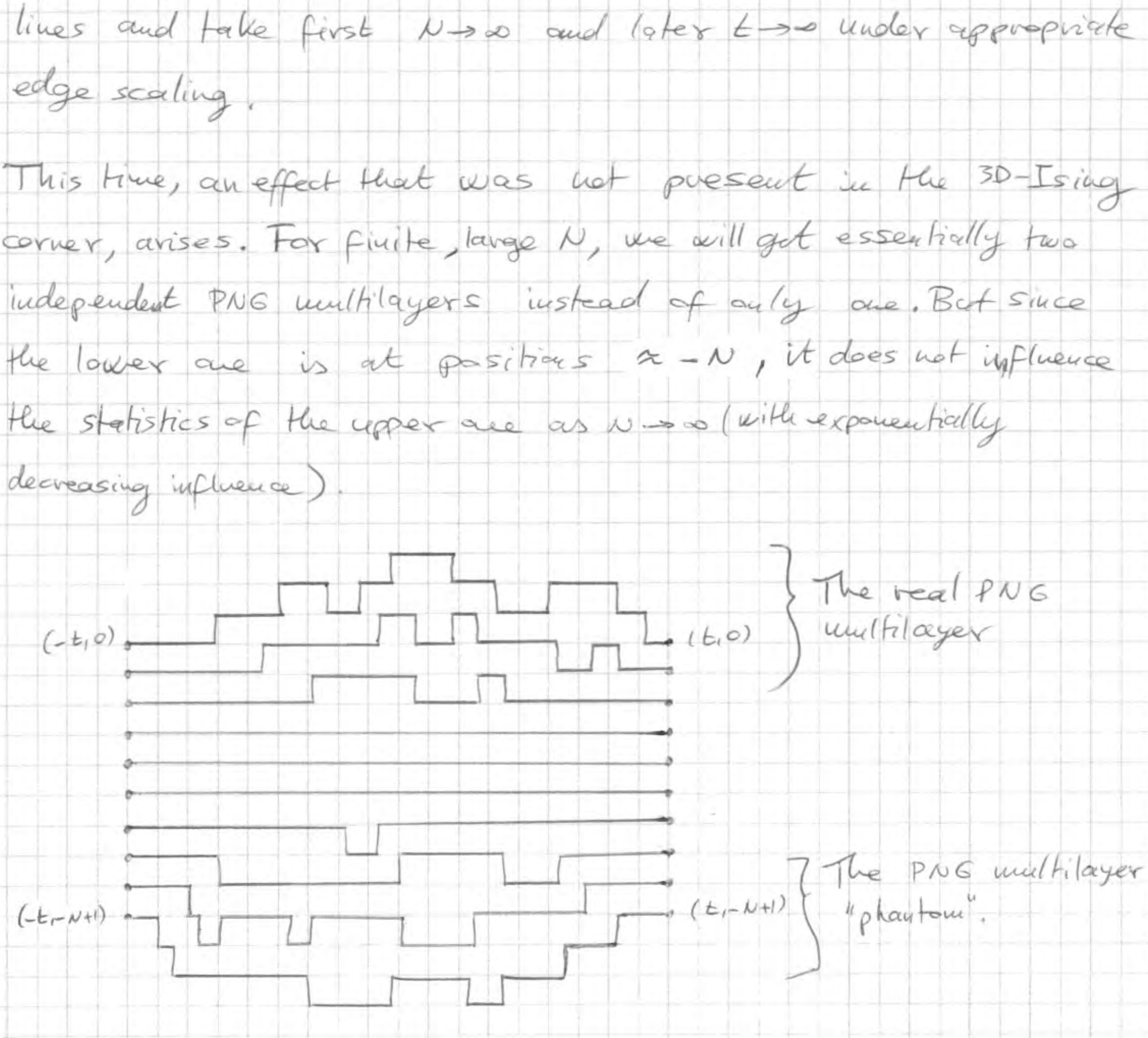
9.4) Non-intersecting lines and extended determinantal point process

. The multilayer PNG is a set of non-intersecting lines with fixed initial and final positions. The measure on the jumps is just Lebesgue measure, i.e., jumps occurs up and down with intensity one (provided the lines do not intersect).

. Moreover, by the RSK construction backwards in time, to any non-intersecting lines it corresponds a set of Poisson points.

. To apply Karlin-McGregor theorem, we start with N non-intersecting lines and take first $N \to \infty$ and later $t \to \infty$ under appropriate edge scaling.

. This time, an effect that was not present in the 3D-Ising corner, arises. For finite, large N, we will get essentially two independent PNG multilayers instead of only one. But since the lower one is at positions $\approx -N$, it does not influence the statistics of the upper one as $N \to \infty$ (with exponentially decreasing influence).



. So, we have to study the system of N non-intersecting lines starting from $(x_i(-t)=-i)_{i=0,1,\dots,N-1}$ and ending at $(x_i(t)=-i)_{i=0,1,\dots,N-1}$, under the non-intersecting constraint but otherwise doing continuous time random walks with jump rates one ($\equiv$ Lebesgue measure on jumps).

. By Karlin-McGregor, we need to determine first the transition probability of a single free path. To go from x to y during a time interval τ, it will be given by $\langle y, e^{-\tau H} x\rangle$, where $H\psi(u) = -[\psi(u+1)+\psi(u-1)]$ for $\psi \in \ell^2(\mathbb{Z})$. In other words,

<u>for $\tau \geq 0$:</u> $P_\tau(x;y) = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dz}{z^{y-x+1}} \cdot e^{\tau\left(z+\frac{1}{z}\right)}$, where Γ_0 is any anticlockwise simple loop around $z=0$.

[To check it, just look $\tau \to 0$: $P_\tau(x;y) = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{y-x+1}}\cdot\left(1+\tau\left(z+\frac{1}{z}\right)+O(\tau^2)\right)$
$$= \delta_{x,y} + \tau\cdot(\delta_{x,y+1}+\delta_{x,y-1}) + O(\tau^2)$$
and use the fact that we have a semi-group].

. Therefore, the point process associated with the line ensemble has kernel given by:

$$K_N(t_1,x_1;t_2,x_2) = -P_{t_1-t_2}(x_1;x_2)\,\mathbb{1}_{[t_2<t_1]} + \sum_{x,y\leq 0} P_{t-t_2}(x_2;x)\cdot[A_N^{-1}]_{x,y}\cdot P_{t+t_1}(y;x_1)$$

where $[A_N]_{x,y} = P_{2t}(x;y)$.

. The $N\to\infty$ limit is as for the 3D-Ising corner model. So, we need to determine the inverse of $A = [P_{2t}(x;y)]_{x,y\leq 0}$.

. We consider A as a matrix on $\ell^2(\mathbb{Z})$ and the inverse to be computed is on $\ell^2(\{\ldots,-2,-1,0\})$ only.

. We use the same decomposition of the 3D-Ising corner, page 14:

let P_+ = projector on $\{1,2,\ldots\}$ and P_- = projector on $\{\ldots,-1,0\}$.

. Denote by $a_+(\tau)$ the matrix with upjumps only, i.e.,

$$[a_+(\tau)]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw\, e^{\tau w}}{w^{y-x+1}}, \qquad \Rightarrow \text{upper-triangular},$$

and by $a_-(\tau)$ the matrix with downjumps only, i.e.,

$$[a_-(\tau)]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw\, e^{\tau/w}}{w^{y-x+1}}. \qquad \Rightarrow \text{lower-triangular}.$$

. The block decompositions are: $a_+(\tau) = \begin{pmatrix} m_1(\tau) & m_2(\tau) \\ 0 & m_3(\tau) \end{pmatrix}$, $a_-(\tau) = \begin{pmatrix} \tilde m_1(\tau) & 0 \\ \tilde m_2(\tau) & \tilde m_3(\tau) \end{pmatrix}$

$$\Rightarrow A = a_+(2t)\,a_-(2t) = a_-(2t)\,a_+(2t) = \begin{pmatrix} \tilde m_1(2t)\, m_1(2t) & * \\ * & * \end{pmatrix}$$

. We need to compute: $\boxed{P_-\left(P_+ + P_- a_-(2t)a_+(2t)P_-\right)^{-1} P_- = a_+(2t)^{-1} P_- a_-(2t)^{-1}}$.

In fact, $a_+(2t)^{-1} = \begin{pmatrix} m_1^{-1} & -m_1^{-1}\cdot m_2 \cdot m_3^{-1} \\ 0 & m_3^{-1} \end{pmatrix}$ and $a_-(2t)^{-1} = \begin{pmatrix} \tilde m_1^{-1} & 0 \\ -\tilde m_3^{-1}\cdot \tilde m_2 \cdot \tilde m_1^{-1} & \tilde m_3^{-1} \end{pmatrix}$

Then, $$a_+(2t)^{-1} P_- a_-(2t)^{-1} = \underbrace{\begin{pmatrix} m_1^{-1} & -m_1^{-1} m_2 m_3^{-1} \\ 0 & m_3^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} m_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} \tilde m_1^{-1} & 0 \\ -\tilde m_3^{-1} \tilde m_2 \cdot \tilde m_1^{-1} & \tilde m_3^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} m_1^{-1}\cdot \tilde m_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and $$P_-(P_+ + P_- a_- a_+ P_-)^{-1} P_- = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde m_1 m_1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} m_1^{-1}\tilde m_1^{-1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} m_1^{-1}\cdot \tilde m_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Inverse of a_+ and a_- on $\ell^2(\mathbb{Z})$:

- $$\left[a_+^{-1}(2t)\right]_{x,y} = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \, \frac{e^{-2tw}}{w^{y-x+1}} \quad \text{and}$$

- $$\left[a_-^{-1}(2t)\right]_{x,y} = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw \; e^{-2t/w}}{w^{y-x+1}}.$$

$$\Rightarrow K(t_1,x_1;t_2,x_2) = -\left[a_-(t_1-t_2)\, a_+(t_1-t_2)\right]_{x_1,x_2} \cdot \mathbb{1}_{[t_2<t_1]}$$

$$+ \sum_{x,y\le 0} \left[a_-(t-t_2)\, a_+(t-t_2)\right]_{x_2,x} \cdot \left[a_+(2t)^{-1} P_- \, a_-(2t)^{-1}\right]_{x,y} \cdot$$

$$\cdot \left[a_-(t+t_1)\, a_+(t+t_1)\right]_{y,x_1}$$

Now, we can extend the sum over all $x,y\in\mathbb{Z}$, since the middle term is zero for x or $y>0$.

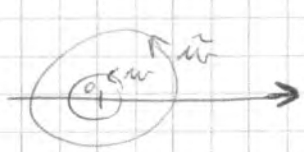
Therefore,

$$K(t_1,x_1;t_2,x_2) = -\left[a_-(t_1-t_2)\, a_+(t_1-t_2)\right]_{x_1,x_2} \cdot \mathbb{1}_{[t_2<t_1]}$$

$$+ \sum_{e\le 0} \left[a_-(t-t_2)\cdot a_+^{-1}(t+t_2)\right]_{x_2,e} \cdot \left[a_-^{-1}(t-t_1)\cdot a_+(t+t_1)\right]_{e,x_1}$$

- $$\left[a_-(t-t_2)\, a_+^{-1}(t+t_2)\right]_{x,y} = \sum_{z\in\mathbb{Z}} \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} \frac{dw\, e^{(t-t_2)/w}}{w^{z-x+1}} \cdot \oint_{\Gamma_0} \frac{d\tilde{w}\, e^{-(t+t_2)\tilde{w}}}{\tilde{w}^{y-z+1}}$$

$$= \sum_{z\le y} \left(\quad '' \quad \right)$$



$$= \sum_{e = y-z \ge 0} \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w} \, \frac{e^{(t-t_2)/w} \cdot e^{-(t+t_2)\tilde{w}}}{w^{y-x+1} \cdot \tilde{w}} \cdot \left(\frac{w}{\tilde{w}}\right)^e$$

$$= \frac{1}{(2\pi i)^2} \underbrace{\oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w}}_{|w|<|\tilde{w}|} \, \frac{e^{(t-t_2)/w} \, e^{-(t+t_2)\tilde{w}}}{w^{y-x+1}} \cdot \underbrace{\sum_{e\ge 0} \left(\frac{w}{\tilde{w}}\right)^e \frac{1}{\tilde{w}}}_{= \frac{1}{\tilde{w}-w}}$$

Residue at $\tilde{w}=w$

$$= \frac{1}{2\pi i} \oint_{\Gamma_0} dw \, \frac{e^{\frac{t-t_2}{w}} \cdot e^{-(t+t_2)w}}{w^{y-x+1}}.$$

and similarly,

$$\left[a_-^{-1}(t-t_1)a_+(t+t_1)\right]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} dw\, \frac{e^{-(t-t_1)/w}\cdot e^{(t+t_1)w}}{w^{y-x+1}}.$$

<u>Parenthesis on Bessel functions:</u> For $b \geq a$ and $n \in \mathbb{Z}$,

(a) $$\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z}\cdot\frac{e^{b(z-\frac{1}{z})}\cdot e^{a(z+\frac{1}{z})}}{z^n} \underset{z=\frac{1}{w}}{=} \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w}\cdot\frac{e^{b(\frac{1}{w}-w)}\cdot e^{a(w+\frac{1}{w})}}{w^{-n}}$$

$$= \left(\frac{b+a}{b-a}\right)^{n/2} J_n\left(2\sqrt{b^2-a^2}\right),$$

where J_n are the standard Bessel functions.

$$\Rightarrow \sum_{\ell\leq 0}\left[a_-(t-t_2)a_+^{-1}(t+t_2)\right]_{x_2,\ell}\cdot\left[a_-^{-1}(t-t_1)a_+(t+t_1)\right]_{\ell,x_1} =$$

$$= \sum_{\ell\leq 0}\left(\frac{t-t_2}{t+t_2}\right)^{\frac{x_2-\ell}{2}}\cdot\left(\frac{t+t_1}{t-t_1}\right)^{\frac{x_1-\ell}{2}}\cdot J_{x_2-\ell}\left(2\sqrt{t^2-t_2^2}\right)\cdot J_{x_1-\ell}\left(2\sqrt{t^2-t_1^2}\right)$$

(b) $$\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z}\cdot\frac{e^{t(z+\frac{1}{z})}}{z^n} = I_{|n|}(2t)$$, where I_n is the modified Bessel function.

(c) One can also compute:

$$\sum_{\ell\in\mathbb{Z}}\left[a_-(t-t_2)a_+^{-1}(t+t_2)\right]_{x_2,\ell}\cdot\left[a_-^{-1}(t-t_1)a_+(t+t_1)\right]_{\ell,x_1} = \frac{1}{2\pi i}\oint_{\Gamma_0} dw\,\frac{e^{(t_1-t_2)(w+\frac{1}{w})}}{w^{x_1-x_2+1}}$$

Therefore the kernel can be written as follows:

$$K(t_1,x_1;t_2,x_2) = \begin{cases} \sum_{\ell\leq 0}\left(\frac{t-t_2}{t+t_2}\right)^{\frac{x_2-\ell}{2}}\cdot\left(\frac{t+t_1}{t-t_1}\right)^{\frac{x_1-\ell}{2}} J_{x_2-\ell}\left(2\sqrt{t^2-t_2^2}\right) J_{x_1-\ell}\left(2\sqrt{t^2-t_1^2}\right), & \text{for } t_2\leq t_1 \\ \sum_{\ell>0} \quad \text{"} & \text{, for } t_2>t_1. \end{cases}$$

This is the form of the extended (Bessel) kernel obtained in the original paper by Prähofer and Spohn '02.

Back to the main track.

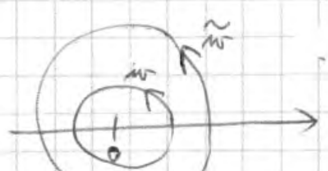
. The main part of the kernel writes then:

$$\sum_{\ell\le 0} (\ldots)_{x_2,\ell}(\ldots)_{\ell,x_1} = \sum_{\ell\le 0}\frac{1}{(2\pi i)^2}\oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w}\, e^{t(\frac{1}{w}-w)}\cdot e^{-t_2(w+\frac{1}{w})}$$

$$\cdot\, e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}\cdot \frac{w^{x_2-1}}{\tilde{w}^{x_1+1}}\left(\frac{\tilde{w}}{w}\right)^{\ell}$$

$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w}}_{|\tilde{w}|\ge|w|}\, e^{t(\frac{1}{w}-w)}\, e^{-t_2(w+\frac{1}{w})}\cdot e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}$$

$$\cdot\,\frac{w^{x_2-1}}{\tilde{w}^{x_1+1}}\underbrace{\sum_{\ell\le 0}\left(\frac{\tilde{w}}{w}\right)^{\ell}}_{=\frac{\tilde{w}}{\tilde{w}-w}}$$



$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w}}_{|\tilde{w}|>|w|}\,\frac{e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}}{e^{t(w-\frac{1}{w})}\cdot e^{t_2(w+\frac{1}{w})}}\cdot\frac{w^{x_2-1}}{\tilde{w}^{x_1}}\cdot\frac{1}{\tilde{w}-w}$$

which can be reexpressed also as

$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w}}_{|w|>|\tilde{w}|}\,\frac{e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}}{e^{t(w-\frac{1}{w})}\cdot e^{t_2(w+\frac{1}{w})}}\cdot\frac{w^{x_2-1}}{\tilde{w}^{x_1}}\,\frac{1}{\tilde{w}-w}$$

$$+\frac{1}{2\pi i}\oint_{\Gamma_0} dw\,\frac{e^{(t_1-t_2)(w+\frac{1}{w})}}{w^{x_1-x_2+1}}.$$

. Remark: the last term is, for $t_1>t_2$, exactly the extra term in the complete kernel. This happens all the time and it is not particular of the PNG model.

- Therefore, the final formula for the Kernel is:

$$\boxed{K(t_1,x_1;t_2,x_2) = \frac{1}{(2\pi i)^2}\oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w}\ \frac{e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}}{e^{t(w-\frac{1}{w})}\cdot e^{t_2(w+\frac{1}{w})}}\ \frac{w^{x_2-1}}{\tilde{w}^{x_1}}\cdot\frac{1}{\tilde{w}-w}}$$

where: $\begin{cases} \underline{\text{for } t_1 \leq t_2}, \text{ the integrand satisfy: } w \text{ inside path } \tilde{w}, \\ \underline{\text{for } t_1 > t_2}, \text{ the integrand satisfy: } \tilde{w} \text{ inside path } w. \end{cases}$

Case: $t_1 \leq t_2$ Case: $t_1 > t_2$

9.5) Edge scaling and convergence to the Airy process.

- let $\eta_t(x,j) = \begin{cases} 1, \text{ a line crosses } (x,j), \\ 0, \text{ otherwise}. \end{cases}$

Then, since we want to analyze $\dfrac{h(ut^{2/3},t) - 2t + u^2 t^{1/3}}{t^{1/3}} \equiv h_t^{resc}(u)$, our point process η_t rescales as:

$$\eta_t^{edge}(u,s) = t^{1/3}\cdot\eta_t\left(ut^{2/3}, [2t - u^2 t^{1/3} + s t^{1/3}]\right),$$

and the associated Kernel as:

$$K^{edge}(u_1,s_1;u_2,s_2) = t^{1/3}\cdot K\left(u_1 t^{2/3}, [2t - u_1^2 t^{1/3} + s_1 t^{1/3}]; u_2 t^{2/3}, [2t - u_2^2 t^{1/3} + s_2 t^{1/3}]\right).$$

let $\begin{cases} f_0(z) \doteq z - \frac{1}{z} - 2\cdot\ln z \\ f_1(z;u) \doteq \left(z + \frac{1}{z}\right)u \\ f_2(z,u,s) \doteq -(s-u^2)\ln z \end{cases}$

- Consider the case $t_1 \leq t_2$, i.e., $\underline{u_1 \leq u_2}$. The case $u_1 > u_2$ follows similarly.

Then, $$K_t^{edge}(u_1,s_1;u_2,s_2) = \frac{t^{1/3}}{(2\pi i)^2} \underbrace{\oint_{\Gamma_0} dw \oint_{\Gamma_0} dz}_{\substack{w \text{ inside} \\ z\text{-path}}} \cdot \frac{e^{t f_0(z) + t^{2/3} f_1(z,u_1) + t^{1/3} f_2(z,u_1,s_1)}}{e^{t f_0(w) + t^{2/3} f_1(w,u_2) + t^{1/3} f_2(w,u_2,s_2)}} \cdot \frac{1}{w(z-w)}.$$

. Asymptotic analysis : u_1 and u_2 remain fixed.

(a) Convergence on bounded set (uniformly): do the asymptotic analysis of K^{edge} for $s_1,s_2 \in [-L,L]$, for $L \gg 1$ fixed.

Needed to control the cvg. of the Fredholm determinants.

(b) Obtain a bound for $(s_1,s_2) \in [-L, \varepsilon t^{2/3}]^2 \setminus [-L,L]$ for L large enough and ε small enough.
Usually one can get : $|K^{edge}| \underset{\substack{\text{up to ev. a} \\ \text{conjugation}}}{\leq} C \cdot e^{-(s_1+s_2)}$.

(c) Obtain a bound for $(s_1,s_2) \in [-L,\infty)^2 \setminus [-L, \varepsilon t^{2/3}]^2$.

. For part (a), see the details below. The leading contribution comes from the neighborhood of the double critical point of $f_0(z)$.

. To get the part (b), one can modify the path used in (a) locally around the old critical points, since now one will have two real critical points of the term proportional to t. One can estimate by taylor series for $\varepsilon \ll 1$.

. To get part (c), one usually can keep the path of (b) for $s_i = \frac{\varepsilon}{2} t^{2/3}$ and see that the difference is exponentially small.

. Now we explain how to do the asymptotic analysis for s_1, s_2 in a bounded set.

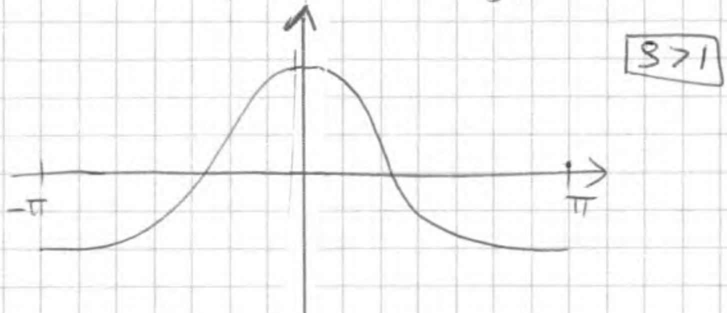
. <u>Step 1</u>: Find integration paths s.t. $\mathrm{Re}[f_0(z)]$ decreases by going away from the critical point and $\mathrm{Re}(-f_0(w))$ too.

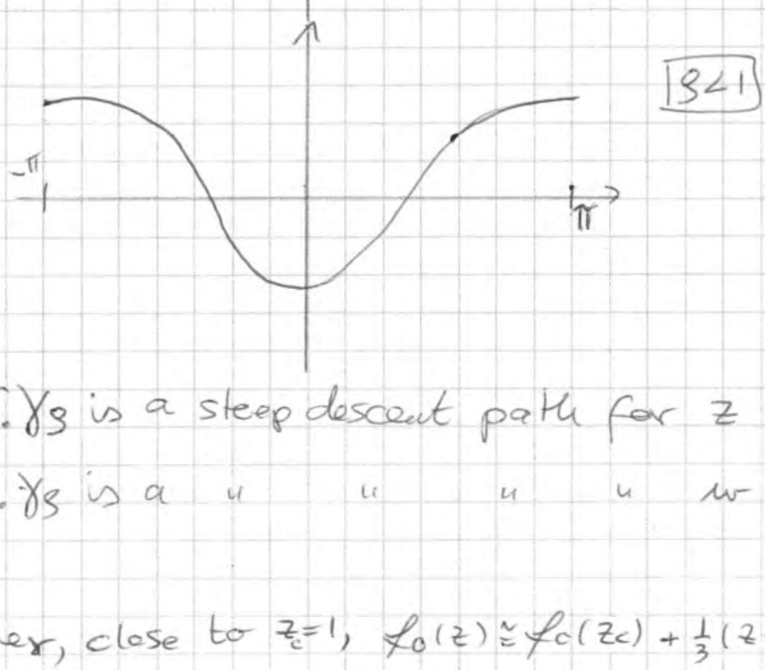
. <u>Critical point</u>: $\frac{df_0(z)}{dz} = 1 + \frac{1}{z^2} - \frac{2}{z} = \frac{1+z^2-2z}{z^2} = \frac{(z-1)^2}{z^2}$

$= 0 \Rightarrow$ <u>$z=1$: double critical point</u>

. Consider the path: $\gamma_\rho = \{z = \rho e^{i\varphi},\ \varphi \in [-\pi, \pi)\}$.

Then, $\mathrm{Re}(f_0(z)) = (\rho - \frac{1}{\rho})\cos\varphi - 2\ln\rho$

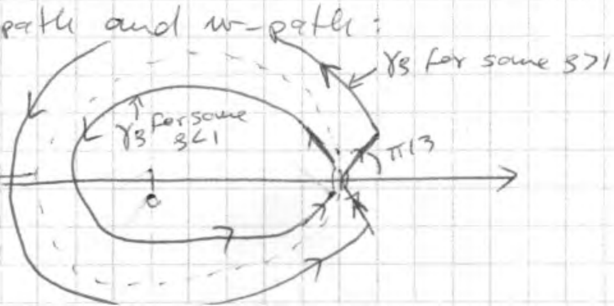




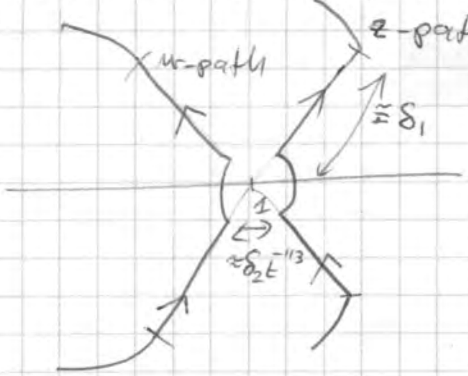
$\Rightarrow \begin{cases} .\ \gamma_\rho \text{ is a steep descent path for } z \text{ if } \rho > 1 \\ .\ \gamma_\rho \text{ is a " " " " } w \text{ if } \rho < 1 \end{cases}$

. Moreover, close to $z_c = 1$, $f_0(z) \simeq f_0(z_c) + \frac{1}{3}(z - z_c)^3$, which means that a path leaving $z = z_c$ is <u>steepest descent</u> $\Rightarrow$ it leaves with an angle $\pm\pi/3$.

. $\Rightarrow$ Choice of z-path and w-path:



. Zoom close to the critical point:



- We need to keep a small distance $\mathcal{O}(t^{-1/3})$ to the critical point because the two paths can't intersecate due to the $\frac{1}{z-w}$ term.

- We can fix a small δ_1 s.t. the $\mathrm{Re}(f_0(z))$ decreases along $e^{i\pi/3}\cdot x$, $0 \le x \le \delta_1$, and we already have that along γ_s, $s>1$, we are fine too.

- Once we get the steep descent path, we can replace the integrals in K^{resc} by the ones restricted to a δ-neighborhood of the critical point. The error can be estimated $\mathcal{O}(e^{-\mu(\delta)t})$ where $0<\mu(\delta) \sim \delta^3$. Denote by $\Gamma_z^\delta, \Gamma_w^\delta$ these paths.

. <u>Step 2:</u> Taylor series around the critical point.

. We have:
$$\begin{cases} f_0(z) = f_0(z_c) + \frac{1}{3}(z-z_c)^3 + \mathcal{O}((z-z_c)^4) \\ f_1(z,u) = f_1(z_c,u) + u\cdot(z-z_c)^2 + \mathcal{O}(|z-z_c)^3) \\ f_2(z,u,s) = f_2(z_c,u,s) + (u^2-s)(z-z_c) + \mathcal{O}((z-z_c)^2) \end{cases}$$

$$\Rightarrow K_t^{edge} \cong \frac{t^{1/3}}{(2\pi i)^2} \cdot \int_{\Gamma_w^\delta} dw \int_{\Gamma_z^\delta} dz \, \frac{e^{\frac{1}{3}t(z-z_c)^3 + t^{2/3}u_1(z-z_c)^2 + t^{1/3}(u_1^2-s_1)(z-z_c)}}{e^{\frac{1}{3}t(w-z_c)^3 + t^{2/3}u_2(w-z_c)^2 + t^{1/3}(u_2^2-s_2)(w-z_c)}} \cdot \frac{1}{(z-w)z_c} \cdot e^{\mathcal{O}(t(z-z_c)^4,\, t^{2/3}(z-z_c)^3,\, t^{1/3}(z-z_c)^2\} \text{ same with } w)} (1+\mathcal{O}(z-z_c))$$

conjugated by
$$\left(\frac{e^{t^{2/3}f_1(z_c,u_1)+t^{1/3}f_2(z_c,u_1,s_1)}}{e^{t^{2/3}f_1(z_c,u_2)+t^{1/3}f_2(z_c,u_2,s_2)}}\right)^{-1}$$

- At this point we use: $|e^{x}-1| \leq |x| \cdot e^{|x|}$ applied for $x = \mathcal{O}(\dots) \Rightarrow$ If we replace by zero the error term, we do an error given by the same main integrand times an extra $\mathcal{O}(\dots)$.

- By the change of variable: $\begin{cases} (z-z_c)t^{1/3} = Z, \\ (w-z_c)t^{1/3} = W, \end{cases}$

one then sees that this error is bounded by $\mathcal{O}(t^{-1/3})$

- So, we obtained:

$$K_t^{edge} \underset{\text{conj.}}{=} \mathcal{O}(t^{-1/3}, e^{-\mu(\delta)t}) + \frac{t^{1/3}}{(2\pi i)^2} \cdot \int_{\Gamma_w^\delta} dw \int_{\Gamma_z^\delta} dz \, \frac{e^{\frac{1}{3}t(z-z_c)^3 + t^{2/3}u_1(z-z_c)^2 + t^{1/3}(u_1^2-s_1)(z-z_c)}}{e^{\frac{1}{3}t(w-z_c)^3 + t^{2/3}u_2(w-z_c)^2 + t^{1/3}(u_2^2-s_2)(w-z_c)}} \cdot \frac{1}{z_c \cdot (z-w)}$$

- By the change of variable indicated above, this last integral becomes ($z_c = 1$)

$$\frac{1}{(2\pi i)^2} \int dW \int dZ \, \frac{e^{\frac{Z^3}{3} + u_1 \cdot Z^2 + (u_1^2-s_1)Z}}{e^{\frac{W^3}{3} + u_2 W^2 + (u_2^2-s_2)W}} \cdot \frac{1}{Z-W}$$

where the integration paths can be continued to:

- for Z: $e^{\pm i\pi/3} \cdot \infty$,
- for W: $e^{\pm 2\pi i/3} \cdot \infty$.

The error made will be once more only $\mathcal{O}(e^{-\mu(\delta)t})$.

- Therefore, we showed that for s_1, s_2 in a bounded set,

$$\lim_{t\to\infty} K_t^{resc}(u_1,s_1;u_2,s_2) \underset{\substack{\uparrow \\ \text{conjugation} \\ \text{too}}}{\equiv} \frac{1}{(2\pi i)^2} \int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} dw \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} dZ \, \frac{e^{\frac{Z^3}{3}+u_1Z^2+(u_1^2-s_1)Z}}{e^{\frac{w^3}{3}+u_2w^2+(u_2^2-s_2)w}} \cdot \frac{1}{Z-w}.$$

- Claim: This is the Airy kernel (up to conjugation)

In fact, since $Re(z-w)>0$, $\frac{1}{z-w} = \int_0^\infty e^{-\lambda(z-w)}\,d\lambda$

$$\Rightarrow \int_0^\infty d\lambda \left(\frac{1}{2\pi i}\int dw\, e^{-\left(\frac{w^3}{3}+u_2w^2+(u_2^2-s_2-\lambda)w\right)}\right) \cdot \left(\frac{1}{2\pi i}\int dz\, e^{\frac{z^3}{3}+u_1z^2+(u_1^2-s_1-\lambda)z}\right)$$

$$\underset{\substack{w=\tilde{w}-u_2 \\ z=\tilde{z}-u_1}}{=} \int_0^\infty d\lambda \left(\frac{1}{2\pi i}\int d\tilde{w}\, e^{-\left(\frac{\tilde{w}^3}{3}-(s_2+\lambda)\tilde{w}\right)} \cdot e^{-\lambda u_2} \cdot e^{\frac{u_2^3}{3}-u_2s_2}\right) \left(\frac{1}{2\pi i}\int d\tilde{z}\, e^{\frac{\tilde{z}^3}{3}-(s_1+\lambda)\tilde{z}} \cdot e^{\lambda u_1} \cdot e^{-\frac{u_1^3}{3}+u_1s_1}\right)$$

↳ in the conjugation

$$= \int_0^\infty d\lambda\, e^{-\lambda(u_2-u_1)} \cdot Ai(s_2+\lambda)\,Ai(s_1+\lambda)$$

where we used:

$$\begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} dw\, e^{-\frac{w^3}{3}+\alpha w} = Ai(\alpha) \\ \text{and } \dfrac{1}{2\pi i}\displaystyle\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} dz\, e^{\frac{z^3}{3}-\beta z} = Ai(\beta) \end{cases}$$

Finished.