

## 11.4) Doob-h transform and conditioning.

(a)

### 11.4.1) Doob-h transform

• let  $X_t$  be a Brownian motion wrt.  $\mathbb{P}$  with  $X_0 = 0$ .

• We say that a strictly positive function  $h: \mathbb{R}_+ \times \mathbb{R} \rightarrow (0, \infty)$  is space-time regular if,  $\forall 0 \leq t, x \in \mathbb{R}$  it satisfies

$$h(s, x) = \int_{\mathbb{R}} P_{t-s}(x, y) h(t, y) dy \quad (1)$$

where  $P_t(x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right)$  is the transition density of the BM.

• Wlog we can set  $h(0, 0) = 1$ .

Example: let  $h(t, x)$  = Probability density that a BM is at 0 at time 1 conditioned that it was at  $x$  at time  $t$  ( $t < 1$  clearly)

$\Rightarrow h(t, x)$  is space-time regular.

• It is easy to see that  $h$  is a  $C^\infty$  function

• Further  $h(t, X_t) =: Z_t$  is a martingale.

Lemma 11.2:  $Z_t := h(t, X_t)$  is a martingale.

Proof.: Take any  $T > t$ .

$$\begin{aligned} &\Rightarrow \text{By (1), } \frac{\partial}{\partial t} h(t, x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} h(t, x) = \\ &= \int_{\mathbb{R}} dy \left[ \frac{d}{dt} P_{T-t}(x, y) h(T, y) + \frac{1}{2} \frac{\partial^2}{\partial x^2} P_{T-t}(x, y) h(T, y) \right] \end{aligned}$$

(b)

$$= \int_{\mathbb{R}} dy h(t, y) \left( \underbrace{\frac{1}{2} \frac{\partial^2}{\partial x^2} P_{T-t}(x, y) - \frac{\partial}{\partial T} P_{T-t}(x, y)}_{=0} \right) = 0.$$

$\Rightarrow$  By Lemma 7.9  $\Rightarrow h(t, X_t) \in \mathcal{M}_{loc}$ .

Since  $t \mapsto h(t, X_t) \geq 0 \Rightarrow$  it is a supermartingale.

Finally,  $E(h(t, X_t)) = \int_{\mathbb{R}} dx P_t(0, x) h(t, x)$

$$\stackrel{\text{def of } h}{=} h(0, 0) \stackrel{\text{def}}{=} 1, \quad t \geq 0.$$

#

Therefore we can define a change of measure  $Q_t$  by setting  $Q_t \ll P$  with

$$\frac{dQ_t}{dP} = Z_t = h(t, X_t). \quad (2)$$

Lemma 11.13) Under  $Q_t$ , the process  $X_t$  is

a Markov process with transition density

$$P_s^h(x, y) = \frac{h(t, y)}{h(s, x)} P_{t-s}(x, y), \quad s \leq t \leq T \quad (3)$$

this is called  
Doob-h transform

Proof: In order to derive the transition density, consider Lemma 11.3 with  $Y = \mathbf{1}_{(X_t \in B)}$ ,  $B \in \mathcal{B}(\mathbb{R})$  and  $F_s = \sigma(X_u, 0 \leq u \leq s)$ .

Then we have; for  $0 \leq s \leq t$ ,

①

$$\mathbb{E}_{\mathbb{Q}_t} (\mathbb{1}_{X_t \in B} | X_s = x) = \frac{1}{h(s, x)} \mathbb{E}_P (h(t, X_t) \mathbb{1}_{X_t \in B} | X_s = x)$$

// def.

$$\int \limits_B P_{s,t}^h(x,y) dy$$

|   ||

$$\frac{1}{h(s, x)} \int \limits_B h(t, y) P_{t-s}(x, y) dy$$

As this holds for any  $B \in \mathcal{B}(\mathbb{R})$ , we have the result. ~~✓~~

Next we want to see what is the effect of introducing the change of measure for the original BM  $X_t$ .

Apply Itô formula to the  $\mathbb{P}$ -martingale  $Z_t$  leads to:

$$dZ_t = \nabla h(t, X_t) dX_t + \left( \dot{h}(t, X_t) + \frac{1}{2} \Delta h(t, X_t) \right) dt$$

$$= Z_t \cdot c_t dX_t \quad \text{where } c_t := \frac{\nabla h(t, X_t)}{h(t, X_t)}.$$

$$\Rightarrow Z_t = \mathcal{E}^{(c \cdot X)_t}$$

$$= \exp \left( \int_0^t \frac{\nabla h(s, X_s)}{h(s, X_s)} ds - \frac{1}{2} \int_0^t \left( \frac{\nabla h(s, X_s)}{h(s, X_s)} \right)^2 ds \right)$$

Girsanov's theorem (Thm 11.6) gives then:

$\forall t > 0$ ,  $\tilde{X}_t := X_t - \int_0^t \frac{\nabla h(s, X_s)}{h(s, X_s)} ds$  is a  $\mathbb{Q}_t$ -BM, (4)  
for  $t \in [0, T]$ .

(d)

$\Rightarrow$  Under  $\mathbb{Q}_t^\varepsilon$ , the process  $X_t$  satisfies the SDE:

$$dX_t = \frac{\nabla h(t, X_t)}{h(t, X_t)} dt + d\tilde{X}_t \quad , 0 \leq t \leq T.$$

$\begin{matrix} 1 \\ \text{BM.} \end{matrix}$

$\Rightarrow$  The change of measure introduces a drift given by

$$b(t, X) = \nabla_{X_t} h(t, X).$$

### Example: Brownian Bridge.

Lemma 11.14) The law of a B.B.  $X$  is the Doob-h transform of the BM measure  $\mathbb{P}$  on  $C([0, 1], \mathbb{R})$ , where  $h(t, x) = \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(\frac{-x^2}{2(1-t)}\right)$ .

Formally,  $h(t, x) = \mathbb{P}^H(B_1 = 0 | B_T = x)$ ,  
(density... actually)  
 which clearly satisfies space-time regularity.

Proof:  $\forall \varepsilon \in (0, 1)$ , let  $\mathbb{Q}_t^\varepsilon$  defined by

$$\frac{d\mathbb{Q}_t^\varepsilon}{d\mathbb{P}} = M_{t(1-\varepsilon)} \quad \text{where } M \text{ is the martingale } M_t = \frac{h(t, X_t)}{h(0, 0)}.$$

By (4),  $\tilde{X}_t^\varepsilon := X_t + \int_0^{t(1-\varepsilon)} \frac{X_s}{1-s} ds$  is a BM on  $[0,1]$ . ②

$\Rightarrow \exists Q_t$  on  $C([0,1], \mathbb{R})$  s.t.  $Q_t = Q_t^\varepsilon \forall t \leq 1-\varepsilon$   
and this  $\neq 0$ .

$$\Rightarrow \tilde{X}_t := X_t - \int_0^t \frac{X_s}{1-s} ds$$

is a BM (under  $Q_t$ ).

But this was the characterisation of the BB we had before.  $\#$

#### 11.4.2) Conditioning on a value at time T.

The first conditioning is as in the BB case.

Consider  $X_t$  being a diffusion process on  $\mathbb{R}^d$

$$dX_t = \sigma(t, X_t) dB_t + b(t, X_t) dt.$$

Let  $D$  be a bounded open domain in  $\mathbb{R}^d$ .

We want to see what effect has on  $X_t$  the following condition:

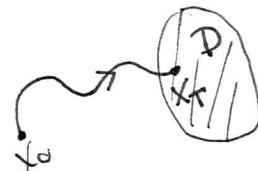
We want that  $X_T \in D$  for some fixed  $T \in \mathbb{R}_+$  (Assume:  $P(X_T \in D) > 0$ ).

For  $0 \leq s < t \leq T$ :

$$P(X_t \in dy | X_s = x, X_T \in D)$$

$$= \frac{P(X_t \in dy, X_T \in D | X_s = x)}{P(X_T \in D | X_s = x)}$$

$$= \frac{P(X_T \in D | X_t = y, X_s = x)}{P(X_T \in D | X_s = x)} \cdot P(X_t \in dy | X_s = x)$$



$$\Rightarrow \text{let } P_{s,t}(x,y) := \mathbb{P}(X_t \in dy | X_s = x)$$

$$\text{and } h(t,x) := \mathbb{P}(X_T \in D | X_t = x).$$

$\Rightarrow$  The transition density of the conditioned process is given by

$$\frac{h(t,y)}{h(s,x)} P_{s,t}(x,y)$$

which is the Doob-h transform of the measure of the diffusion  $X_t$ .

### 11.4.3) Conditioning on exit values.

First we consider a variation of Lemma 11.B.

Lemma 11.15) Let  $B$  be a BM in a bounded open domain  $D \subset \mathbb{R}^d$  killed at  $\partial D$  and

$$F_s = \sigma(B_u, 0 \leq u \leq s).$$

Let h be a strictly positive harmonic function, i.e.,  $Dh(x) = 0$ , for  $x \in D$ .

Define the measure  $\mathbb{P}^h$  s.t.  $\forall Y F_t$ -meas,

$$\mathbb{E}_x^h(Y) = \frac{1}{h(x)} \mathbb{E}_x(h(B_t)Y).$$

Then, under  $\mathbb{P}^h$ , the process  $B$  is the solution of the SDE:

$$dB_t = \frac{\nabla h(B_t)}{h'(B_t)} dt + dW_t$$

where  $W_t$  is a BM.

The proof is left as exercise for the reader.

  $\mathbb{E}_x$  is the measure of the killed BM, not of the BM.

(g)

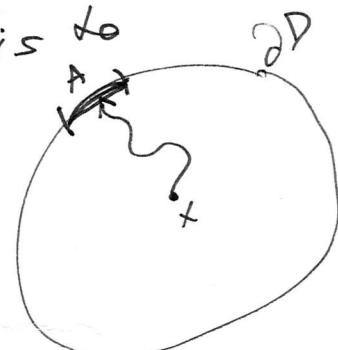
Consider now  $T_{\partial D} := \inf \{t \geq 0 \mid B_t \in \partial D\}$  and start the BM in  $x \in D$ .

Thm 11.16) The BM conditioned on the event

$\{B_{T_{\partial D}} \in A\}$  for some subset  $A \subset \partial D$   
is the Doob h-transform with  
 $h(x) = \mathbb{P}_x(B_{T_{\partial D}} \in A).$

The first step to get this result is to show that  $h(x)$  is harmonic.

Lemma 11.17)  $h(x)$  is harmonic in  $D$ .



Proof: Consider the Cauchy problem

$$\begin{cases} (\Delta f)(x) = 0 & \text{for } x \in D, \\ f(x) = g(x) & \text{for } x \in \partial D. \end{cases}$$

Let  $X_t$  the BM starting at  $x$  and let  $g(x)$  bounded.

Then,  $M_t := f(X_t) - f(x) - \frac{1}{2} \int_0^t f''(X_s) ds$  is a (local) martingale with  $\mathbb{E}(M_0) = 0$ .

Since  $T_{\partial D} < \infty$  a.s. (because  $D$  is bounded)  
we can apply the optional sampling thm and get

$$0 = \mathbb{E}_x(f(X_{T_{\partial D}})) - f(x)$$

$$\Rightarrow f(x) = \mathbb{E}_x(g(X_{T_{\partial D}})).$$

In the particular case of  $g(x) = \mathbb{I}_{x \in A}$ ,  
the solution of  $\begin{cases} (\Delta f)(x) = 0 & \text{for } x \in D \\ f(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in \partial D \setminus A, \end{cases} & \end{cases}$

$$\begin{cases} f(x) = \mathbb{P}_x(X_{T_{\partial D}} \in A) & \end{cases}$$

As the solution of the Cauchy problem is unique, we have that  $\Delta f(x) = 0$  for  $x \in D$ .

- And  $f(x) = h(x)$   $\square$

. Proof of Thm 11.16) Now we can apply Lemma 11.15. (h)

By Lemma 11.17 & Lemma 11.15, if  $Y$   $\mathcal{F}_{\mathbb{T}_{\text{end}}}$ -measurable function, the Doob h-transform is given by

$$\begin{aligned} \mathbb{E}_x^h(Y) &= \frac{\mathbb{E}_x(\mathbb{P}_{B_{\mathbb{T}_{\text{end}}}}(B_{\mathbb{T}_{\text{end}}} \in A) Y)}{\mathbb{P}_x(B_{\mathbb{T}_{\text{end}}} \in A)} \\ &= \frac{\mathbb{E}_x(\mathbf{1}_{B_{\mathbb{T}_{\text{end}}} \in A} Y)}{\mathbb{P}_x(B_{\mathbb{T}_{\text{end}}} \in A)} \\ &= \mathbb{E}_x(Y | B_{\mathbb{T}_{\text{end}}} \in A), \end{aligned}$$

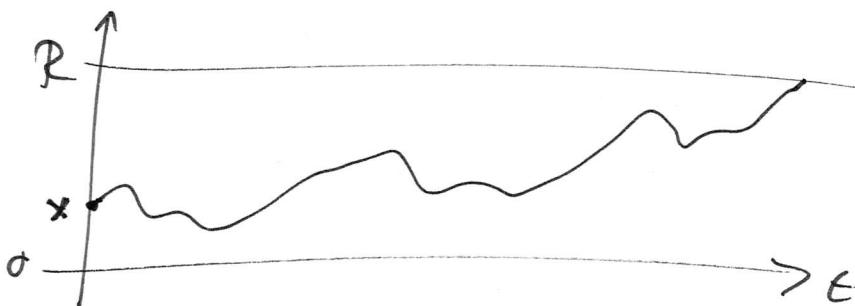
which is exactly the measure conditioned on the event  $\{B_{\mathbb{T}_{\text{end}}} \in A\}$ . #

⇒ the BM conditioned to exit D at a given place can be represented as a solution of an SDE with particular drift.

Now let us consider some examples.

Example 1: let  $D = (0, R)$  in dimension  $d=1$ .

Consider the BM conditioned to leave  $D$  at  $R$ .



Then, a simple computation gives!

$$h(x) = \mathbb{P}_x(B_{T_D} = R) = \frac{x}{R}.$$

$\Rightarrow h(x) = \frac{x}{R}$  and the conditioned BM

solves the SDE

$$dX_t = \frac{dt}{X_t} + dW_t \text{ on } (0, R). \quad (\text{The process is killed when reaching } R).$$

Remark that we can take  $R \nearrow \infty$  without changing the SDE. In particular, the BM conditioned on never reaching  $0$

is the solution of the SDE

$$dX_t = \frac{dt}{X_t} + dW_t \text{ on } \mathbb{R}_+,$$

which is the Bessel(3) process.

\* Remark first that  $h(x)$  is harmonic implies (for BM),  $h(x) = a + bx$ . Then, since  $h(0) = 0$ ,  $a = 0$ ; and since  $h(R) = 1$ ,  $b = \frac{1}{R}$ .

Example 2: BM in the Weyl chamber. ④

- The Weyl chamber in  $N$ -dimensional is the set  $W_N := \{x \in \mathbb{R}^N \mid x^{(1)} < x^{(2)} < \dots < x^{(N)}\}$ .
- Consider a  $N$ -dimensional BM,  $X_t$ , with  $X_0 \in W_N$ . We want to describe the process  $X_t$  conditioned on staying in  $W_N$  forever.
- This is equivalent to study a system of  $N$  one-dimensional BM conditioned to stay ordered forever.
- As in the previous example, we have to find a harmonic function vanishing at  $\partial W_N$ , and strictly positive inside  $W_N$ .

Lemma 11.18) The function

$$h(x^{(1)}, \dots, x^{(N)}) := \prod_{1 \leq i < j \leq N} (x^{(i)} - x^{(j)})$$

Satisfies:

- (a)  $h > 0$  in  $W_N$
- (b)  $h = 0$  on  $\partial W_N$
- (c)  $\sum_{k=1}^N \frac{\partial^2}{(\partial x^{(k)})^2} h(x) = 0, x \in W_N$ .

Proof.: Consider the following representation  
 (easy to see by induction) of  $h(x)$   
 (aka. Vandermonde determinant).

$$h(x) = \prod_{1 \leq i < j \leq n} (x^{(j)} - x^{(i)}) = \det \left( (x^{(i)})^{j-1} \right)_{1 \leq i, j \leq n}$$

$$= \text{const. } \det \left( H_{j-1}(x^{(i)}) \right)_{1 \leq j \leq n} \text{ if } \begin{cases} \text{family of} \\ \text{polynomials,} \\ \{H_j\} \text{ of degree } j. \end{cases}$$

Choose  $H_k(x)$  = Hermite polynomials

$$= (-1)^k e^{x^2} \frac{d^2}{dx^k} e^{-x^2} \Big|_1$$

which satisfy:  $\frac{d^2}{dx^2} H_k(x) = 4k(k-1) H_{k-1}(x)$ .

$$\Rightarrow h(x) = \text{const} \cdot \sum_{\sigma \in S_N} (-1)^{|\sigma|} \cdot \sum_{i=1}^N \frac{\partial^2}{(\partial x^{(\sigma(i))})^2} \prod_{j=1}^N H_{\sigma_{j-1}}(x^{(\sigma(j))})$$

\* For the permutations:

$$\begin{aligned} & \text{S.t. } \tau_i = k, \tau_j = k-2 \\ \Rightarrow & \left( \prod_{\substack{1 \leq i \leq k \\ i \neq j}} H_{\tau_{i-1}}(x^{(i)}) \right) 4(k-1)(k-2) H_{k-3}(x^{(j)}) H_{k-3}(x^{(j)}) \\ & \quad (\text{cancel terms}) \\ & \quad \frac{\partial^2}{(\partial x^{(i)})^2} \end{aligned}$$

The same term appears when  $\frac{\partial}{(\partial x^{(i)})^2}$

for the permutation with  $\tilde{x}_i = k-2$ ,  $\tilde{x}_j = k$

an otherwise  $\tilde{\sigma}_{e=5e}$ . But  $\tilde{\sigma} = \sigma_{tot}$  with

$$C = (k - z_k k) \Rightarrow (-1)^{\tilde{m}} = (-1)^m \cdot (-1) \Rightarrow \text{cancel's out } -1$$

(2)

. As a consequence, the process  $X_t$  conditioned to stay forever in  $W_N$  is the Doob h-transformation of the N-dim BM with h-function given by the Vandermonde determinant.

. In particular, it satisfies the SDE's:

$$\left\{ \begin{array}{l} dX_t^{(k)} = \frac{\frac{\partial}{\partial X_t^{(k)}} \prod_{1 \leq i \leq N} (X_t^{(i)} - X_t^{(k)})}{\prod_{1 \leq i \leq N} (X_t^{(0)} - X_t^{(i)})} dt + dB_t^k, \quad 1 \leq k \leq N \\ \\ = \sum_{\substack{i=1 \\ i \neq k}}^N \frac{1}{X_t^{(i)} - X_t^{(k)}} dt + dB_t^k, \quad 1 \leq k \leq N. \end{array} \right.$$

. This process is also known as Dyson's Brownian Motion and originally arises in a Random Matrix Diffusion process introduced by Dyson 1962.

. As an exercise you could compute the h-function of the BM conditioned to stay in  $(0,1)$  forever.

(m)

Now we want to see what is the transition probability density of Dyson's Brownian Motion.

By Lemma 11.15 and taking  $Y = \prod_{B_t \in B} \mathbb{1}_{B_t \in B}$ ,  $\# B \in \mathcal{B}(W_N)$ ,

$$\mathbb{E}_x^h(\mathbb{1}_{B_t \in B}) = \frac{1}{h(x)} \mathbb{E}_x(h(B_t) \mathbb{1}_{B_t \in B})$$

By considering  $B$  infinitesimal <sup>(around)</sup> we get the transition probability density:

$$\mathbb{P}_x^h(B_t \in dy) \equiv P_t^h(x, y) dy \quad \text{and let } p_t^{(x,y)} dy = \mathbb{P}_x^h(B_t \in dy),$$

with 
$$P_t^h(x, y) = \frac{h(y)}{h(x)} \cdot P_t(x, y).$$

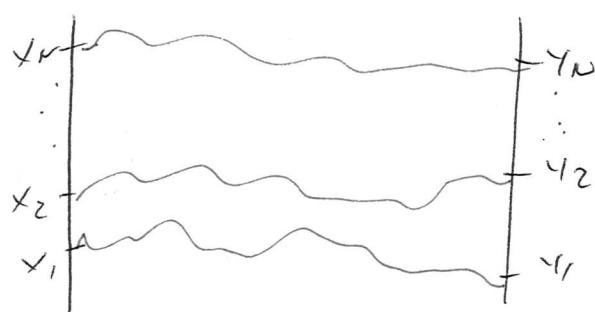
Now we have to compute  $P_t^{(x,y)}$ , which is the transition probability density for the killed BM (at  $\partial W_N$ ).

Lemma 11.16 (Karlin-Mc Gregor type formula).

let  $\phi_t(x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right).$

Then the transition density of  $N$  Brownian motions killed whenever two of them collide is given by:

$$P_t(x_1, \dots, x_N; y_1, \dots, y_N) = \det_{1 \leq i, j \leq N} (\phi_t(x_i, y_j))$$



Proof.: We have

(h)

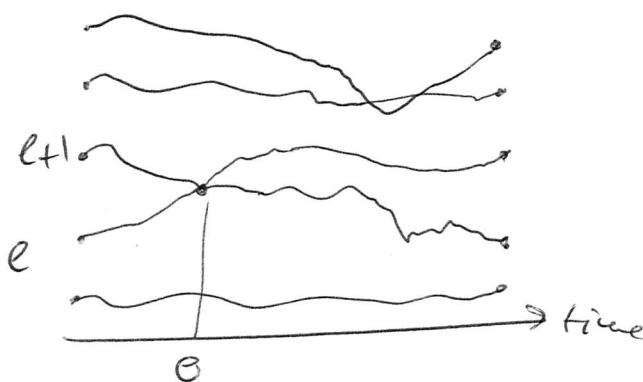
$$\mathbb{P}_x(B_t \in dy) = \mathbb{P}_x^{\text{free BM}}(B_{t \wedge \tau} \in dy, \tau > t)$$

where  $\tau = \inf\{s \geq 0 \mid B_s \notin \mathbb{W}_N\}$ .

- let us start from the r.h.s.

$$\det \left( \phi_t(x_i, y_j) \right)_{1 \leq i, j \leq N} = \\ = \sum_{\sigma \in S_N} (-1)^{|\sigma|} \prod_{k=1}^N \phi_t(x_k, y_{\sigma(k)})$$

- Consider the case that there is a crossing, i.e.,  $t \leq \tau$ . Then, let  $\theta$  be the time of first crossing and  $e$  s.t.  $B_\theta^{(e)} = B_\theta^{(\tau)}$ .



- Since the BM after the stopping time  $\theta$  are still BM (Strong MP, see also corollary of Dambis-Schwarz)  $\Rightarrow$  By exchanging the paths of  $B^{(e)}$  and  $B^{(\tau)}$  prior  $\theta$ , i.e.,

$$\tilde{B}_t^{(k)} := \begin{cases} B_t^{(k)}, & k \neq \{e, e+1\} \text{ or } t > \theta, \\ B_\theta^{(\tau)} & , k=e, t \leq \theta, \\ B_\theta^{(e)} & , k=e+1, t \leq \theta. \end{cases}$$

$\Rightarrow$  the contribution to the transition probability in the free case of  $B$  and  $\tilde{B}$  are the same. But  $\tilde{B}$  corresponds to a permutation  $\sigma$  with  $\tilde{\sigma} = (0, 1, \dots, N)$

①

⇒ The contributions of  $B$  and  $\tilde{B}$

in  $\det \left( \phi_t(x_i, y_j) \right)_{\substack{1 \leq i, j \leq N}}$  cancels out (pairwise)

⇒ Only the contributions of  $B_M$  which do not intersect until time  $t$  remains. #

Conclusion:  $P_t^y(x_1, \dots, x_N; y_1, \dots, y_N) = \frac{\prod_{1 \leq i < j \leq N} (y_j - y_i)}{\prod_{1 \leq i < j \leq N} (x_j - x_i)} \cdot \det \left( \phi_t(x_i, y_j) \right)_{\substack{1 \leq i, j \leq N}}$

for  $x = (x_1, \dots, x_N)$  and  $y = (y_1, \dots, y_N) \in W_N$ .