}$$

$$\overset{(*)}{=} \frac{1}{Z_s}\tilde{M}_s Z_s \Rightarrow \tilde{M}_s \in \tilde{M}^0_{loc,T} \tag{11.32}$$

□

11.3 The Theorem of Girsanov

Let W be a d-dimensional BM and X a d-dimensional adapted process with

$$\mathbb{P}\left(\int_0^T (X_t^k)^2 dt < \infty\right) = 1 \forall 1 \leq k \leq d, T < \infty \tag{11.33}$$

Then define

$$L_t := (X \bullet W)_t \equiv \sum_{k=1}^d \int_0^t X_s^k dW_s^k \tag{11.34}$$

and

$$Z_t := \mathcal{E}^{L_t} = \exp\left(\sum_{k=1}^d \int_0^t X_s^k dW_s^k - \frac{1}{2}\sum_{k=1}^d \int_0^t (X_s^k)^2 ds\right) \tag{11.35}$$

$\Rightarrow (Z_t)_{t\geq 0}$ is a local cont. martingale with $Z_0 = 1$.

Theorem 11.5 (Girsanov)**.**
Assume that Z_t defined above is a martingale. Set

$$\tilde{W}_t^k = W_t^k - \int_0^t X_s^k ds, \; k = 1, ..., d; t \geq 0 \tag{11.36}$$

Then $\forall T \in [0, \infty)$, the process $\tilde{W} = (\tilde{W}_t)_{t\in[0,T]} = (\tilde{W}_t^1, ..., \tilde{W}_t^d)_{t\in[0,T]}$ is a d-dimensional BM w.r.t. $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T)}, \mathbb{Q}_T)$ with $\mathbb{Q}_T = Z_T\mathbb{P}$

[15.01.2013]
[18.01.2013]

Proof. Theorem 11.4 gives us

$$W_t - \langle W, L\rangle \in \tilde{\mathcal{M}}^0_{loc,T} \tag{11.37}$$

We compute

$$W_t^k - \langle W^k, L\rangle_t = W_t^k - \langle W^k, \sum_{l=1}^d (X^l \bullet W^l)_t\rangle \tag{11.38}$$

$$\underset{\text{Watanabe}}{\overset{\text{Kunita}}{=}} W_t^k - \sum_{l=1}^d (X_k \bullet \underbrace{\langle W^k, W^l\rangle}_{=\delta_{kl}t})_t \tag{11.39}$$

$$= W_t^k - \int_0^t X_s^k ds \tag{11.40}$$

$$= \tilde{W}_t^k \tag{11.41}$$

And thus $\tilde{W}_t^k \in \tilde{M}_{loc,T}^0$. Further, Theorem 11.4 implies

$$\langle \tilde{W}^k \rangle_t = \langle W^k \rangle_t = t \tag{11.42}$$

and with polarisation

$$\langle \tilde{W}^k, \tilde{W}^l \rangle_t = \langle W^k, W^l \rangle_t = \delta_{kl} t \tag{11.43}$$

Levy gives that $\tilde{W}$ is a BM. □

Theorem 11.6 (Novikov)**.**
Define $Z := \mathcal{E}^L \equiv e^{L - \frac{1}{2}\langle L \rangle}$. If

$$\mathbb{E}\left[e^{\frac{1}{2}\langle L \rangle_t}\right] < \infty, \forall t \geq 0 \tag{11.44}$$

then Z is a martingale.

Let W be a 1-dimensional BM w.r.t. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and for a $b \neq 0$, let

$$T_b := \inf\{s \geq 0 : W_s = b\} \tag{11.45}$$

Proposition 11.7.

- $\mathbb{P}(T_b \in dt) = \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} dt$
- $\mathbb{E}\left[e^{-\alpha T_b}\right] = \exp(-|b|\sqrt{2\alpha}), \alpha > 0$

Proof. 1) already computed.
2)

$$\mathbb{E}\left[e^{-\alpha T_b}\right] = \int_0^\infty e^{-\alpha t} \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} dt \tag{11.46}$$

$$\overset{t=\frac{b^2}{2u^2}}{=} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} e^{-\frac{\alpha |b|^2}{2u^2}} du \tag{11.47}$$

$$= \frac{2}{\sqrt{\pi}} e^{-\sqrt{2\alpha}|b|} \int_0^\infty e^{-(u-\frac{c}{u})^2} du \tag{11.48}$$

with $c = \sqrt{\frac{\alpha}{2}}|b|$.

Remains to show $F(c) := \int_0^\infty e^{-(u-\frac{c}{u})^2} du = \sqrt{\frac{\pi}{2}}$ For $c = 0$ ✓. Then take

$$\frac{dF(c)}{dc} = \ldots = 2F(c) - 2\int_0^\infty dx e^{-(\frac{c}{x}-x)^2} = 0. \tag{11.49}$$

□

Consider the process

$$\tilde{W} := (\tilde{W}_t)_{t\geq 0} = (W_t - \mu t)_{t \geq 0} \tag{11.50}$$

where μ is a constant. Girsanov gives, that $\tilde{W}$ is a BM w.r.t.

$$\mathbb{P}^\mu := Z_t \mathbb{P} \tag{11.51}$$

with

$$Z_t = e^{\mu W_t - \frac{1}{2}\mu^2 t}. \tag{11.52}$$

Here we have $L_t = \mu W_t$ and $\langle L \rangle_t = \mu^2 t$.
$\Rightarrow W_t = \mu t + \tilde{W}_t$ is a BM with drift μ w.r.t. $\mathbb{P}^\mu$. ($\tilde{W}_t$ is a BM with drift $-\mu$ w.r.t. $\mathbb{P}$.)

Proposition 11.8.

$$\mathbb{P}^\mu(T_b \in dt) = \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{(b-\mu t)^2}{2t}} dt \tag{11.53}$$

$$\mathbb{E}^\mu(e^{-\alpha T_b}) = \exp(\mu b - |b| \sqrt{\mu^2 + 2\alpha}), \alpha > 0 \tag{11.54}$$

Proof.

$$\begin{aligned}
\mathbb{P}^\mu(T_b \le t) &= \mathbb{E}^\mu(\mathbb{1}_{[T_b \le t]}) &(11.55)\\
&\overset{\mathbb{P}^\mu = Z_t \mathbb{P}}{=} \mathbb{E}(\mathbb{1}_{[T_b \le t]} Z_t) &(11.56)\\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{[T_b \le t]} Z_t | \mathcal{F}_{T_b \wedge t}\right]\right] &(11.57)\\
&= \mathbb{E}\left[\mathbb{1}_{[T_b \le t]} \mathbb{E}\left[Z_t | \mathcal{F}_{T_b \wedge t}\right]\right] &(11.58)\\
&\underset{\text{Opt. Sampl.}}{\overset{\text{Novikov}}{=}} \mathbb{E}\left[\mathbb{1}_{[T_b \le t]} Z_{T_b \wedge t}\right] &(11.59)\\
&= \mathbb{E}(\mathbb{1}_{[T_b \le t]} \underbrace{Z_{T_b}}_{e^{\mu b - \frac{1}{2}\mu^2 T_b}}) &(11.60)\\
&= \mathbb{E}\left[\mathbb{1}_{[T_b \le t]} e^{-\frac{1}{2}\mu^2 T_b} e^{\mu b}\right] &(11.61)\\
&= \int_0^t e^{-\frac{1}{2}\mu^2 s} e^{\mu b} \frac{|b|}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} ds &(11.62)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{P}^\mu(T_b \in dt) &= \left(\frac{d}{dt}\mathbb{P}^\mu(T_b \le t)\right) dt &(11.63)\\
&= e^{-\frac{1}{2}\mu^2 t} e^{\mu b} \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} dt &(11.64)\\
&= \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{(b-\mu t)^2}{2t}} dt &(11.65)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^\mu(e^{-\alpha T_b}) &= \int_0^\infty e^{-\alpha s} \frac{e^{-\frac{1}{2}\frac{(b-\mu s)^2}{2s}} |b|}{\sqrt{2\pi s^3}} &(11.66)\\
&\overset{\tilde{\alpha} = \alpha + \frac{\mu^2}{2}}{=} e^{\mu b} \underbrace{\int_0^\infty ds \frac{e^{\tilde{\alpha} s} e^{-\frac{b^2}{2s}} |b|}{\sqrt{2\pi s^3}}}_{= \mathbb{E}\left[e^{-\tilde{\alpha} T_b}\right]} &(11.67)\\
&\overset{\text{Prop 11.7}}{=} e^{\mu b} e^{-|b|\sqrt{2\alpha + \mu^2}} &(11.68)
\end{aligned}$$

□

$$\mathbb{P}^\mu(T_b \le t) = .. = \int_0^t e^{\mu b - \frac{\mu^2}{2} s} \mathbb{P}(T_b \in ds) = e^{\mu b} \mathbb{E}\left[e^{-\frac{\mu^2}{2} T_b} \mathbb{1}_{[T_b \le t]}\right] \tag{11.69}$$

Corollary 11.9.

$$\mathbb{P}^\mu(T_b < \infty) = \exp(\mu b - |\mu b|) \tag{11.70}$$

$$= \begin{cases} 1 & \text{if } sgn(\mu) = sgn(b) \\ \exp(-2|\mu b|) & \text{if } sgn(\mu) = -sgn(b) \end{cases} \tag{11.71}$$

Proof. From (11.69) we have

$$\mathbb{P}^{\mu}(T_b \leq t) = e^{\mu b}\mathbb{E}\left[e^{-\frac{\mu^2}{2}T_b}\right] \tag{11.72}$$

$$\overset{11.8}{\underset{\alpha=\frac{\mu^2}{2}}{=}} e^{\mu b}\exp(-|b|\sqrt{2\frac{\mu^2}{2}}) \tag{11.73}$$

$$= \exp(\mu b - |\mu b|) \tag{11.74}$$

□

Corollary 11.10.
Let $\mu > 0, W_* = \inf_{t>0} W_t$. Then

$$\mathbb{P}^{\mu}(-W_* \in db) = 2\mu e^{-2\mu b}db, \text{ for } b > 0 \tag{11.75}$$

$$\mathbb{P}^{\mu}(-W_* < 0) = 0 \tag{11.76}$$

Proof. Let $b > 0$.

$$\mathbb{P}^{\mu}(-W_* \leq b) = \mathbb{P}^{\mu}(T_{-b} < \infty) = e^{-2\mu b} \tag{11.77}$$

Then differentiate by b to see

$$\mathbb{P}^{\mu}(-W_* \in db) = 2\mu e^{-2\mu b}db, \text{ for } b > 0 \tag{11.78}$$

(11.79)

□

[18.01.2013]
[22.01.2013]

12 Local time

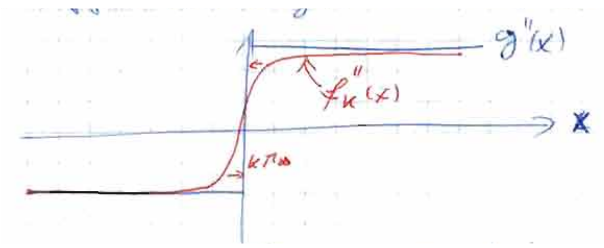
Q.: If $g \in C^2$ and B is a BM, then,

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds \tag{12.1}$$

What happens if g is not C^2, but maybe $g \in C^2(\mathbb{R} \setminus \{z_1, ..., z_k\})$?

Lemma 12.1.
Let $(B_t)_{t\geq 0}$ be a 1-dimensional BM. Then, the Itô-Formula still holds for $Y_t = g(B_t)$ if g is C^1 everywhere and C^2 except for finite # of points $z_1, ..., z_k$, if g'' is (locally) bounded for $x \notin \{z_1, ..., z_k\}$

Proof. C^2 approximation as in the picture



Choose $f_n \in C^2$ s.t. $f_n \to g$, $f_n' \to g'$ uniformly in n and $f_n'' \to g''$ on $\mathbb{R} \setminus \{z_1, ..., z_k\}$ and $|f_n''(x)| \leq M$ for x in a neighbourhood of $\{z_1, ..., z_k\}$ Now use Itô on f_n:

$$f_n(B_t) = f_n(B_0) + \int_0^t f_n'(B_s)dB_s + \frac{1}{2}\int_0^t f_n''(B_s)ds \tag{12.2}$$

This equation converges in L^2 as $n \to \infty$ towards

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds \tag{12.3}$$

□

Theorem 12.2 (Tanaka)**.**
Let B be a 1-d BM and λ the Lebesgue-measure. Then,

$$L_t := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\lambda(\{s \in [0,t] : B_s \in [-\varepsilon, \varepsilon]\}) \tag{12.4}$$

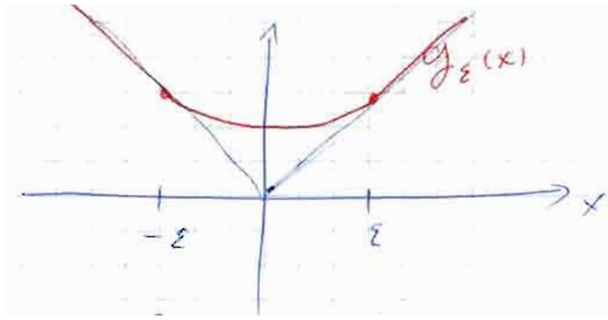
exists in $L^2(\Omega, \mathbb{P})$ and it is given by

$$L_t = |B_t| - |B_0| - \int_0^t sgn(B_s)dB_s \tag{12.5}$$

Remark: *L_t is called the local time of the BM at 0*

Proof. Let us consider the function

$$g_\varepsilon(x) = \begin{cases} |x| & ,|x| \geq \varepsilon \\ \frac{1}{2}(\varepsilon + \frac{x^2}{\varepsilon}) & ,|x| < \varepsilon \end{cases} \tag{12.6}$$



Then we have $g_\varepsilon \in C^2(\mathbb{R} \setminus \{-\varepsilon, \varepsilon\}), g_\varepsilon \in C^1(\mathbb{R})$.

$$g'_\varepsilon(x) = \begin{cases} 1 & , x > \varepsilon \\ -1 & , x < -\varepsilon \\ \frac{x}{\varepsilon} & , |x| < \varepsilon \end{cases} \tag{12.7}$$

By the previous Lemma

$$\underbrace{\frac{1}{2}\int_0^t g''_\varepsilon(B_s)ds}_{=\frac{1}{2\varepsilon}\lambda(\{s\in[0,t]:B_s\in(-\varepsilon,\varepsilon)\})\to L_t} = g_\varepsilon(B_t) - g_\varepsilon(B_0) - \int_0^t g'_\varepsilon(B_s)dB_s \tag{12.8}$$

since $g''(\varepsilon)(x) = \frac{1}{\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)}(x), x \notin \{-\varepsilon, \varepsilon\}$.
$g_\varepsilon(B_t) \overset{\varepsilon\to 0}{\longrightarrow} |B_t|$

$$\|\int_0^t (g'_\varepsilon(B_s) - sgn(B_s))dB_s\|^2 = \|\int_0^t \mathbb{1}_{(B_s\in(-\varepsilon,\varepsilon))}(\underbrace{g'_\varepsilon(B_s)}_{=\frac{B_s}{\varepsilon}} - sgn(B_s))dB_s\|^2 \tag{12.9}$$

$$\overset{Ito}{\underset{isom}{=}} \mathbb{E}\left[\int_0^t \mathbb{1}_{(B_s\in(-\varepsilon,\varepsilon))}(\underbrace{\frac{B_s}{\varepsilon} - sgn(B_s))^2}_{\leq 1} ds)\right] \tag{12.10}$$

$$\leq \int_0^t \mathbb{P}\left(B_s \in (-\varepsilon, \varepsilon)\right) ds \overset{\varepsilon\to 0}{\longrightarrow} 0 \tag{12.11}$$

□

Remark: *For $f \in C^2$: $|f(t)| - |f(0)| - \int_0^t sgn(f(s))f'(s)ds = 0$, but $d|B_t| \neq sgn(B_t)dB_t$ since*

$$|B_{t+\Delta t} - B_t| \neq sgn(B_t)(B_{t+\Delta t} - B_t) \tag{12.12}$$

e.g. is $B_t < 0$ and $B_{t+\Delta t} > 0$. Thus the L_t can be viewed as a correction term.

13 Representation of local martingale as stochastic integral

Let B be a BM and denote by $\mathcal{F}^B$ the Brownian filtration. i.e. $(F_t^0 := \sigma(B_s, 0 \leq s \leq t))$ + rightcontinuous + complete $\Rightarrow \mathcal{F}^B$.

Theorem 13.1.
Let $(\mathcal{F}_t^B)_{t\geq 0}$ be the Brownian filtration. Then, each local $(F_t^B)_{t\geq 0}$-martingale M has continuous version with stochastic integral representation:

$$M_t = M_0 + \int_0^t H_s dB_s \tag{13.1}$$

where M_0 and $H \in L^2(\Omega \times \mathbb{R}_+, \mathbb{P} \otimes \text{Leb})$ are uniquely determined by M. Moreover, if M is a continuous martingale, then

$$H_t = \frac{d}{dt}\langle M, B\rangle_t \tag{13.2}$$

Remark:

$$\begin{aligned} d\langle M, B\rangle_t &= dM_t dB_t && (13.3)\\ &= H_t dB_t dB_t && (13.4)\\ &= H_t dt && (13.5)\\ \Rightarrow \langle M, B\rangle_t &= \int_0^t H_s ds && (13.6) \end{aligned}$$

Remark: *$\exists(\mathcal{F}_t^B)_{t\geq 0}$-martingale M s.t. the BM B can not be written as $B_0 + \int_0^t A_s dM_t$. Recall: $L_t = |B_t| - |B_0| - \int_0^t sgn(B_s)dB_s$. Let $\beta_t := \int_0^t sgn(B_s)dB_s$. β is adapted to $\mathcal{F}^B$ (β has indep. incr.). What is $\langle\beta\rangle_t$?*

$$\begin{aligned} d\beta_t &= sgn(B_t)dB_t && (13.7)\\ \Rightarrow d\langle\beta\rangle_t &= (sgn(B_t))^2 d\langle B\rangle_t = dt && (13.8)\\ \Rightarrow \langle\beta\rangle_t &= t && (13.9) \end{aligned}$$

Thus β is a $\mathcal{F}^B$-BM. Assume that $\exists A_t, \mathcal{F}^B$-measurable s.t.

$$B_t = \int_0^t A_s d\beta_s \tag{13.10}$$

$\Rightarrow B_t$ is $\mathcal{F}_t^\beta$-measurable $\Rightarrow \mathcal{F}_t^B \subset \mathcal{F}_t^\beta$. Now: $\beta_t = |B_t| - L_t$. One can prove that L_t is a r.v. w.r.t. $\sigma(|B_s|, 0 \leq s \leq t) \Rightarrow \beta_t \in \mathcal{F}_t^{|B|} \Rightarrow \mathcal{F}_t^B \subset \mathcal{F}_t^\beta \subset \mathcal{F}_t^{|B|}$ but this is wrong, it holds $\mathcal{F}_t^{|B|} \subsetneq \mathcal{F}_t^B$

[22.01.2013]
[25.01.2013]

14 Connection between SDE's and PDE's

$$b : \mathbb{R}^d \to \mathbb{R}^d \tag{14.1}$$
$$\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times r}(\text{Lipschitz, bounded, measurable}) \tag{14.2}$$

$a = \sigma\sigma^T, a_{ij} = \sum_{k=1}^r \sigma_{ik}\sigma_{jk}$ Let $(B_t)_{t\geq 0}$ be a BM. Let X_t^x be the solution of

$$\begin{cases} dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dB_t \\ X_0^x = x \end{cases} \tag{14.3}$$

Theorem 14.1.
Let $f \in C_b(\mathbb{R}^d), u \in C_b([0,\infty)\times\mathbb{R}^d) \cap C_b^2((0,\infty)\times\mathbb{R}^d)$ s.t. u solves the Cauchy Problem, i.e.

$$\frac{\partial}{\partial t}u(t,x) = Au(t,x) \text{ for all } t \geq 0, x \in \mathbb{R}^d \tag{14.4}$$
$$u(0,x) = f(x) \text{ for all } x \in \mathbb{R}^d \tag{14.5}$$

where

$$Au(t,x) = \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}u(t,x) + \frac{1}{2}\sum_{i,j=1}^d \sigma_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(t,x). \tag{14.6}$$

Then

$$u(t,x) = \mathbb{E}\left[f(X_t^x)\right] \tag{14.7}$$

Proof. (From now on write $X_t = X_t^x$.) Fix $T > 0$ and use 'time reversal',

$$M_t = u(T-t, X_t). \tag{14.8}$$

Then, by Itô's Formula,

$$M_t = M_0 + \int_0^t \sum_{i=1}^d \frac{\partial}{\partial x_i}u(T-s,X_s)dX_s^{(i)} - \int_0^t \frac{\partial}{\partial t}u(T-s,X_s)ds + \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i\partial x_j}u(T-s,X_s)d\langle X^{(i)},X^{(j)}\rangle_s \tag{14.9}$$

$$= M_0 + \int_0^t \sum_{i=1}^d b_i(X_s)\frac{\partial}{\partial x_i}u(T-s,X_s)ds + \int_0^t \sum_{i=1}^d\sum_{j=1}^r \sigma_{ij}(X_s)\frac{\partial}{\partial x_i}u(T-s,X_s)dB_s^{(j)} - \int_0^t \frac{\partial}{\partial t}u(T-s,X_s)ds + \tag{14.10}$$

$$= M_0 + \underbrace{\int_0^t \sum_{i=1}^d\sum_{j=1}^r \sigma_{ij}(X_s)\frac{\partial}{\partial x_i}u(T-s,X_s)dB_s^{(j)}}_{loc.Mart.} + \int_0^t \underbrace{(A - \frac{\partial}{\partial t})u(T-s,X_s)}_{=0}ds \tag{14.11}$$

Use that $d\langle X^{(i)},X^{(j)}\rangle_s = \sum_{k,l}\sigma_{ik}\sigma_{jl}d\langle B^{(k)},B^{(l)}\rangle_s = \sum_k \sigma_{ik}\sigma_{jk}ds = a_{ij}ds$.

Thus we have that $(M_t)_{t\geq 0}$ is a local martingale. u bounded $\Rightarrow$ $(M_t)_{0\leq t<T}$ is bounded. Hence $(M_t)_{0\leq t<T}$ is a true martingale. For any $\varepsilon > 0$

$$u(T,x) = u(T-0, X_0^x) = M_0 = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[u(\varepsilon, X_{T-\varepsilon}^x)\right] \overset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}\left[u(0, X_T^x)\right] = \mathbb{E}\left[f(X_T^x)\right] \tag{14.12}$$

because u is bounded continuous. Thus $u(T,x) = \mathbb{E}\left[f(X_T^x)\right]$ □

Theorem 14.2.
Let $D \subset \mathbb{R}^d$ be open, $Z = (\{0\}\times D)\cup([0,\infty)\times\partial D)$, $f \in C_b(Z)$, $u \in C_b([0,\infty)\times\bar{D})\cap C_b^2((0,\infty)\times D)$ s.t.

$$\frac{\partial}{\partial t}u = Au \text{ in } (0,\infty)\times D \tag{14.13}$$

$$u = f \text{ on } Z \tag{14.14}$$

Then

$$u(t,x) = \mathbb{E}\left[f(t - t\wedge\tau_D, X_{t\wedge\tau_D}^x)\right] \tag{14.15}$$

where τ_D is the exit time from D,

$$\tau_D = \inf\{t > 0 : X_t^x \notin D\} \tag{14.16}$$

Proof. Fix $T > 0$, set $M_t = u(T-t, X_t^x)$. As before, M is a martingale.

$$\Rightarrow M_{T\wedge\tau_D} = u(T - T\wedge\tau_D, X_{T\wedge\tau_D}^x) \tag{14.17}$$

$$= \begin{cases} u(0, X_T^x) & , T < \tau_D \\ u(T-\tau_D, X_{\tau_D}^x & , T > \tau_D \end{cases} \tag{14.18}$$

$$= f(T - T\wedge\tau_D, X_{T\wedge\tau_D}) \tag{14.19}$$

$\Rightarrow u(T,x) = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_{T\wedge\tau_D}\right] = \mathbb{E}\left[f(T - T\wedge\tau_D, X_{T\wedge\tau_D})\right]$ □

Theorem 14.3.
Let $D \subset \mathbb{R}^d$ be open, $\tau_D < \infty$ a.s., $f \in C_b(D), u \in C_b(\bar{D}) \cap C_b^2(D)$, s.t. u solves the Dirichlet problem, i.e.

$$Au = 0 \text{ in } D \tag{14.20}$$

$$u = f \text{ on } \partial D \tag{14.21}$$

then

$$u(x) = \mathbb{E}\left[f(X_{\tau_D}^x)\right]. \tag{14.22}$$

Proof. Let $v(t,x) := u(x)$ for all $t \geq 0$. Then v solves

$$\underbrace{\frac{\partial}{\partial t}v(t,x)}_{=0} = \underbrace{Av(t,x)}_{=0} \tag{14.23}$$

$$v = f \text{ on } [0,\infty)\times\partial D \tag{14.24}$$

$$v = u \text{ on } \{0\}\times D \tag{14.25}$$

$\Rightarrow u(x) = v(t,x) = \mathbb{E}\left[f(X_{\tau_D}^x)\mathbb{1}_{\{\tau_D<t\}}\right] + \mathbb{E}\left[f(X_{\tau_D}^x)\mathbb{1}_{\{\tau_D\geq t\}}\right]$. Take the limit $r \to \infty$: since $\tau_D < \infty$ a.s. and f, u are bonded, we get

$$u(x) = \mathbb{E}\left[f(X_{\tau_D}^x)\right] + 0 \tag{14.26}$$

□

Remark: *It is usually not trivial to chek $\tau_D < \infty$. A sufficient condition would be: D bounded & $\sum_{i=1}^d a_i i \geq \lambda > 0$ for some λ.*

Theorem 14.4.
Let $D \subset \mathbb{R}^d$ be open, $\mathbb{E}[\tau_D] < \infty$, $g \in C_b(D)$, $u \in C_b(\bar{D}) \cap C_b^2(D)$ s.t. u solves the Poisson problem, i.e.

$$-Au = g \text{ in } D \tag{14.27}$$

$$u = 0 \text{ on } \partial D. \tag{14.28}$$

Then,

$$u(x) = \mathbb{E}\left[\int_0^{\tau_D} g(X_s^x) ds\right]. \tag{14.29}$$

Proof. Consider $M_t = u(X_t) + \int_0^t g(X_s) ds$. For $t < \tau_D$: By Itô's-formula,

$$M_t = M_0 + \int_0^t \sum_{i=1}^d b_i(X_s) \frac{\partial}{\partial x_i} u(X_s) ds + \int_0^t \sum_{i=1}^d \sum_{j=1}^r \sigma_{ij}(X_s) \frac{\partial}{\partial x_i} u(X_s) dB_s^{(j)} \tag{14.30}$$

$$+ \frac{1}{2} \int_0^t \sum_{i,j=1}^d a_{ij}(X_s) \frac{\partial^2}{\partial x_i \partial x_j} u(X_s) ds + \int_0^t g(X_s) ds \tag{14.31}$$

$$= M_0 + \text{ local martingale } + \underbrace{\int_0^t Au(X_s) + g(X_s) ds}_{=0 (by assumption)} \tag{14.32}$$

$\Rightarrow (M_t)_{0 \leq t < \tau_D}$ is a martingale. $\Rightarrow (M_{t \wedge \tau_D})_{t \geq 0}$ is a martingale.

$$\Rightarrow (u(x) = \mathbb{E}[M_0] = \mathbb{E}[M_{\tau_D}] = \mathbb{E}\left[u(X_{\tau_D}^x) + \int_0^{\tau_D} g(X_s) ds\right] = \mathbb{E}\left[\int_0^{\tau_D} g(X_s) ds\right] \tag{14.33}$$

□

Corollary 14.5.
If $-Au = g$ in D, $u = f$ on ∂D, then $u(x) = \mathbb{E}\left[f(X_{\tau_D}) + \int_0^{\tau_D} g(X_s) ds\right]$.

[25.01.2013]
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