

2.8) Some basic tools

In this section we are going to see some important tools for the construction of generators and Feller processes.

2.8.1) Construction of generators.

Usually one is not able to give the domain of the generator completely. However, one can propose a generator on a nice sets of function and then take the closure.

Def. 2.23). A linear operator \mathcal{L} on $C(S)$ is closed if its graph

$$\{(f, \mathcal{L}f) : f \in D(\mathcal{L})\}$$

is a closed subset of $C(S) \times C(S)$.

The closure $\bar{\mathcal{L}}$ of \mathcal{L} is the linear operator whose graph is the closure of the graph of \mathcal{L} , provided that this closure is the graph of a linear operator.

Rem.: Not every linear operator has a closure (see exercises): If $S = [0, 1]$ and

$$D(\mathcal{L}) = \{f \in C(S) : f'(0) \text{ exists}\},$$

$$\mathcal{L}f = f'(0) \text{ for } f \in D(\mathcal{L}).$$

Then the closure of the graph of \mathcal{L} is not the graph of a linear operator.

This can't happen for probability generators.

Recall:

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\mathcal{L} is a linear operator

Def. 2.6: A proba. generator \mathcal{L} on $C(S)$ satisfies:

- (a) $D(\mathcal{L})$ dense in $C(S)$
- (b) $f \in D(\mathcal{L}), \lambda > 0, f - \lambda \mathcal{L}f = g \Rightarrow \inf_{x \in S} f(x) \geq \inf_{x \in S} g(x)$
- (c) $R(\mathbb{I} - \lambda \mathcal{L}) = C(S)$ for λ small enough
- (d) $\mathcal{L}1 = 0$ if S compact
• If S locally compact: for $\lambda \ll 1, \exists f_n \in D(\mathcal{L})$ s.t.
 $g_n := f_n - \lambda \mathcal{L}f_n$ satisfies: $\sup_n \|g_n\|_1 < \infty$, $f_n, g_n \rightarrow 1$ pointwise

Prop. 2.24: (a) If \mathcal{L} satisfies (a), (b) of Def. 2.6 $\Rightarrow \bar{\mathcal{L}}$ exists and satisfies (a) and (b).

- (b) If \mathcal{L} satisfies (a), (b), (c) of Def. 2.6 $\Rightarrow \mathcal{L}$ is closed.
- (c) If \mathcal{L} satisfies (b), (c) of Def. 2.6.
 $\Rightarrow R(\mathbb{I} - \lambda \mathcal{L}) = C(S'), \forall \lambda > 0$.
- (d) If \mathcal{L} is closed and satisfies (b) of Def. 2.6.
 $\Rightarrow R(\mathbb{I} - \lambda \mathcal{L})$ is a closed subset of $C(S)$.

Rem: As we will see in the applications to interacting particle systems, the real challenge is to see part (c) of Def. 2.6: we will define an operator \mathcal{L} of a set of fct. and then we will need to see that its closure $\bar{\mathcal{L}}$ is a probability generator.

The usefulness of Prop 2.24 is, for example, that if we verify that (a), (b) are verified for an operator \mathcal{L} (not yet a probability generator), then we know that its closure $\bar{\mathcal{L}}$ also inherit these two properties.

Proof.: @ We want to verify that for $f \in \overline{D(G)}$, $\bar{f} = \lim_{n \rightarrow \infty} f_n$, $f_n \in D(G)$, it exists a dense \bar{g} s.t. $\lim_{n \rightarrow \infty} g f_n = \bar{g} \bar{f}$.

By linearity, it is enough to show that if $f_n \in D(G)$ with $f_n \rightarrow 0$ and $L f_n \rightarrow h$, then necessarily $h=0$.

• Choose any $g \in D(G)$, then by Lemma 2.7,

$$\|f_n + \lambda g\| \leq \|(\mathbb{1} - \lambda L)(f_n + \lambda g)\|, \quad \lambda > 0.$$

$$= \|f_n + \lambda g - \lambda L f_n - \lambda^2 L g\|.$$

Take now $n \rightarrow \infty$ and divide by λ

$$\Rightarrow \|g\| \leq \|g - h - \lambda L g\|.$$

Next, take $\lambda \downarrow 0 \Rightarrow \|g\| \leq \|g - h\|$.

Finally, by taking $\lambda \rightarrow 0$ (here we use the fact that $D(G)$ is dense in $C(S)$, i.e., property @).

$\Rightarrow h = 0 \Rightarrow$ The closure \bar{L} exists.

• Property @ for \bar{L} is clear since $D(\bar{L}) \subseteq D(G) \subseteq C(S)$ and $D(\bar{L})$ is dense in $C(S)$.

• Property ⑥ for \bar{L} : Suppose $f \in D(\bar{L})$, $\lambda \geq 0$ with $f - \lambda \bar{L} f = g$.

Then, by def. of the closure, $\exists f_n \in D(G)$ s.t. $f_n \rightarrow f$ and $L f_n \rightarrow \bar{L} f$.

But Property ⑥ for L says:

$$\inf_{x \in S} f_n(x) > \inf_{x \in S} g_n(x) \quad \text{with } g_n = f_n - \lambda L f_n.$$

Taking $n \rightarrow \infty$ gives:

$$\inf_{x \in S} f(x) > \inf_{x \in S} (f - \lambda \bar{L} f)(x) \equiv \inf_{x \in S} g(x).$$

(b) By part (a) we know that \bar{L} exists.

If $f \in D(\bar{L})$, $\lambda > 0$ is small, by Property (c) for L ,

$\exists h \in D(L)$ s.t. $h - \lambda L h = f - \lambda \bar{L} f$ (since $f - \lambda \bar{L} f \in C(S')$).

$\cancel{\exists}(h-f) - \lambda \bar{L}(h-f) = 0$. By Lemma 2.7, $h = f$ (one needs only Prop (b) for the used statement)
 $Lh = \bar{L}h$
 $\Rightarrow \bar{L}f = Lh \Rightarrow \bar{L} = L$.

(c) It is to be proven that for $\lambda \in (0, \gamma)$, $R(I-\lambda L) = C(S)$ implies $R(I-\lambda \bar{L}) = C(S')$, since for λ small enough it is part of Property (c) of Def. 2.6.

Let $g \in G(S)$ and define

$$\Pi: C(S) \rightarrow D(\bar{L})$$

$$h \mapsto \Pi h = \frac{\lambda}{\gamma} (I - \lambda \bar{L})^{\dagger} g + (I - \frac{\lambda}{\gamma}) (I - \lambda L)^{-1} h.$$

As $R(I - \lambda L) = C(S)$, Π is well-defined.

By Lemma 2.7,

$$\begin{aligned} \|\Pi h_1 - \Pi h_2\| &= \left\| \left(I - \frac{\lambda}{\gamma} \right) \cdot \left((I - \lambda \bar{L})^{-1} (h_1 - h_2) \right) \right\| \\ &\leq \left(1 - \frac{\lambda}{\gamma} \right) \cdot \|h_1 - h_2\|, \end{aligned}$$

i.e., Π is a strict contraction $\Rightarrow \exists!$ fixed point of Π
 Denote by f this fix-point. Then, $f \in D(\bar{L})$ (since $\Pi f \in D(\bar{L})$)
 and $\Pi f = f$ gives, after multiplying it by $(I - \lambda L)$,

$$(I - \lambda L) f = \frac{\lambda}{\gamma} g + (I - \frac{\lambda}{\gamma}) f$$

$$\hookrightarrow f - \lambda \bar{L} f = g, \text{i.e., } g \in R(I - \lambda \bar{L})$$

as we had to show.

(d) Suppose that $g_n \in R(\mathcal{L} - \lambda g)$ with $-g_n \rightarrow g$.

Then, set $f_n \in D(\mathcal{L})$ as

$$f_n - \cancel{\lambda} f_n = g_n. \quad \cancel{\text{for}}$$

$$\Rightarrow (f_n - f_m) - \cancel{\lambda} (f_n - f_m) = g_n - g_m$$

and again by Lemma 2.7, $\|f_n - f_m\| \leq \|g_n - g_m\|$. \oplus

By since $g_n \rightarrow g$, g_n is a Cauchy sequence,
so by \oplus also f_n is a Cauchy sequence.

Denote by f the limit of f_n . Then, $f_n \rightarrow f$, $g_n \rightarrow g$
 $\Rightarrow \oplus$ implies

$$\lim_{n \rightarrow \infty} \mathcal{L} f_n \text{ exists too.}$$

But we assumed that \mathcal{L} is closed. Therefore

$$\lim_{n \rightarrow \infty} \mathcal{L} f_n = \mathcal{L} f$$

and by \oplus : $f - \lambda \mathcal{L} f = g$, i.e., $g \in R(\mathcal{L} - \lambda g)$ as wanted. $\#$

Def. 2.25) A set $D \subset D(\mathcal{L})$ is a core for \mathcal{L} if

$$\text{closure } (\mathcal{L}|_D) = \mathcal{L}.$$

Rem.: This means that \mathcal{L} is determined by its values on D .

However, this does not mean that \mathcal{L} is determined by its restriction to any dense subset of $D(\mathcal{L})$!

Corollary:

If \mathcal{L} satisfies \oplus , \odot and \oplus of Def 2.6 and the weaker form of \odot that states only $R(\mathcal{L} - \lambda g)$ is dense in $C(S)$ for sufficiently small positive λ , then \mathcal{L} is a probability generator.

Indeed, one need that D is dense in $C(S)$ (that follows from D dense in $D(\mathcal{L})$ since $D(\mathcal{L})$ is dense in $C(S)$ and $R(\mathcal{L} - \lambda g|_D)$ dense in $C(S)$ for some $\lambda > 0$). Then, D is a core for \mathcal{L} .

• A useful characterization of a core is the following.

Lemma 2.26) probability let \mathcal{L} be a generator with associated semigroup $T(t)$ on $C(S)$ (a Banach space).
 let D be a dense subset of $D(\mathcal{L})$.
 If, for all $t \geq 0$, $T(t): D \rightarrow D$, then D is a core.
 (In fact, it is enough to have a dense subset $D_0 \subset D$ such that $T(t)$ maps D_0 into D).

Proof: let $f \in D_0$ and set

$$f_n := \frac{1}{n} \sum_{k=0}^{n^2} e^{-\lambda k/n} T\left(\frac{k}{n}\right) f$$

By hypothesis, $f_n \in D$. Thus, by strong continuity,

$$\begin{aligned} \lim_{n \rightarrow \infty} (\lambda - \mathcal{L}) f_n &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n^2} e^{-\lambda k/n} T\left(\frac{k}{n}\right) (\lambda - \mathcal{L}) f \\ &= \int_0^\infty d\lambda e^{-\lambda t} T(t)(\lambda - \mathcal{L}) f \\ &= \underbrace{U(\lambda)}_{\text{Resolvent}} (\lambda - \mathcal{L}) f = f \\ (\lambda - \mathcal{L}) f &= \lambda(1 - \frac{1}{\lambda} \mathcal{L}) f = g \\ \Leftrightarrow f &= U(\lambda) g. \end{aligned}$$

Therefore, $\forall f \in D_0$, \exists sequence of functions $(\lambda - \mathcal{L}) f_n \in R(\lambda - \mathcal{L}|_D)$ that converges to f .

Thus, the closure of $R(\lambda - \mathcal{L}|_D)$ contains D_0 .

But since D_0 is dense in $C(S)$, the statement follows. \blacksquare

Example: Consider \mathcal{L} the generator of the Brownian motion.

Then, $C^\infty(\mathbb{R}^d)$ is a core for \mathcal{L} and \mathcal{L} is the closure of $\frac{1}{2} \Delta$ with this domain. ($D_{\frac{1}{2}\Delta}$ is not closed for $d \geq 2$).

Indeed, as $C^\infty(\mathbb{R}^d)$ is dense in $C(\mathbb{R}^d)$, one needs to show that $T(t)$ maps $C^\infty(\mathbb{R}^d)$ into $C^\infty(\mathbb{R}^d)$, which is obvious by the explicit formulae for the transition probability of

2.8.2) Martingales

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A second tool to construct Markov processes is via the martingale problem.

Thm. 2.27) Let $X(t)$ be a Feller process with semigroup $T(t)$ and generator \mathcal{L} . Then, $f \in D(\mathcal{L})$,

$$M(t) := f(X(t)) - \int_0^t \mathcal{L} f(X(s)) ds$$

is a martingale relative to \mathbb{P}^x , $\forall x \in S$.

Proof.: $\mathbb{E}^x(M(t)) = T(t)f(x) - \int_0^t T(s) \mathcal{L}f(x) ds$

$\stackrel{\substack{T(s)\mathcal{L} = \frac{d}{ds}T(s) \\ (\text{Thm 2.10})}}{=} T(t)f(x) - \int_0^t \frac{d}{ds}T(s) f(x) ds \quad \#$

$$= f(x).$$

Now, choose $0 \leq s \leq t$ and recall the MP: $\mathbb{E}^x(Y \circ \theta_s | \mathcal{F}_s) = \mathbb{E}^{X(s)}(Y)_{\mathbb{P}^x \text{-as.}}$

$$\begin{aligned} \Rightarrow \mathbb{E}^x(M(t) | \mathcal{F}_s) &= \\ &= \mathbb{E}^x \left(f(X(t-s)) \circ \theta_s - \int_0^s \mathcal{L} f(X(u)) du \right. \\ &\quad \left. - \int_0^{t-s} \mathcal{L} f(X(u)) \circ \theta_s du \mid \mathcal{F}_s \right) \end{aligned}$$

Martingale Property $\mathbb{E}^{X(s)}(f(X(t-s))) - \int_0^s \mathcal{L} f(X(u)) du$

$$- \mathbb{E}^{X(s)} \left(\int_0^{t-s} \mathcal{L} f(X(u)) du \right)$$

$$= \mathbb{E}^{X(s)}(M(t-s)) - \int_0^s \mathcal{L} f(X(u)) du$$

$$\stackrel{\#}{=} f(X(s)) - \int_0^s \mathcal{L} f(X(u)) du \stackrel{\text{def.}}{=} M(s). \quad \#$$

As it was the case for Brownian motion, also in this more general framework the process can be characterized by the martingale.

Thm 2.28) Let $X(t)$ be a Feller process with semigroup $T(t)$ and generator \mathcal{L} .

If \mathbb{P} is a probability measure on $(\mathcal{E}, \mathcal{F})$ such that $\mathbb{P}(X(0)=x)=1$ for some $x \in S$, and $M(t) = f(X(t)) - \int_0^t \mathcal{L}f(X(s)) ds$ is a martingale relative to \mathbb{P} , $f \in D(\mathcal{L})$, then $\mathbb{P} = \mathbb{P}^X$.

Proof: For any given $g \in C(S)$, $\alpha > 0$, define $\varphi \in D(\mathcal{L})$ by $\varphi - \alpha^{-1} \mathcal{L}\varphi = g$.

$M(t)$ is a martingale, thus \mathbb{E}^φ

$$\Rightarrow \mathbb{E}^\varphi \left(f(X(t)) - f(X(s)) - \int_s^t \mathcal{L}f(X(u)) du \mid \mathcal{F}_s \right) = 0$$

$$\Rightarrow \int_s^\infty dt \alpha e^{-\alpha t} \left(\dots \right) = 0$$

$$\Leftrightarrow \int_s^\infty dt \alpha e^{-\alpha t} \mathcal{L}f(X(s)) = \int_s^\infty dt \alpha e^{-\alpha t} \mathbb{E}^\varphi \left(f(X(t)) \mid \mathcal{F}_s \right) - \int_s^\infty dt \alpha e^{-\alpha t} \mathbb{E}^\varphi \left(\frac{1}{\alpha} \mathcal{L}f(X(t)) \mid \mathcal{F}_s \right)$$

$$= \mathbb{E}^\varphi \left(\int_s^\infty dt \alpha e^{-\alpha t} g(X(t)) \mid \mathcal{F}_s \right). \quad \text{⊕}$$

$$\int_s^\infty dt \alpha e^{-\alpha t} \int_u^\infty du \mathbb{E}^\varphi(g(X(u))) =$$

$$= \int_s^\infty du \alpha e^{-\alpha u} \int_1^\infty dt \alpha e^{-\alpha t}$$

$$= \int_s^\infty du e^{-\alpha u} \int_1^\infty dt e^{-\alpha t}$$

For any $A \in \mathcal{F}_s$, compute

$\mathbb{E}^\varphi(A e^{\alpha s} \oplus)$ on both sides.

$$\Rightarrow \mathbb{E}^\varphi(f(X(s)) A) = \int_s^\infty dt \alpha e^{-\alpha t} \mathbb{E}^\varphi \left(g(X(s+t)) A \right). \quad \text{⊗}$$

Now, \mathbb{P}^* satisfies the hypothesis of Thm 2.28 by Theorem 2.27. Thus,

$\textcircled{*}$ holds also with \mathbb{E}^* instead of \mathbb{E} .

- In particular, for $s=0$, $A=\mathbb{R}$,

$$\int_0^\infty dt \alpha e^{-\alpha t} \mathbb{E}[g(X(t))] = \int_0^\infty dt \alpha e^{-\alpha t} \mathbb{E}^*[g(X^{(\epsilon)})] = \mathcal{L}g.$$

- Now, $\mathbb{E}[g(X(t))]$ and $\mathbb{E}^*[g(X(t))]$ are right-continuous, so by the 1:1 correspondence between P.V. and Laplace transforms, $\forall t \geq 0$, $\mathbb{E}(g(X(t))) = \mathbb{E}^*(g(X(t)))$.

- This equality being true $\forall g \in C(0, \infty) \rightarrow X(t)$ has the same distribution under the measures \mathbb{P} and \mathbb{P}^* .

- Next, we want to verify that also ^{the} joint distributions are the same under \mathbb{P} and \mathbb{P}^* :

- Assume that $(X(t_1), \dots, X(t_n))$ has the same distribution under \mathbb{P} and \mathbb{P}^* , for $0 < t_1 < \dots < t_n$.

Then, by $\textcircled{*}$ to $s=t_n$ and A any set depending on $X(t_1), \dots, X(t_n)$ we have:

$$\mathbb{E}(\mathcal{L}(X(t_n)) \mathbb{1}_A) = \underbrace{\int_0^\infty dt \alpha e^{-\alpha t} \mathbb{E}(g(X(t_n+t)) \mathbb{1}_A)}_{\text{||}} \\ \mathbb{E}^*(\mathcal{L}(X(t_n)) \mathbb{1}_A) = \underbrace{\int_0^\infty dt \alpha e^{-\alpha t} \mathbb{E}^*(g(X(t_n+t)) \mathbb{1}_A)}$$

$$\Rightarrow \mathbb{E}(g(X(t_n+t)) \mathbb{1}_A) = \mathbb{E}^*(g(X(t_n+t)) \mathbb{1}_A)$$

$\Rightarrow (X(t_1), \dots, X(t_n), X(t_n+t))$ has also the same joint distr. under \mathbb{P} and \mathbb{P}^* .

By Kolmogorov-Daniell theorem $\Rightarrow \mathbb{P} \equiv \mathbb{P}^*$. #

Remark: Thm 2.27 and 2.28 can be used as a basis to construct the Feller Processes associated with a generator.

Instead of constructing from it the semigroup (this is also known as Hille-Yosida theorem) and then construct the process, one proves existence and uniqueness of probability measures μ^t satisfying the hypothesis of Thm 2.28. In general, the difficulty lies in the uniqueness.

2.8.3) Stationary distributions.

The next question we want to address is the existence and the structure (if not unique) of stationary measures.

Given a ^(probable) measure μ or δ at time $t=0$, the measure at time t is then given by $\mu T(t)$ and it satisfies the relation

$$\int_S f d(\mu T(t)) = \int_S T(t) f d\mu, \quad \forall f \in C(S).$$

$$\text{Equivalently, } E^{\mu} (f(x(t))) = \int S E^x (f(x(t))) \mu(dx).$$

Def. 2.29) A probability measure μ on S is stationary for the Feller process with semigroup $T(t)$ if $\mu T(t) = \mu$, $\forall t \geq 0$, i.e.,

$$\int_S T(t) f d\mu = \int_S f d\mu, \quad \forall f \in C(S), t \geq 0.$$

Lemma 2.30) Denote by \mathbf{I} the class of stationary distributions for a Feller process.
Then \mathbf{I} is convex.

Proof: It follows directly from the eq. in Def. 2.29. $\#$

Notation: We denote by \mathbf{I}_e the set of extreme points of \mathbf{I} .

Theorem 2.31) Suppose that D is a cone for the generator \mathcal{L} .
Then, a probability measure μ on S is stationary for the corresponding process
 $\Leftrightarrow \int Lf d\mu = 0, \forall f \in D.$

Proof: $\Rightarrow: \forall f \in D(\mathcal{L}),$

$$\begin{aligned} \int Lf d\mu &= \int \lim_{\epsilon \downarrow 0} \frac{T(\epsilon)f - f}{\epsilon} d\mu \\ &= \lim_{\epsilon \downarrow 0} \frac{\int T(\epsilon)f d\mu - \int f d\mu}{\epsilon} = 0. \end{aligned}$$

\Leftarrow : Since D is a cone,
 $\forall f \in D(\mathcal{L})$, then $\exists f_n \in D$ s.t. $f_n \rightarrow f$
and $Lf_n \rightarrow Lf$.

Thus, the assumption $\int Lf d\mu = 0, \forall f \in D$,
implies $\int Lf d\mu = 0$ for all $f \in D(\mathcal{L})$.

Now, if $f \in D(\mathcal{L})$ with $f - \lambda Lf = g$, then

$$\int f d\mu - \lambda \int Lf d\mu = \int g d\mu, \text{i.e., } \int g d\mu = \int (\lambda - \lambda^2)g d\mu.$$

By iteration, $\int (\lambda - \lambda^2)^n g d\mu = \int g d\mu$.

By Theorem 2.10 \circlearrowleft , i.e., $\lim_{n \rightarrow \infty} (\lambda - \frac{\lambda}{n} L)^n f = T(\lambda) f$, it
then follows that $\int T(\lambda)g d\mu = \int g d\mu, \forall f \in C(S), \lambda > 0$. $\#$.

- Generically the question of existence (and characterisation) of stationary distributions is not easy. For compact S we can however prove that a stationary distribution exists.

Thm. 2.32) If S is compact, then $\mathcal{I} \neq \emptyset$.

Proof: let μ be a probability measure on S and consider the time average of its evolution, i.e.,

$$\nu_n := \frac{1}{n} \int_0^n \mu T(r) dr.$$

For any $f \in C(S)$,

$$\begin{aligned} \int_S f d(\nu_n T(t)) &= \int_S T(t) f d\nu_n \\ &= \frac{1}{n} \int_0^n dr \int_S T(t+r) f du \\ &\quad \xrightarrow{\text{summing}} \\ &= \frac{1}{n} \int_t^{t+n} dr \left(\int_S T(r) f du \right). \end{aligned}$$

$$\Rightarrow \int_S f d\nu_n - \int_S T(t) f d\nu_n = \frac{1}{n} \int_0^n dr \int_S f du - \frac{1}{n} \int_t^{t+n} dr \int_S T(r) f du$$

$$\underbrace{\int_0^n - \int_t^{t+n} = \int_0^t - \int_{t+n}}_{\text{(just bounded)}} = \frac{1}{n} \left[\int_0^t \int_S T(r) f du - \int_t^{t+n} \int_S T(r) f du \right] \xrightarrow{n \rightarrow \infty} 0$$

- Now, by the fact that S is compact, it exists a converging subsequence n_k s.t.

$$\nu_{n_k} \Rightarrow \nu$$

for some probability measure ν on S . (this follows by Prokhorov's theorem and the fact that for S compact, all families of proba. measures are tight; see Thm A.21 and Def-A.20 of Liggett's book).

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Since $T(\epsilon)f \in C(S)$, we can take the limit in \oplus
along the sequence n_k and obtain

$$\int f d\omega = \int T(\epsilon) f d\omega, \quad \forall f \in C(S),$$

i.e., $\nu T(\epsilon) = \omega$. #

Here are some further properties:

Lemma 2.33) Let $T(\epsilon)$ be a probability semigroup of a Feller process. Then, if S is compact,

- ① If I is a compact subset of the probability measures on S , $M_1(S)$.
- ② If $\mu = \lim_{t \rightarrow \infty} \nu T(t)$ exists for some $\nu \in M_1(S)$, then $\mu \in I$.
- ③ If $\mu = \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} \nu T(t) dt$ exists for some $\nu \in M_1(S)$ and some $t_n \nearrow \infty$, then $\mu \in I$.

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Proof: See exercises.

Lemma 2.34) For a Feller process with semigroup $T(\epsilon)$, let μ be a probability measure on S and $\mu T(\epsilon) \Rightarrow \nu$ as $\epsilon \rightarrow 0$. Then ν is stationary.

Proof: $\mu T(\epsilon) \Rightarrow \nu$ means that $\forall f \in C(S)$,

$$\lim_{\epsilon \rightarrow 0} \int_S f d(\mu T(\epsilon)) = \int_S f d\nu. \quad \oplus$$

$$\Rightarrow \int_S f d\nu - \int_S T(s) f d\nu =$$

$$= \lim_{\epsilon \rightarrow 0} \int_S f d(\mu T(\epsilon)) - \lim_{\epsilon \rightarrow 0} \int_S T(s) f d(\mu T(\epsilon))$$

$$\stackrel{\text{Semigroup}}{=} \lim_{\epsilon \rightarrow 0} \underbrace{\int_S \mathbb{E}^x (\ell(X(\epsilon))) d\mu(x)}_{\in C(S)} - \lim_{\epsilon \rightarrow 0} \underbrace{\int_S \mathbb{E}^x (\ell(X(\epsilon+s))) d\mu(x)}_{\in C(S)}$$

$$\stackrel{\oplus}{=} \int_S f d\nu - \int_S f d\nu = 0. \quad \text{by the Feller property}$$

2.8.4) Duality

The usefulness of duality will be clear in the applications to interacting particle systems. First let us define it.

Def. 2.35) Let $X_1(t)$ and $X_2(t)$ be Feller processes

or nonexplosive Markov chains on S_1 and S_2 respectively. Given a bounded function H on $S_1 \times S_2$, the two processes are said to be dual with respect to H

if

$$\mathbb{E}^{x_1}(H(X_1(\epsilon), x_2)) = \mathbb{E}^{x_2}(H(x_1, X_2(\epsilon))),$$

$\forall t \geq 0, x_1 \in S_1, x_2 \in S_2$.

Rem. • The first \mathbb{E}^{x_1} is the expectation for the process $X_1(\epsilon)$ starting from x_1 , while \mathbb{E}^{x_2} is for the process $X_2(\epsilon)$ starting from x_2 .

• The function $H(x_1, x_2) \equiv 1$ is a trivial duality function, which however does not provide any information.

If H is in the domain of the generators, then it is useful to verify duality from them. If this is not the case, we will have to use the semigroups directly.

Rem. In what follows, when one of the processes is a Markov chain, \mathcal{L} will be taken to be

$$\mathcal{L}f(x) = \sum_{y \in S} q(x, y) f(y)$$

for all function f s.t. the series converges for all x (even if it is not a Feller process).

Theorem 2.36) Assume that

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$\mathcal{L}_1 H(x_1, x_2)$ and $\mathcal{L}_2 H(x_1, x_2)$
are well-defined and equal $\forall x_1 \in S'_1, x_2 \in S'_2$.
Then $X_1(\epsilon)$ and $X_2(\epsilon)$ are dual with
respect to H .

Proof: We do the proof for $X_1(\epsilon)$ a Feller process
with generator \mathcal{L}_1 and semigroup $T_1(t)$
and $X_2(\epsilon)$ a Markov chain with Q-matrix
 $q(x, y)$ and transition probabilities $p_t(x, y)$.
The proofs for the MC or two Feller processes
are analogues.

$$\begin{aligned} \text{Let } U(t, x_1, x_2) &:= \mathbb{E}^{x_1} (H(X_1(\epsilon), X_2)) \\ &= T_1(\epsilon) H(x_1, x_2). \end{aligned}$$

By Theorem 2.10(b),

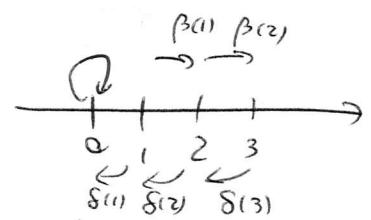
$$\begin{aligned} \frac{d}{dt} U(t, x_1, x_2) &= T_1(t) \mathcal{L}_1 H(x_1, x_2) \\ &\stackrel{\text{defn.}}{=} T_1(t) \mathcal{L}_2 H(x_1, x_2) \\ &= \sum_{y \in S'_2} T_1(t) q(x_2, y) H(x_1, y) \\ &= \sum_{y \in S'_2} q(x_2, y) \underbrace{T_1(t) H(x_1, y)}_{= U(t, x_1, y)} \end{aligned}$$

By Exercise 2(a), sheet 2, $\forall x_1 \in S'_1$, the unique
solution of this system of diff. eq. with I.C. $U(0, x_1, x_2) = H(x_1, x_2)$
is given by $\sum_{y \in S'_2} p_t(x_2, y) H(x_1, y) = \mathbb{E}^{x_2} (H(X_1(\epsilon), X_2))$.

$$\Rightarrow \mathbb{E}^{x_1} (H(X_1(\epsilon), X_2)) = \mathbb{E}^{x_2} (H(X_1, X_2(\epsilon))). \quad \#$$

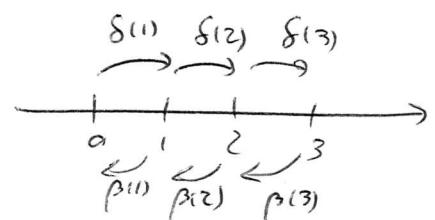
Example: let $X_1(e)$ and $X_2(e)$ be two M.C. on $\mathbb{Z} = \{0, 1, \dots\}$ with Q-matrices:

$$q_1(k, e) = \begin{cases} \beta(k), & \text{if } e = k+1, \\ \delta(k), & \text{if } e = k-1, \\ 0, & \text{otherwise,} \end{cases}$$



and

$$q_2(e, k) = \begin{cases} \delta(k+1), & \text{if } k = e+1, \\ \beta(e), & \text{if } k = e-1, \\ 0, & \text{otherwise,} \end{cases}$$



and $\beta(0) = \delta(0) = 0$.

Assume that the votes are s.t. the two chains are nonexplosive (See Thm 1.17 and Cor. 1.18).

Set $H(k, e) := \prod_{\{k \leq e\}} q_i(k, e)$. Then,

$$\begin{aligned} \sum_k H(k, e) &= \sum_{m \in \mathbb{Z}_+} q_1(k, m) H(m, e) \\ &= \beta(k) H(k+1, e) + \delta(k) H(k-1, e) \\ &\quad - (\beta(k) + \delta(k)) \cdot H(k, e) \end{aligned}$$

$$\begin{aligned} \sum_k H(k, e) &= \beta(k) \prod_{\{k+1 \leq e\}} + \delta(k) \prod_{\{k-1 \leq e\}} \\ &\quad - (\beta(k) + \delta(k)) \prod_{\{k \leq e\}} = -\beta(k) \delta_{k+1, e} + \delta(k) \delta_{k-1, e} \\ \text{and } \sum_k H(k, e) &= \sum_{m \in \mathbb{Z}_+} q_2(e, m) H(k, m) \\ &= \beta(e) \delta_{k, e-1} + \delta(e+1) \delta_{k, e+1} \\ &= \delta(e+1) H(k, e+1) + \beta(e) H(k, e-1) \\ &\quad - (\beta(e) + \delta(e+1)) H(k, e) \\ &= \beta(e) \prod_{\{k \leq e-1\}} + \delta(e+1) \prod_{\{k \leq e+1\}} \\ &\quad - (\beta(e) + \delta(e+1)) \prod_{\{k \leq e\}} \\ &= -\beta(e) \delta_{k, e-1} + \delta(e+1) \delta_{k, e+1} \end{aligned}$$

$\Rightarrow X_1$ and X_2 are dual w.r.t. H . Rem: "0" is a trap for X_1 (but not for $X_2(e)$)

Rem.: The duality is often particularly useful if the process is self-dual or if the dual process does not guess.

Ex.: If we constantly study the n-point correlation function of a system with infinitely many particles by the study of a system of n particles evolving as the dual process. (→ see next chapter).

2.8.5) Superposition of processes

The natural question when we want to construct complicated processes is whether we can do it by modeling them by separate "microscopic" processes and then superimpose them, i.e., consider as generator of the complex system to be just the sum of the generators of the simpler processes.

Thm. 2.37) Let $T_1(\epsilon), T_2(\epsilon), T(\epsilon)$ be probability semigroups with generators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$ respectively.

Let D be a core for \mathcal{L} and assume that $D \subseteq D(\mathcal{L}_1)_n(D_2)$ and $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ on D . Then, for each $f \in C(S)$,

$$\lim_{n \rightarrow \infty} \left[T_1\left(\frac{\epsilon}{n}\right) T_2\left(\frac{\epsilon}{n}\right) \right]^n f = T(\epsilon)f, \quad \forall \epsilon > 0.$$

We are going to prove a weaker statement for $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$ bounded operators. The proof of Thm 2.37 is the content of Sect I.6 of Ref. 20 in Liggett Book.

Thm 2.38) Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$ be bounded generators corresponding to the semigroups $T_1(\epsilon), T_2(\epsilon), T(\epsilon)$.

If $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, then

$$\lim_{n \rightarrow \infty} \left(T_1\left(\frac{\epsilon}{n}\right) T_2\left(\frac{\epsilon}{n}\right) \right)^n f = T(\epsilon)f, \quad \forall \epsilon \in ((S)).$$

Proof: The simplification due to the boundedness of the operators is that we can use the series to define the semigroups:

$T(t)f = \sum_{m=0}^{\infty} \frac{(2t)^m}{m!} \left(\frac{f}{2}\right)^m$, which can also be rewritten as:

$$T(t)f = e^{2t} E\left[\left(\frac{f}{2}\right)^Z\right], \text{ with } Z \sim \text{Poi}(2t) \text{ random variable.}$$

$$= e^{2t} E\left[\left(\frac{X_1 + Y_1}{2}\right)^Z f\right]. \quad \textcircled{*}$$

$$\text{Also, } \left(T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right)\right)^n f = \left(\sum_{k, e=0}^{\infty} \frac{\left(\frac{t}{n}\right)^k}{k!} \frac{\left(\frac{t}{n}\right)^e}{e!} L_1^k L_2^e f\right)^n$$

$$= e^{2t} E\left[L_1^{X_1} L_2^{X_2} \dots L_1^{Y_n} L_2^{Y_n} f\right] \quad \textcircled{+}$$

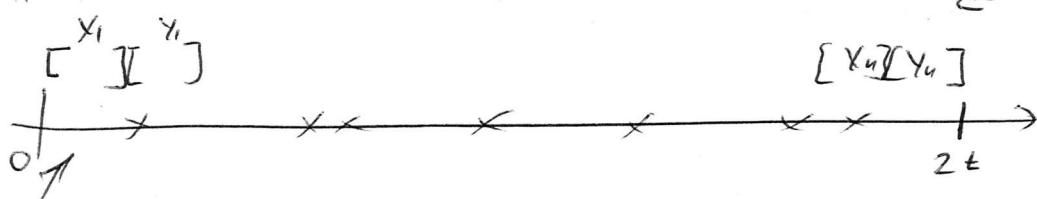
where X_1, \dots, X_n and Y_1, \dots, Y_n are iid. random variables with $Y_i, X_i \sim \text{Poi}\left(\frac{t}{n}\right)$.

We would like to take the $n \rightarrow \infty$ limit of $\textcircled{+}$ and obtain $\textcircled{*}$.

First notice that

$$\|L_1^{X_1} L_2^{X_2} \dots L_1^{Y_n} L_2^{Y_n} f\| \leq (\max(\|L_1\|, \|L_2\|))^{X_1 + \dots + X_n + Y_1 + \dots + Y_n} \|f\|.$$

Let $X_1, Y_1, \dots, X_n, Y_n$ be the thinning of Z .



Poisson P.P. with intensity 1 $\Rightarrow Z = \# \text{pts. in } [0, 2t]$

$$\left\{ \begin{array}{l} X_1 = \# \text{pts. in } [0, \frac{t}{n}] \\ Y_1 = \# \text{pts. in } [\frac{t}{n}, \frac{2t}{n}] \\ \vdots \\ Y_n = \# \text{pts. in } [2t - \frac{t}{n}, 2t] \\ \Rightarrow Z = X_1 + Y_1 + \dots + X_n + Y_n. \end{array} \right.$$

Next, we show:

$$\lim_{n \rightarrow \infty} \mathbb{E} (\mathcal{L}_1^{X_1} \mathcal{L}_2^{Y_1} \cdots \mathcal{L}_n^{X_n} \mathcal{L}_2^{Y_n}) \underset{\text{XXX}}{=} \frac{1}{2^n} (\mathcal{L}_1 + \mathcal{L}_2)^n \mathcal{L}.$$

By the fact that the PPP is simple,

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_i > 2 \text{ or } Y_i > 2 \text{ for some } i \leq n) = 0.$$

\Rightarrow the sequence $(X_1, Y_1, \dots, X_n, Y_n)$, conditioned on the event $\{Z = n\}$, is (as $n \rightarrow \infty$) a sequences of 0's and 1's, with exactly n 1's.

The subsequence of X 's and Y 's that takes values 1 is random and each one has proba. $\frac{1}{2^n}$.

$$\Rightarrow \text{l.h.s. of } \text{XXX} = \frac{1}{2^n} \sum_{\substack{I \\ I_1, \dots, I_m \in \mathcal{Z}_{1,2}}}^n \prod_{i=1}^m \mathcal{L}_{I_i} \mathcal{L} = \frac{1}{2^n} (\mathcal{L}_1 + \mathcal{L}_2)^n \mathcal{L}.$$

$$\begin{aligned} \Rightarrow \text{Thus, } \lim_{n \rightarrow \infty} \left[T_1\left(\frac{t}{n}\right) T_2\left(\frac{t}{n}\right) \right]^n \mathcal{L} &= e^{2t} \sum_{n \geq 0} \mathbb{P}(Z=n) \left(\frac{\mathcal{L}_1 + \mathcal{L}_2}{2} \right)^n \mathcal{L} \\ &= e^{2t} \mathbb{E} \left[\left(\frac{\mathcal{L}_1 + \mathcal{L}_2}{2} \right)^Z \mathcal{L} \right]. \end{aligned}$$

Corollary 2.39) Consider the context of Theorem 2.38. Then, if K is a strongly closed subset of $C(S)$, or a weakly closed set of probability measures on S' , that is invariant under $T_1(\epsilon)$ and $T_2(\epsilon)$, then K is invariant under $T(\epsilon)$ as well.

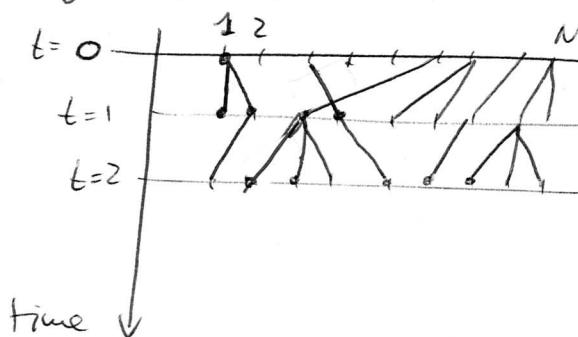
2.9) Applications to diffusion processes

2.9.1) Brownian motion with speed change.

- let $X(t)$ be a standard B.M. Its generator is given by $\frac{1}{2}D$. If we change time $t \rightarrow ct$, then the generator of $Y(t) := X(ct)$ is given by

$$\mathcal{L}_y f(x) = \frac{c}{2} f''(x).$$

- If we want to define a process that locally behaves like B.M. but around x it moves at a relative speed $c(x)$ compared with B.M., then the generator should be given by $\mathcal{L}_y f(x) = \frac{c(x)}{2} f''(x)$ (provided that it defines a generator!).
- This process arises naturally as scaling limit of systems like the following biologically-motivated model.



$\bullet N$ individuals.

M.C.: Link individuals of generation $t+1$ with one randomly chosen of generation t .

- let $X(t)$ be the fraction of a subset of individuals of $\{1, \dots, N\}$.
 - Then, $\mathbb{P}(X(t)=y | X(t-1)=x) =$
- $$= \binom{N}{yN} \cdot x^{yN} \cdot (1-x)^{(1-y)N}$$
- with $x, y \in \{0, \frac{1}{N}, \dots, 1\}$.

\Rightarrow The analogue of the generator for the discrete time M.C. is $\mathcal{L}_N \varphi(x) \equiv \mathbb{E}^x (\varphi(X_1)) - \varphi(x)$

$$= \sum_{y \in \{0, \frac{1}{N}, \dots, 1\}} \underbrace{\binom{N}{yN} x^{yN} (1-x)^{(1-y)N}}_{\text{relevant for } y-x = O(N^{-1/2})} \cdot (\varphi(y) - \varphi(x))$$

For $\varphi \in C^2([0, 1])$, $\varphi(y) = \varphi(x) + \varphi'(x) \cdot (y-x) + \frac{1}{2} \varphi''(x) \cdot (y-x)^2 + O((y-x)^3)$

(elementary computations) $\Rightarrow \lim_{N \rightarrow \infty} 2N \mathcal{L}_N \varphi(x) = \frac{1}{2} x (1-x) \varphi''(x).$

\Rightarrow If the limit process exists (as $n \rightarrow \infty$) it should have state space $S = [0, 1]$ and the generator (its closure)

$$\mathcal{L} f(x) = \frac{1}{2}x(1-x)f''(x), \text{ for } f \text{ polynomials, } \quad \textcircled{*}$$

(as they are dense in $C([0, 1])$). This is known as Wright-Fisher diffusion.

Theorem 2.40) (a) The closure of \mathcal{L} given in $\textcircled{*}$ is a probability generator.

Let $X(t)$ be the Feller process with generator $\bar{\mathcal{L}}$. Then,

(b) $X(t)$ has continuous paths,

(c) if $\tau := \inf\{t \geq 0 : X(t) = 0 \text{ or } 1\}$, then

$$\mathbb{E}^X(\tau) = -2x \ln x - 2(1-x) \ln(1-x),$$

$$P^X(X(\tau) = 1) = x, \text{ and}$$

$$\mathbb{E}^X \left(\int_0^\tau dt X(t)(1-X(t)) \right) = x(1-x)$$

Proof: Part (a): $D(\mathcal{L}) = \{f \in C([0, 1]) \mid f \text{ are polynomials}\}$.

Property (a) of Def. 2.6: $D(\mathcal{L})$ dense in $C([0, 1])$: ✓.

Property (b) of Def. 2.6: let $f \in D(\mathcal{L})$ with $f - \lambda \mathcal{L} f = g$ for some $\lambda \geq 0$. Let x_0 be a point in $[0, 1]$ where f is minimal. Then,

$$\begin{aligned} &\rightarrow \text{if } x_0 \in (0, 1) \Rightarrow f''(x_0) \geq 0 \\ &\rightarrow \text{if } x_0 \in \{0, 1\} \Rightarrow \mathcal{L} f(x_0) \geq 0 \end{aligned} \Rightarrow \mathcal{L} f(x_0) \geq 0$$

$$\Rightarrow \min_{x \in [0, 1]} f(x) = f(x_0) \geq g(x_0) \geq \min_{x \in [0, 1]} g(x). \quad \checkmark$$

Property (c) of Def. 2.6: $f(1) = 0$. ✓.

By Prop. 2.24 $\bar{\mathcal{L}}$ exists and satisfies (a), (b) of Def. 2.6.

Concerning Property (c) of Def. 2.6: let $g(x) = \sum_{k=0}^n a_k x^k$ and look

for $f \in D(\mathcal{L})$ s.t. $f - \lambda \mathcal{L} f = g$. If we write $f(x) = \sum_{k=0}^n b_k x^k$,

$$\text{then } f - \lambda \mathcal{L} f = g \Leftrightarrow \sum_{k=0}^n x^k \left[b_k - \frac{1}{2} x(1-x) k(k-1) \frac{b_k}{x^2} - a_k \right] = 0$$

$$\Leftrightarrow b_k - \frac{\lambda}{2} \cdot k(k+1) b_{k+1} + \frac{\lambda}{2} k(k-1) b_k = a_k, \quad 0 \leq k \leq 4,$$

where $b_{n+1} = 0$.

\Rightarrow Given $a_0, \dots, a_n \Rightarrow b_0, b_1, \dots, b_4$ are recovered.

$\Rightarrow R(\mathbb{I} - \lambda \bar{L})$ contains all polynomials \Rightarrow dense in $C([0, 1])$.

By Prop. 2.24 (d), $R(\mathbb{I} - \lambda \bar{L}) = C([0, 1])$.

\Rightarrow Part (d) is proved.

1. Rem.: $D(\bar{L})$ contains $C^2([0, 1])$ since $\forall f \in C^2([0, 1]), \exists f_n \in D(\bar{L})$ s.t.
 $f_n, f'_n, f''_n \xrightarrow{\text{unif.}} f, f', f''$.

Part (e) We apply Thm. 2.27 (the martingale problem) to various choices of f .

Let $f(x) = x \Rightarrow Lf = 0 \Rightarrow X(t) = f(X(t))$ is a bounded martingale

$\Rightarrow X(\infty) := \lim_{t \rightarrow \infty} X(t)$ exists

and $\mathbb{E}^X(X(\infty)) = x$. \oplus

Let $f(x) = x(1-x) \Rightarrow Lf(x) = -x(1-x)$

$$\Rightarrow M(t) = f(X(t)) - \int_0^t ds L f(X(s))$$

$$= X(t)(1-X(t)) + \int_0^t ds X(s)(1-X(s))$$

is a non-negative martingale [Thm 3.15 of Karatzas-Shreve book applied to $-M(t)$].

$M(\infty) := \lim_{t \rightarrow \infty} M(t)$ exists a.s. and $\mathbb{E}(M(\infty)) < \infty$.

$$\Rightarrow \mathbb{P}^X(X(\infty) = 0 \text{ or } 1) = 1 \text{ and by (d), } \mathbb{P}^X(X(\infty) = x) = x.$$

$$\text{Taking } \lim_{t \rightarrow \infty} \mathbb{E}^X(M(t)) = x(1-x) = \mathbb{E}\left(\int_0^\infty ds X(s)(1-X(s))\right).$$

Let $f(x) = 2x \ln x + 2(1-x) \ln(1-x)$. Although formally $Lf = 1$,

$f \notin C^2([0, 1])$ and also $f \notin D(\bar{L})$. We can however smooth out f by setting $f_\varepsilon \in C^2([0, 1])$ s.t. $f_\varepsilon = f$ on $[\varepsilon, 1-\varepsilon]$ and define

$\tau = \text{ hitting time of } \mathbb{F}_{\varepsilon, 1-\varepsilon}$

Then, $\mathcal{L}_\varepsilon(X(t)) - \int_0^t \mathcal{L} \mathcal{L}_\varepsilon(X(s)) ds$ is a Martingale.

$\Rightarrow \mathcal{L}(X(\tau_{\varepsilon \wedge t})) - \tau_{\varepsilon \wedge t}$ is a \mathbb{P}^x -martingale if $x \in [\varepsilon, 1-\varepsilon]$.

Therefore, $\underbrace{\mathbb{E}^x[\mathcal{L}(X(\tau_\varepsilon))]}_{t \rightarrow \infty} - \underbrace{\mathbb{E}^x(\tau_\varepsilon)}_{t \rightarrow \infty} = \mathcal{L}(x)$ for $x \in [\varepsilon, 1-\varepsilon]$.

Letting $\varepsilon \downarrow 0$, $\mathbb{E}^x(\tau) = -\mathcal{L}(x)$ for $x \in (0, 1)$.

Part ④: We use Thm 2.22.

Fix $y \in [0, 1]$ and let $\mathcal{L}(x) := (x-y)^2$.

Then $\mathcal{L}\mathcal{L}(x) = x(1-x)$. Thus,

$(X(t)-y)^2 - \int_0^t X(s)(1-X(s)) ds$ is a martingale.

It follows:

$$\mathbb{E}^y((X(t)-y)^2) = \int_0^t \mathbb{E}^y(X(s)(1-X(s))) ds \leq \frac{t}{4}.$$

For $g(x) = (x-y)^4$, $\mathcal{L}g(x) = 6x(1-x)(x-y)^2$.

$$\begin{aligned} \Rightarrow \mathbb{E}^y((X(t)-y)^4) &= \mathbb{E}^y \left[\int_0^t \mathbb{E}^y[X(s)(1-X(s)) \cdot (X(s)-y)^2] ds \right] \\ &\leq \frac{3}{2} \int_0^t \mathbb{E}^y(X(s)-y)^2 ds \leq \frac{3}{16} \cdot t^2. \end{aligned}$$

By the Markov property,

$$\mathbb{E}^y[(X(t)-X(s))^4 | F_s] = \mathbb{E}^{\substack{\tilde{X}(s)=X(s) \\ (t>s)}}[(\tilde{X}(t-s)-\tilde{X}(s))^4] \leq \frac{3}{16} (t-s)^2$$

$$\Rightarrow \mathbb{E}^y((X(t)-X(s))^4) \leq \frac{3}{16} (t-s)^2. \Rightarrow \text{The hypothesis of Thm 2.22 holds with } \gamma=4, \beta=1, C=\frac{3}{16}.$$

A similar proof for path continuity holds for generic Brownian motions with speed change.

Let us assume for a moment that the generator

$$\mathcal{L}\mathcal{L}(x) = \frac{1}{2} \operatorname{curl}^H(x)$$

has a closure that is a probability generator.

Then the following holds (see Thm 3.68 for some sufficient conditions):

Thm 2.41) let $C(\cdot) \in C(\mathbb{R})$, $0 \leq C(x) \leq K$, $\forall x \in \mathbb{R}$.

Assume that $X(t)$ is a Feller process with generator $\mathcal{L}f(x) = \frac{1}{2}C(x)f''(x)$

when restricted to C^2 functions with compact support.
Then, $X(t)$ is a diffusion process.

Proof: What we need to show is the path continuity
(see Def. 2.21 and Thm. 2.22).

• let $f \in C_0^2(\mathbb{R})$ with $f(a) = 0$ for some $a \in \mathbb{R}$.

Then,

$$T(t)f(a) - f(a) = \int_0^t ds \frac{d}{ds} T(s)f(a)$$

$$\stackrel{\text{Thm 2.10}}{=} \int_0^t ds T(s) \mathcal{L}f(a) \quad \oplus$$

$$\Rightarrow |T(t)f(a)| \leq t \cdot \frac{K}{2} \cdot \|f''\|$$

• To compute $\mathbb{E}^a(X(t)-a)^2$ one approximate $(X-a)^2$
by $f \in C_0^2(\mathbb{R})$ pointwise with $\|f''\| \leq 2$ and obtain:

$$\mathbb{E}^a(X(t)-a)^2 \leq tK.$$

• Next, for $f \in C_0^2(\mathbb{R})$ with $f(a) = 0$ and $|f''(x)| \leq (x-a)^2/K$,

$$|T(t)f(a)| \stackrel{\oplus}{\leq} \frac{K}{2} \int_0^t ds \mathbb{E}^a(X(s)-a)^2 \leq \frac{K^2 t^2}{4}.$$

• Approximate then $\frac{(x-a)^4}{12}$ by such f_S^4 (pointwise), leads to

$$\mathbb{E}^a(X(t)-a)^4 \leq 3K^2 t^2.$$

• Finally, by the Markov Property,

$$\mathbb{E}^x(X(t)-X(s))^4 = \mathbb{E}^x \left(\mathbb{E}^{\tilde{X}(s)} \tilde{X}(t-s) - \tilde{X}(s) \right)^4 \leq 3K^2(t-s)^2. \quad \#$$

\tilde{X} an indep. copy of X .

2.9.2) Brownian motion with absorption

- For the Wright-Fisher diffusion the boundary behavior was determined by the form of the generator and the process remains at 0 or 1 once it reaches them.
- Here we will see situations where the form of the generator is like the one of standard Brownian motion, but the boundary behavior is reflected into the domain of the generator.

Def. 2.42) The Brownian motion on \mathbb{R}_+ with absorption at 0 is defined as follows.

Let $\tau = \inf \{t \geq 0 \mid X(t) = 0\}$, where $X(t)$ is a standard B.M. Then, the BM with absorption, X_α , is defined by

$$X_\alpha(t) = \begin{cases} X(t), & \text{for } t < \tau, \\ 0, & \text{for } t \geq \tau. \end{cases}$$

Denote by \mathcal{L}_α and $T_{\alpha(t)}$ its generator and semigroup.

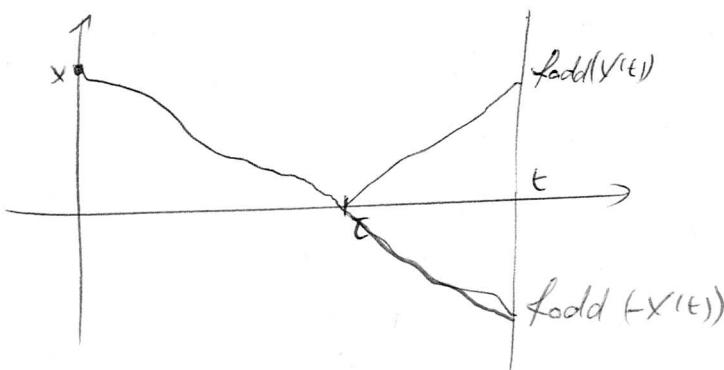
For $f \in C(\mathbb{R}_+)$, let f_{odd} be given by:

$$f_{\text{odd}}(x) = \begin{cases} f(x), & x \geq 0 \\ 2f(0) - f(-x), & x < 0 \end{cases}, \quad x \in \mathbb{R}.$$

Using the reflection principle,

$$\mathbb{E}^x (f_{\text{odd}}(X(\tau)) \mathbb{1}_{t \geq \tau}) = \mathbb{E}^x (f_{\text{odd}}(-X(\tau)) \mathbb{1}_{t \geq \tau})$$

$$\begin{aligned} &= \frac{1}{2} \mathbb{E}^x (f_{\text{odd}}(X(\tau)) + f_{\text{odd}}(-X(\tau))) \mathbb{1}_{t \geq \tau} \\ &= f(0) \mathbb{E}^x (\mathbb{1}_{t \geq \tau}) \\ &= f(0) \mathbb{P}^x (t \geq \tau). \end{aligned}$$



Thus, for $x > 0$,

$$\begin{aligned} T_\alpha(t) f(x) &= \mathbb{E}^x (f(X(\epsilon)) \mathbb{I}_{(t < \tau)} + f(0) \mathbb{I}_{(t \geq \tau)}) \\ &= \mathbb{E}^x (f_{\text{odd}}(X(t))). \end{aligned}$$

So, for $x > 0$,

$$\begin{aligned} \mathcal{L}_\alpha f(x) &= \lim_{\epsilon \downarrow 0} \frac{T_\alpha(\epsilon) f(x) - f(x)}{\epsilon} = f'_{\text{odd}}(x) \\ &= \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}^x (f_{\text{odd}}(X(\epsilon))) - f(x)}{\epsilon} \quad \text{for } x > 0 \\ &= (\mathcal{L} f_{\text{odd}})(x) = \frac{1}{2} f''_{\text{odd}}(x) = \frac{1}{2} f''(x) \\ &\quad \text{generator of } X(t). \end{aligned}$$

The domain of the generator is the one where the above limit exists strongly.

By Ex 2, Sheet 4, $D(\mathcal{L}_\alpha) = \{f \in C(\mathbb{R}_+) \mid f'_\text{odd}, f''_\text{odd} \in C(\mathbb{R})\}$

Since $f'_\text{odd}(x) = \begin{cases} f'(x), & x > 0 \\ f'(-x), & x < 0 \end{cases} \quad (\Rightarrow \text{continuous at } 0)$

and $f''_\text{odd}(x) = \begin{cases} f''(x), & x > 0 \\ -f''(-x), & x < 0 \end{cases} \quad \Rightarrow \text{continuous at } 0$
 $\quad \text{if } f'(0) = 0$

it follows that $D(\mathcal{L}_\alpha) = \{f \in C(\mathbb{R}_+) \mid f'_\text{odd}, f''_\text{odd} \in C(\mathbb{R}), f''(0) = 0\}$

and $\mathcal{L}_\alpha f(x) = \frac{1}{2} f''(x).$

2.9.3) Brownian motion with reflection.

Def. 2.43) Let $X(t)$ be a standard BM on \mathbb{R} . We define the BM on \mathbb{R}_+ with reflection at 0 by

$$X_r(t) = |X(t)|.$$

- Denote by \mathcal{L}_r and $T_{\mathcal{L}_r}(t)$ its generator and semigroup.
- For $f \in C(\mathbb{R}_+)$, define f_{even} by

$$f_{\text{even}}(x) = \begin{cases} f(x), & x \geq 0, \\ f(-x), & x < 0. \end{cases}$$

Then, for $x \geq 0$,

$$\begin{aligned} T_r(t) f(x) &\equiv \mathbb{E}^x(f(|X(t)|)) \\ &= \mathbb{E}^x(f_{\text{even}}(X(t))) \end{aligned}$$

$$\begin{aligned} \text{Thus, for } x \geq 0, \quad &\lim_{t \downarrow 0} \frac{T_r(t)f(x) - f(x)}{t} = \\ &= \lim_{t \downarrow 0} \frac{\mathbb{E}^x(f_{\text{even}}(X(t))) - f_{\text{even}}(x)}{t} \\ &= (\mathcal{L} f_{\text{even}})(x). \end{aligned}$$

$$\Rightarrow \mathcal{L}_r f(x) = \mathcal{L} f_{\text{even}}(x) = \frac{1}{2} f''(x) \quad \text{for } x \geq 0.$$

By Ex 2, Sheet 4, $D(\mathcal{L}_r) = \{f \in C(\mathbb{R}_+) \mid f'_{\text{even}}, f''_{\text{even}} \in C(\mathbb{R})\}$

Since $f'_{\text{even}}(x) = \begin{cases} f'(x), & x \geq 0, \\ -f'(-x), & x < 0, \end{cases}$ and $f''_{\text{even}}(x) = \begin{cases} f''(x), & x \geq 0, \\ +f''(-x), & x < 0, \end{cases}$

we need to have $f'(0)=0$.

$$\Rightarrow D(\mathcal{L}_r) = \{f \in C(\mathbb{R}_+) \mid f', f'' \in C(\mathbb{R}_+), f'(0)=0\}.$$

and $\mathcal{L}_r f(x) = \frac{1}{2} f''(x)$

Remark: let \mathcal{L} be defined by $Lf(x) = \frac{1}{2}f''(x)$ or

$$\{ f \in C(\mathbb{R}_+) \mid f, f' \in C(\mathbb{R}_+), f'(0) = f''(0) = 0 \}.$$

Then, this is not a probability generator, since \mathcal{L}_a and \mathcal{L}_r are two strict extensions of \mathcal{L} and two generators can not be such that one is a strict extension of the other.

- Intuitively this is clear, that we can not at the same time reflect and absorb the process $X(t)$ at 0 (what one can do is "absorb with some probability and reflect otherwise," see also Example 3.59 on sticky boundary at 0 of the book).

B.M. and duality: Taking an appropriate limit of the dual Markov chains at page 74, one gets the B.M. with absorption and the reflected B.M. Thus it is natural to think that the duality survives.

Lemma 2.44) $X_\alpha(t)$ and $X_t(\epsilon)$ are dual with respect to $H(X, Y) = \mathbb{E} \sum_{x \leq y} \delta_x$.

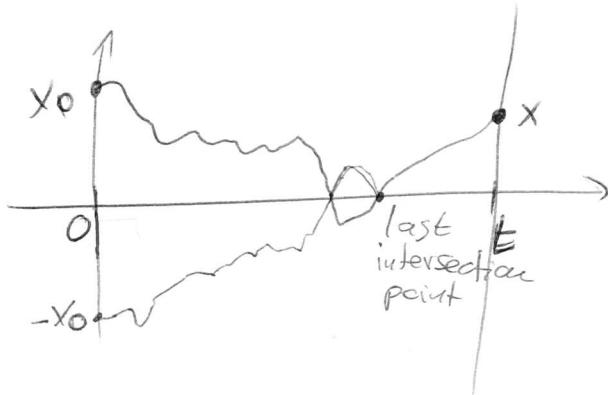
Lemma 2.45) For $x_0 > 0$, $x > 0$,

$$\mathbb{P}^{x_0}(X_\alpha(t) \in dx) = (\phi_t(x - x_0) - \phi_t(x + x_0)) dx.$$

with $\phi_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$.
Therefore,

$$\begin{aligned} \mathbb{P}^{x_0}(X_\alpha(t) \in dx) &= (\phi_t(x - x_0) - \phi_t(x + x_0)) dx \\ &\quad + \left(1 - \int_{\mathbb{R}_+} dy \phi_t(y - x_0) + \int_{\mathbb{R}_+} dy \phi_t(y + x_0)\right) \delta_0(dx) \end{aligned}$$

Proof:

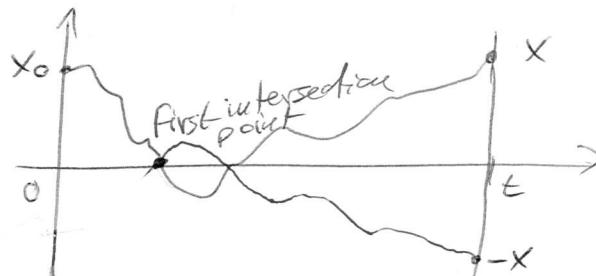


Using the reflection principle, we immediately have, for $x > 0$, $\mathbb{P}^{x_0}(X_a(\epsilon) \in dx) = (\phi_t(x-x_0) - \phi_t(x-(-x_0)))dx$.

Further, $1 - \int_0^x (\phi_t(x-x_0) - \phi_t(x+x_0)) dx = \mathbb{P}^{x_0}(X_a(\epsilon) = 0)$ from which the second statement follows.

Lemme 2.46) For $x_0, x > 0$,

$$\mathbb{P}^{x_0}(X_r(\epsilon) \in dx) = (\phi_t(x-x_0) + \phi_t(x+x_0))dx.$$



By the reflection principle,

$$\begin{aligned} \mathbb{P}^{x_0}(X_r(\epsilon) \in dx) &= (\underbrace{\phi_t(x-x_0) + \phi_t(-x-x_0)}_{= \phi_t(x+x_0)}) dx \\ &\quad \# . \end{aligned}$$

Proof of Lemme 2.44) We can not apply Thee 2.36 directly, since H is not in $D(L_a)$ and $D(L_r)$.

To show duality we need to prove:

$$\mathbb{E}^{x_1}(H(X_a(\epsilon), x_2)) = \mathbb{E}^{x_2}(H((x_1, X_r(\epsilon)))) , \forall \epsilon > 0, x_1, x_2 \in \mathbb{R}_+.$$

With $H(x, y) = \mathbb{I}(x \leq y)$ it is to be provee:

$$\mathbb{P}^{x_1}(X_a(\epsilon) \leq x_2) = \mathbb{P}^{x_2}(X_r(\epsilon) \geq x_1).$$

By Lemma 2.45,

$$\begin{aligned}
 \mathbb{P}^{x_1}(X_{\tau(\epsilon)} \leq x_2) &= \int_0^{y_2} dx \phi_t(x-x_1) - \int_0^{x_2} dx \phi_t(x+x_1) \\
 &\quad + 1 - \int_0^{\infty} dx \phi_t(x-x_1) + \int_0^{\infty} dx \phi_t(x+x_1) \\
 &= \underbrace{\int_{-x_1}^{x_2-x_1} dy \phi_t(y)}_{-\infty} - \underbrace{\int_{x_1}^{x_2+x_1} dy \phi_t(y)}_{\mathbb{R}} + \underbrace{\int_{-\infty}^{\infty} dy \phi_t(y) dy}_{\mathbb{R}} \\
 &\quad - \underbrace{\int_{-x_1}^{\infty} dy \phi_t(y)}_{-\infty} + \underbrace{\int_{x_1}^{\infty} dy \phi_t(y)}_{\mathbb{R}} \\
 &= - \underbrace{\int_{x_2-x_1}^{\infty} dy \phi_t(y)}_{-\infty} + \underbrace{\int_{x_2+x_1}^{\infty} dy \phi_t(y)}_{\mathbb{R}} + \underbrace{\int_{-\infty}^{\infty} dy \phi_t(y) dy}_{\mathbb{R}} \\
 &= \int_{x_1-x_2}^{\infty} dy \phi_t(y) + \underbrace{\int_{-\infty}^{x_2-x_1} dy \phi_t(y)}_{-\infty} \\
 &= \int_{x_2-x_1}^{\infty} dy \phi_t(y) \\
 &= \int_{x_1}^{\infty} dx \phi_t(x-x_2) + \int_{x_1}^{\infty} dx \phi_t(x+x_2) = \mathbb{P}^{x_2}(X_{\tau(\epsilon)} \geq x_1).
 \end{aligned}$$

Markov Chains

- (A) Transition (right-ctd., M.P.)
- (B) Tr. fct. (Kolmogorov)
- (C) Q-matrix. (Kolmogorov Backwards Equation)
- (A) \Rightarrow (B) \Rightarrow (C) ✓.

(C) \Rightarrow (B) : Non-uniqueness due to possible explosion.

. Probab. construction
↳ Gives the meaning of $P_t^{(n)}$.

(B) \Rightarrow (A) : Thm. 1.19: If $P_t^{(n)}$ (x,y)
min. sol. of KBE stoch. \Rightarrow ! M.C.

Feller processes

- (A) Process (cadlag, M.P., Feller prop.)
- (B) Semigroup (Properties: $T(t)$: contractive on $C_b(S)$
 $L \mapsto T(t)L$ contin.)
Resolvent $U(\lambda)$
 \hookrightarrow for $\Re(\lambda) > R_{\text{ess}}$

(C) Generator: (Prop.: $(L-\lambda I)^{-1}$ contractive for $\Im(\lambda) < 1$ or $\lambda > 0$).

(A) \Rightarrow (B) \Rightarrow (C) ✓.
Thm 2.10

(C) \Rightarrow (B) : Approx. L by $L_\varepsilon = L(I - \varepsilon J)^{-1}$ (bd).

(B) \Rightarrow (A) : Main point: check 3 cadlag process + martingale results.

Diff-process \hookrightarrow Feller + path continuity.

Tools:

- How to close a linear operator to make it into a probab. generator
(check on a stable dense subset,
i.e., one, is enough).
(w.r.t. semigroup)

- Martingale problem (\rightarrow used to estimate stochasticity).
- Stationary distib. (enough to be convex, for S' compact, not empty).
- Duality
- Superposition of processes.