

12.2) Reflected BM.

(1)

- From theorem 12.2 we have that

$$|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s) dB_s + L_t.$$

- We call $X_t := |B_t|$ the BM reflected at 0.

- The goal of this part is to show the famous result of Paul Lévy (1948) where he proved:

$$\left\{ \max_{0 \leq s \leq t} B_s - B_t, 0 \leq t < \infty \right\}$$

$$\text{and } \left\{ |B_t|, 0 \leq t < \infty \right\}$$

have the same laws (as paths!) under \mathbb{P}_0 , i.e.,
the BM measure with $B_0 = 0$.

- After this we are going to give a different way of representing reflected BM by any continuous functions, in particular BM reflected by another BM.

Remark: We will also obtain that

$$\left\{ \max_{0 \leq s \leq t} B_s, 0 \leq t < \infty \right\} \text{ and } \left\{ L_t(0), 0 \leq t < \infty \right\}$$

have the same laws under \mathbb{P}_0 .

Lemma 12.3) (Skorohod equation ('61)).

(2)

Let $x_0 \geq 0$ be a given number and $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ a continuous function with $f(0)=0$.

Then $\exists!$ function $k: \mathbb{R}_+ \rightarrow \mathbb{R}$ s.t.

$$\textcircled{a} \quad X(t) := x_0 + f(t) + k(t) \geq 0, \quad \forall t \geq 0,$$

\textcircled{b} $k(0)=0$ and it is non-decreasing,

\textcircled{c} k is flat off $\{t \geq 0 \mid X(t)=0\}$, i.e., $\int_0^\infty \mathbf{1}_{\{X(s)>0\}} d k(s) = 0$.

This function is given by:

$$k(t) = \max \left\{ 0, \max_{0 \leq s \leq t} \{-x_0 - f(s)\} \right\}, \quad 0 \leq t < \infty.$$

Proof: let us start with uniqueness.

Suppose k, \tilde{k} satisfy \textcircled{a}, \textcircled{b} and \textcircled{c}, with X and \tilde{X} corresponding to k, \tilde{k} .

Assume $\exists T > 0$ s.t. $X(T) > \tilde{X}(T)$ and

$$\text{set } \tau := \max \{0 \leq t < T \mid X(t) - \tilde{X}(t) = 0\}$$

$$\Rightarrow X(\tau) > \tilde{X}(\tau) \stackrel{\textcircled{a}}{\geq} 0, \quad \forall \tau \in (\tau, T].$$

By \textcircled{c} $\Rightarrow k(\tau) = \tilde{k}(\tau)$, thus it follows that

$$0 < X(T) - \tilde{X}(T) = k(T) - \tilde{k}(T)$$

$$\stackrel{\textcircled{b}}{\leq} k(\tau) - \tilde{k}(\tau) = X(\tau) - \tilde{X}(\tau) \stackrel{\text{def. } \tau}{=} 0 \Rightarrow 0.$$

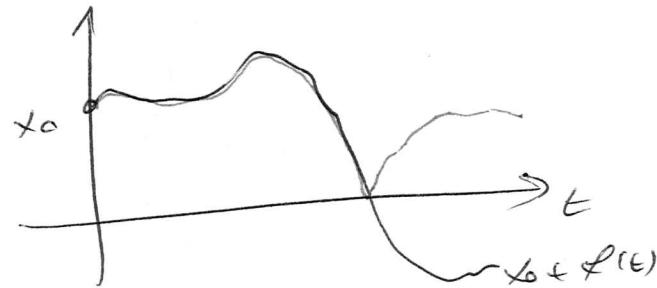
$$\Rightarrow X(T) \leq \tilde{X}(T).$$

$$\text{Similarly, } X(T) > \tilde{X}(T).$$

Next we have to verify that the given function satisfies (a), (b) and (c). (3)

(a) Clearly holds since

$$x(\epsilon) = x_0 + \ell(\epsilon) + \max\left(0, -\min_{0 \leq s \leq \epsilon} (x_0 + \ell(s))\right)$$



(b) Clear.

(c) We need to show: $\int_0^\infty \mathbb{1}_{\{x(s) > \epsilon\}} dk(s) = 0, \forall \epsilon > 0$.

let (t_1, t_2) a component of the open set

$$\{s > 0 \mid x(s) > \epsilon\}.$$

Then, $\forall s \in [t_1, t_2]$,

$$-x_0 - \ell(s) \stackrel{(a)}{\equiv} k(s) - x(s) \stackrel{(b)}{\leq} k(t_2) - \epsilon.$$

$$\Rightarrow k(t_2) = \max \left\{ k(t_1), \max_{t_1 \leq s \leq t_2} (-x_0 - \ell(s)) \right\}$$

$$\leq \max \left\{ k(t_1), k(t_2) - \epsilon \right\}$$

~~$\Rightarrow k(t_2) = k(t_1).$~~ #

Notations: $\forall x_0 > 0, \ell: \mathbb{R}_+ \rightarrow \mathbb{R}$ continuous with $\ell(0) = 0$,
 let \mathcal{K} the class of functions k satisfying (a) & (b) of
 Lemma 12.3. Define the mappings:

$$T_\ell(x_0, \ell) := \max \left\{ 0, \max_{0 \leq s \leq \epsilon} \{-x_0 - \ell(s)\} \right\}$$

$$R_\ell(x_0, \ell) := x_0 + \ell(\epsilon) + T_\ell(x_0, \ell)$$

(4)

Remarks: For the local time L_t of the BM B (under \mathbb{P}_x) it can be shown that

$$\int_0^\infty \mathbb{1}_{B \in \{\cdot\}}(B_s) dL_s = 0, \quad \mathbb{P}_x\text{-a.s.} \quad \text{(*)}$$

$W_t := - \int_0^t \operatorname{sgn}(B_s) dB_s$ is a std. BM (use Lévy-characterization).

Lemma 12.4) Let $x_0 \geq 0$, \tilde{B} a std. BM. on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ with $\tilde{\mathbb{P}}(\tilde{B}_0 = 0) = 1$.

Assume 3 continuous process \tilde{K} adapted to $\tilde{\mathcal{F}}$ s.t. for $\tilde{\mathbb{P}}$ -a.e. $\omega \in \tilde{\Omega}$ it holds:

(a) $X_t(\omega) := x_0 - \tilde{B}_t(\omega) + \tilde{K}_t(\omega) \geq 0, \quad t \geq 0,$

(b) $\tilde{K}_0(\omega) = 0$, $K_t(\omega)$ is non-decreasing,

(c) $\int_0^\infty \mathbb{1}_{(0, \infty)}(X_s(\omega)) d\tilde{K}_s(\omega) = 0.$

Then, X under $\tilde{\mathbb{P}}$ has the same law as $|B|$ under \mathbb{P}_{x_0} (where B is a BM under \mathbb{P}).

Proof: Given \tilde{B} , \tilde{K} and X are determined by Lemma 12.3 as $\tilde{K}_t(\omega) = T_t(x_0, -\tilde{B}_t(\omega))$ and $X_t(\omega) = R_t(x_0, -\tilde{B}_t(\omega))$.

To show: on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where B is a BM, \exists process W_t that is a BM and a continuous non-decreasing process K s.t. \mathbb{P}_{x_0} -a.e. $\omega \in \Omega$,

(a) $|B_t(\omega)| = x_0 - W_t(\omega) + K_t(\omega),$

(b) $K_0(\omega) = 0$, $K_t(\omega)$ non-decreasing

(c) $\int_0^\infty \mathbb{1}_{B \in \{\cdot\}}(B_s(\omega)) dK_s(\omega) = 0.$

We have such object, by Thm 12.2 and (a), (*) with $K_t = L_t - *$

(5)

Thm 12.5) (Lévy '48) Let $M_E^B := \max_{0 \leq s \leq t} B_s$. Then

the pairs $\{(M_E^B - B_t, M_E^B), t \geq 0\}$ and

$\{(B_E t, L_E), t \geq 0\}$ have the same laws under P_0 .

Proof: By uniqueness of the Skorohod equation with $x_0 = 0 \Rightarrow$ for P_0 -a.e. $w \in \Omega$,

$$L_E(w) = \max(a, \max_{0 \leq s \leq t} (W_s(w))) = \max_{0 \leq s \leq t} (W_s(w)) = M_E^W(w)$$

$$\& |B_E(w)| = L_E(w) - W_E(w) = M_E^W(w) - W_E(w).$$

$$\Rightarrow (|B_E(t)|, L_E) \stackrel{D}{=} (M_E^W - W_E, M_E^W) \stackrel{D}{=} (M_E^B - B_E, M_E^B)$$

since both W and B are std. BM starting from a .

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Conclusions: ①. let us consider the process $X(t)$, driven by the BM $B(t)$, starting from $X(0)$ and being reflected at some continuous function $f(t)$ with $f(0) < X(0)$, defined by:

$$X(t) := X(0) + B(t) + \max\{a, \max_{0 \leq s \leq t} \{X(s) - B(s)\}\}$$

$$= \max\{X(0) + B(t), \max_{0 \leq s \leq t} \{f(s) + B(t) - B(s)\}\}$$

$X(t)$ is called the BM. starting from $X(0)$ and reflected by $f(t)$