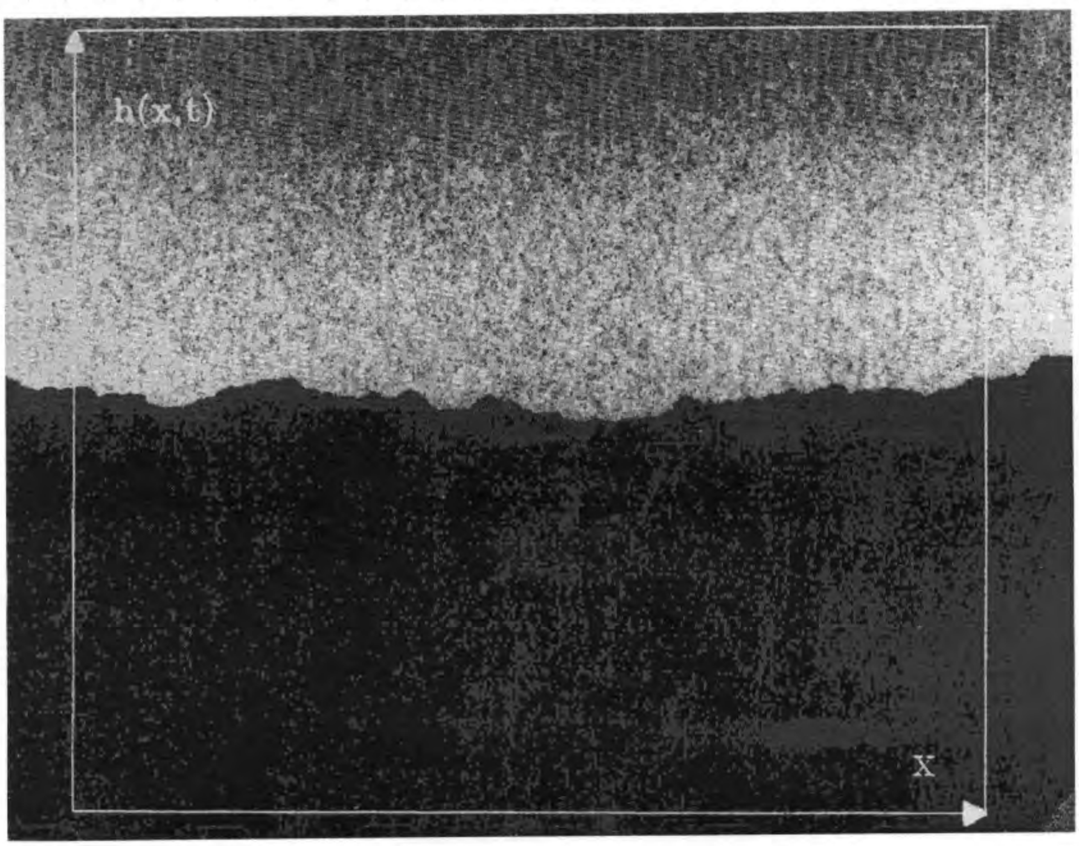
9) Application to the "PNG droplet".

. With "PNG droplet" we mean a corner growth model, which we will define later. First a few words on the class of model to which it belongs.

9.1) Generalities, KPZ class.

. There are a lot of different kind of growth processes. For example, a crystal can grow due to atomic deposition, or when a porous medium is put in contact to some liquid, then the growing quantity is the wetted region.

. As illustration, below there is a piece of paper burned from below (black region). The interface is clearly visible.



[From Barabasi-Stanley book "Fractal Concepts in Surface Growth"]

. On a macroscopic scale the interface is roughly flat, but we would like to say something about its roughness.

. Universality picture: The statistical properties of the interface, for large growth time t, should depend only on a few global properties of the dynamics like:

- substrate dimension,
- locality of growth,
- symmetries,
- conservation laws.

. The model we consider belongs to the Kardar-Parisi-Zhang (KPZ) universality class in 1+1 dimension. Kardar, Parisi, and Zhang wrote down a macroscopic equation which describes a stochastically growing interface $x \mapsto h(x,t)$:

$$\frac{\partial h(x,t)}{\partial t} = \nu \cdot \Delta h(x,t) + \frac{1}{2}\lambda \cdot (\nabla h(x,t))^2 + \eta(x,t),$$

where :
- $\nu \Delta h$ (with $\nu > 0$) is smoothening (surface tension)
- $\lambda > 0$: lateral growth
- η : space-time uncorrelated white noise.

. This equation is the simplest continuous equation for an irreversible, local, non-linear random growth.

. The smoothening makes the surface "macroscopically deterministic", i.e.,

$$\lim_{t\to\infty} \frac{h(\xi \cdot t, t)}{t} = h_{ma}(\xi) \text{ is non-random.}$$

. Thus, we focus at the fluctuations:

$$H(x,t) = h(x,t) - t \cdot h_{ma}(x/t).$$

. Question: On which scale we have to focus to see non-trivial fluctuations / correlations?

. Fluctuation exponent: α s.t. $H(x,t) \simeq t^{\alpha}$

. Correlation exponent: β s.t. in order to have $|H(x,t)-H(x',t)| \simeq t^{\alpha}$ we need to move apart of $|x-x'| \simeq t^{\beta}$.

. KPZ exponent in 1+1 dimension: $\alpha = 1/3$, $\beta = 2/3$

. Therefore, one will have to rescale the height as follows:

$$h_t^{resc}(u) \doteq \frac{h(a\cdot t + u\cdot t^{2/3}, t) - t\cdot h_{ma}(a + u\cdot t^{-1/3})}{t^{1/3}}$$

. Question: Can we determine the limit process describing the height fluctuations, i.e., $\lim_{t\to\infty} h_t^{resc}$? Is it depending on initial conditions or not?

. To try to answer to these questions, we consider a simplified model, the "polynuclear growth (PNG) model".

. As far we know now, starting with an interface without fluctuations, say $h(x,0)=0$ for example, then the limit process should be:

- Airy$_2$ process if h_{ma} is curved, [Airy$_2$ $\equiv$ Airy process].
- Airy$_1$ process if h_{ma} is flat.

[See math.PR/0105240 and 0707.4207 on the www.arXiv.org].

. Now I'll define the PNG model and analyze the particular geometry "corner growth" which leads to a curved limit shape $\Rightarrow$ Airy$_2$ process.

9.2) The polynuclear growth model; droplet case.

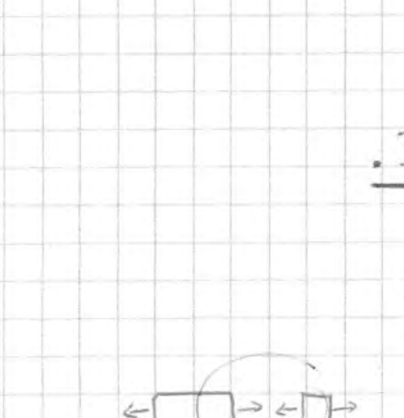
. The model: . Configurations: The configurations are given by integer-valued functions $x \mapsto h(x,t) \in \mathbb{Z}$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$.

$\Rightarrow$ We can describe the configurations by telling the positions of the up- and down-jumps [by convention, h is upper semi-continuous].

. If at the same position we have an up- and a down-jump, we call it nucleation.

. Dynamics: It consists in a deterministic part and a stochastic part:

ⓐ Deterministic part:
- . up-steps move to left, unit speed
- . down-steps move to right, " "
- . when they meet, they merge

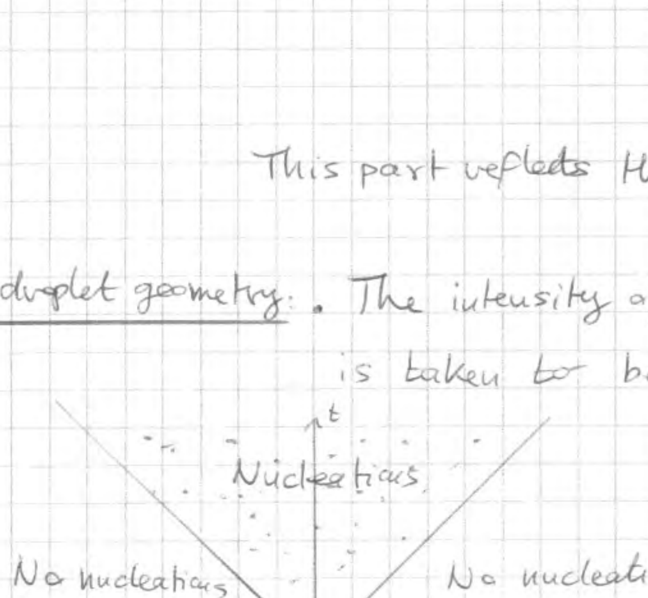


This part reflects both the term $\nu \Delta h$ and $\frac{1}{2}\lambda(\nabla h)^2$ in the KPZ equation.

ⓑ Stochastic part: . nucleations are added as a space-time Poisson process with some intensity $\varrho(x,t)$.

This part reflects the noise term η in the KPZ equation.

. PNG droplet geometry: . The intensity of nucleations for the PNG droplet is taken to be: $\varrho(x,t) = \begin{cases} 2, & |x| \le t, \\ 0, & \text{otherwise.} \end{cases}$



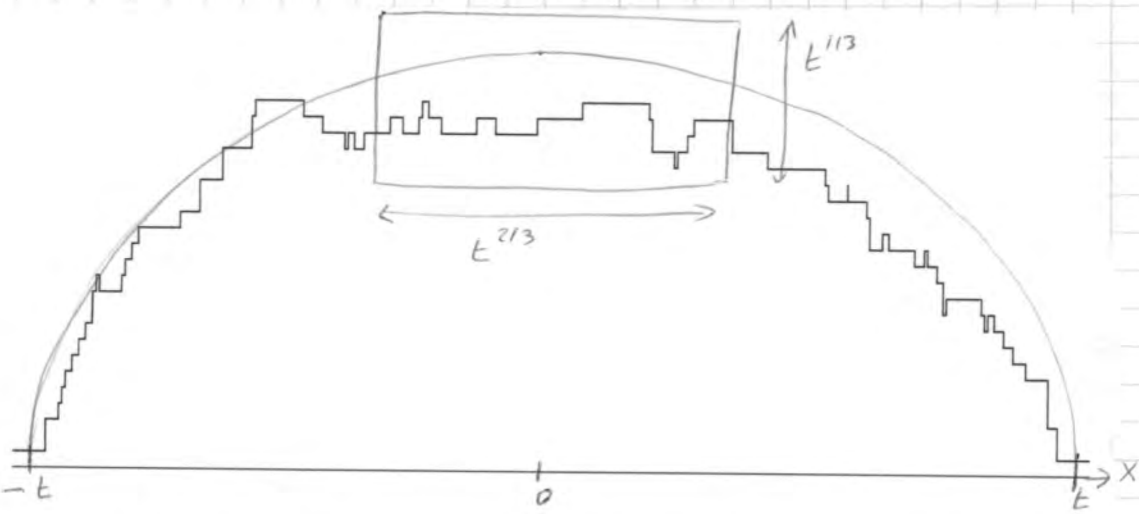
. Equivalently, the PNG droplet is obtained by nucleating with fixed intensity <u>above</u> a first island starting at the origin $(x,t)=(0,0)$.

. <u>Theorem [Prähofer, Spohn '02]</u> . Let $h(x,0)=0$ and $g(x,t)=\begin{cases}2, & |x|\le t,\\ 0, & \text{otherwise}.\end{cases}$

In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty} \frac{h(u\cdot t^{2/3},t)-t\, h_{ma}(u t^{-1/3})}{t^{1/3}} = \mathcal{A}_2(u),$$

with $h_{ma}(\xi) = 2\sqrt{[1-\xi^2]_+}$, and $\mathcal{A}_2$ is the Airy process.



. Snapshot of the PNG droplet
[double jumps at a position are just due to the pixel-based animation, in continuous space they do not occur].

. How was this result obtained?

The answer is first the model is extended to a multilayer version and <u>for this particular geometry</u> (i.e., for this choice of poisson points density) it turns out that it is equivalent to the non-intersecting condition.
So, one can then use the methods we learned so far.

9.3) Multilayer PNG droplet and RSK construction.

. The extension to multilayer PNG is as follows.
Instead of a single height function $h(x,t)$, we have a collection of lines $\{h_\ell(x,t)\}_{\ell\leq 0}$, where $h_0(x,t) \equiv h(x,t)$ and $h_\ell(x,t)$ evolves as follows:

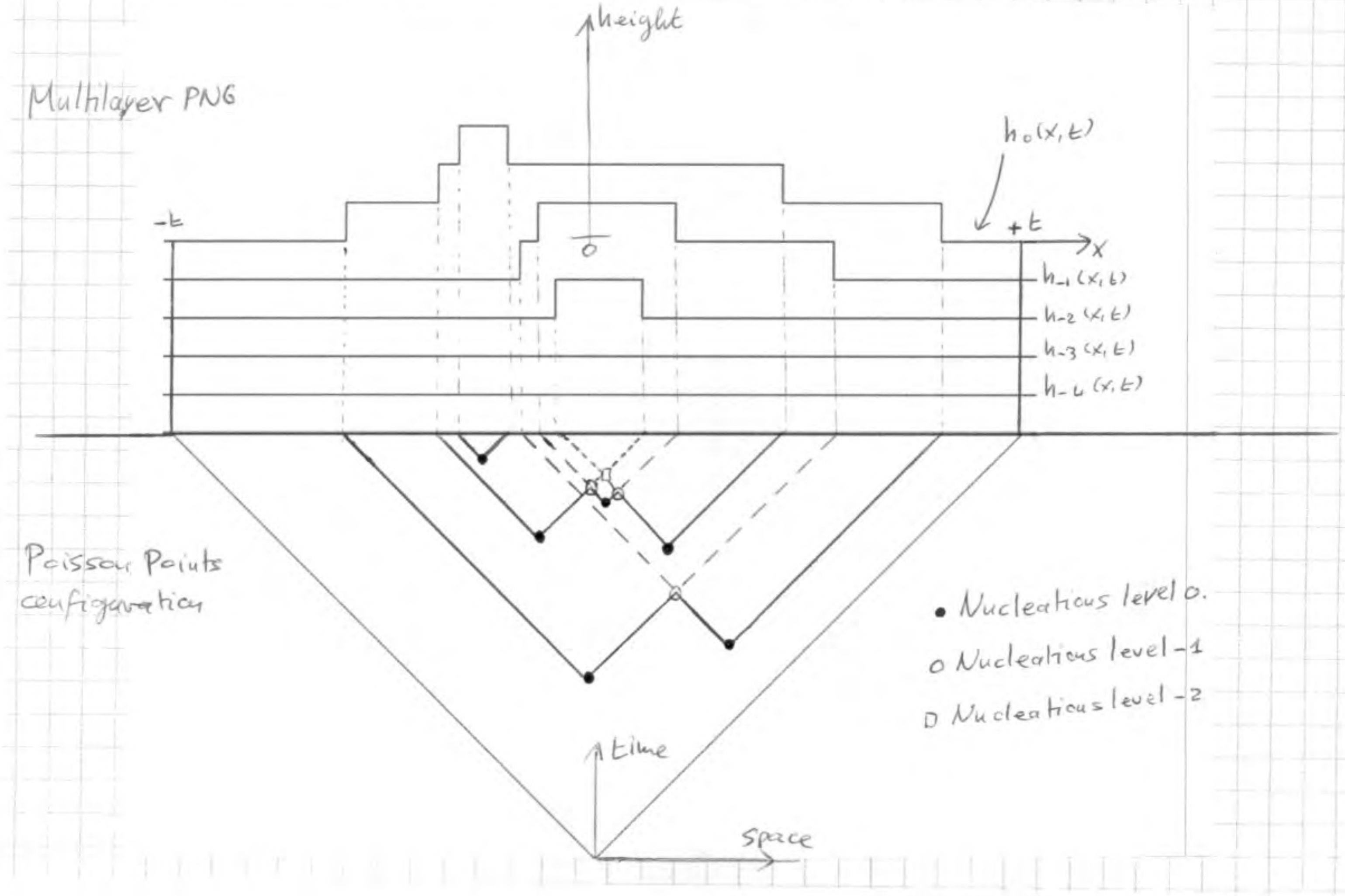
. Initial conditions: . $h_\ell(x,0) = \ell$, $\ell = 0, -1, -2, \dots$

. Evolution: . $h_0(x,t)$ evolves as PNG,

. $h_\ell(x,t)$ has the same deterministic PNG evolution, but the nucleations of level ℓ occurs in space-time when there is a merging at level $\ell+1$.

. The important property is that the set of lines, by construction: . are non-intersecting: $h_\ell < h_{\ell+1}$.

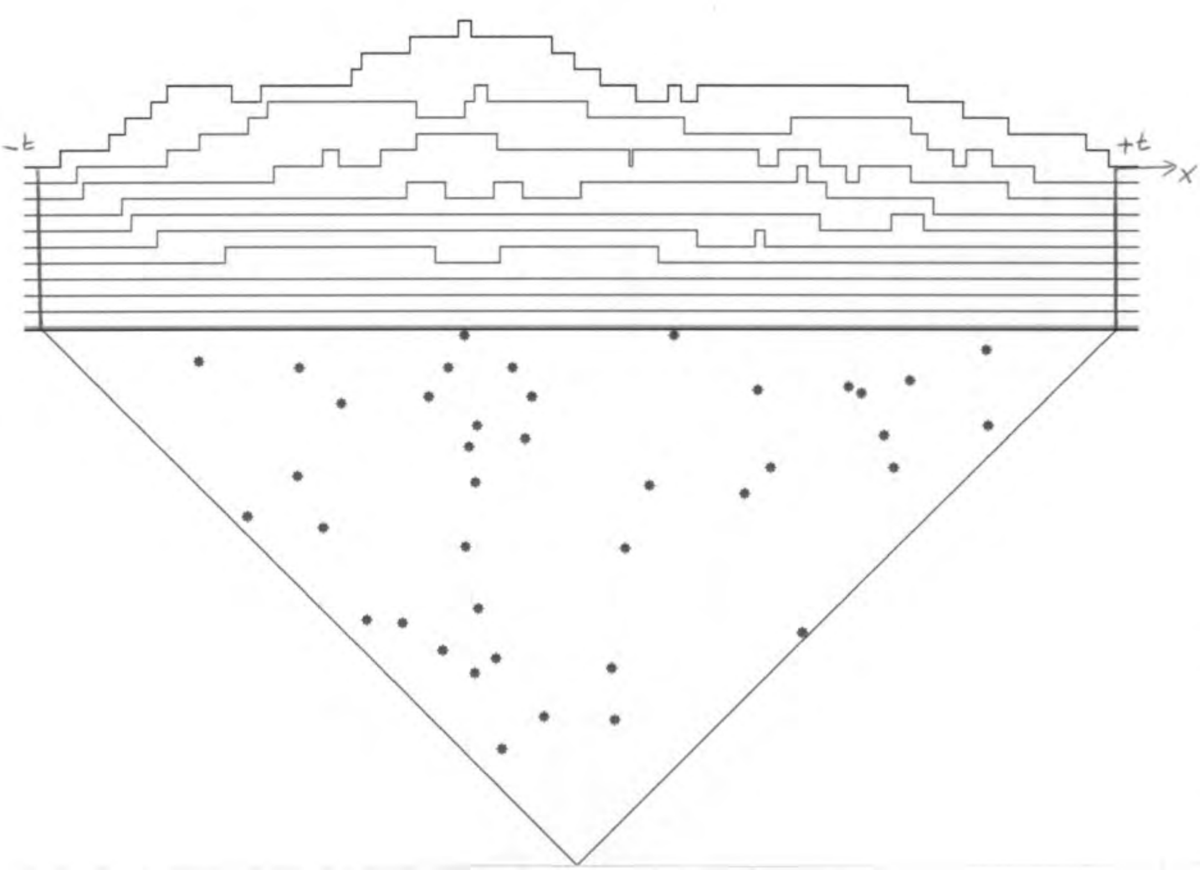
. contains the same information as the Poisson points (the space-time locations of the nucleations).



- To construct the multilayer from the nucleation points one can do the following geometric construction (which is equivalent to the RSK algorithm on permutation, therefore it was given the name RSK construction; RSK = Robinson, Schensted, Knuth).

- One draws the forward light cones of the nucleations until they intersect. These are the space-time positions of the up and down steps of h_0.
- Then, iteratively, the intersection points for level $\ell+1$ becomes nucleations for level ℓ.
- The multilayer PNG at some given time t, is then easily constructed by looking at the positions of the light-cones at time t. An example is shown in the previous picture (page ⑥).
- <u>Remark</u>: From the multilayer, it is easy to recover the positions of the nucleations, by doing backwards in time the construction starting from the lowest "excited" line.

<u>. A larger snapshot of PNG multilayer.</u>



. In their paper "Scale invariance of the PNG droplet and the Airy process", Prähofer and Spohn prove the following.

. Denote by $-t < Y^+_{1,e} < Y^+_{2,e} < \dots < Y^+_{n_e,e} < t$ the positions of the up-jumps of h_e and by $-t < Y^-_{1,e} < Y^-_{2,e} < \dots < Y^-_{n_e,e} < t$ the positions of the down-jumps.

. Set $\vec{n} = (n_0, n_{-1}, n_{-2}, \dots)$ and $|\vec{n}| = \sum_{e \leq 0} n_e$.

. By the RSK construction, $|\vec{n}| = \#$ Poisson points. Since we use density two, $\mathbb{P}(|\vec{n}| = k) = e^{-a_t} \frac{a_t^k}{k!}$, with $a_t = 2 \cdot t^2$, so $|\vec{n}|$ is a.s. finite.

. Denote by $\Gamma_t(\vec{n})$ the set of all step configurations $((Y^+_{j,e}, Y^-_{j,e})_{1 \leq j \leq n_e})_{e \leq 0}$ resulting from an admissible line configuration $(h_e(x,t))_{e \leq 0} \in \Lambda_t$ ($\Lambda_t \equiv$ set of line configurations). $\Gamma_t(0) \cong \phi$ and $\Gamma_t(\vec{n})$ in naturally embedded in $[-t,t]^{2|\vec{n}|}$. Then, $\Gamma_t = \bigcup_{|\vec{n}| < \infty} \Gamma_t(\vec{n})$.

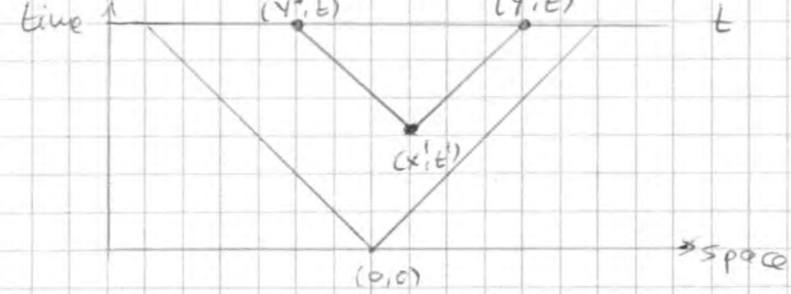
. By the RSK construction, we have a <u>bijective map</u> $S : \Lambda_t \to \Gamma_t$

<u>Theorem:</u> Let w_t be the uniform measure on Γ_t, i.e., $w_t(\Gamma_t(0)) = 1$, and $w_t|_{\Gamma_t(\vec{n})}$ is the $2|\vec{n}|$-dimensional Lebesgue measure on $\Gamma_t(\vec{n})$.

Then, $\int w_t \equiv Z(t) = \exp(2 \cdot t^2)$ and $\mu_t = \frac{w_t}{Z(t)}$ is a proba. measure on Γ_t. If the height functions $\{h_e\}_{e \leq 0}$ evolves by the RSK dynamics, then μ_t is the joint distribution of $\{h_e(x,t), x \in \mathbb{R}, e \leq 0\}$ under the map S.

. To see that this theorem holds, we have just to see that the measure (on the step positions) inherited by the Poisson points measure on $\{(x', t'), 0 \leq t' \leq t, |x'| \leq t'\}$ is μ_t. For this is enough to see that we have the right measure for every N, the number of Poisson points.

. To be precise, to a Poisson point at (x',t'), it corresponds an up-step at $Y^+ = x' - (t-t')$ and a down-step at $Y^- = x' + (t-t')$.



. To a configuration of N Poisson points it corresponds a set of up- and down-jumps $\{(Y_i^+, Y_i^-)_{i=1,\dots,N}\}$ such that

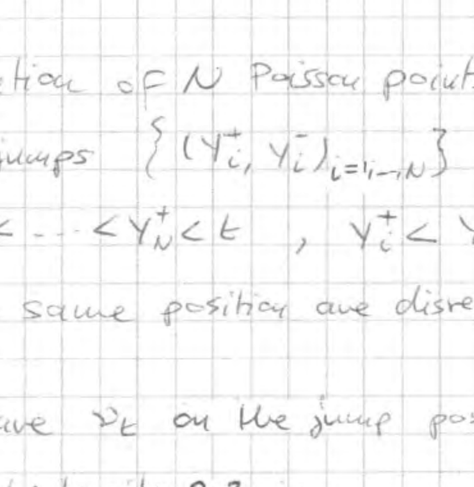
$$-t < Y_1^+ < \dots < Y_N^+ < t \quad , \quad Y_i^+ < Y_i^- , \; i = 1,\dots,N$$

(two jumps at the same position are disregarded, since they occur with probability 0).

. Then, the measure ν_t on the jump positions induced by the Poisson process with intensity $\rho=2$ is:

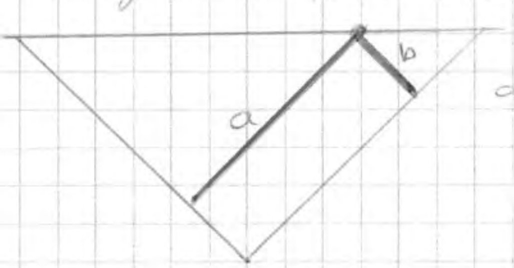
$$. \; \nu_t(\text{no jumps}) = e^{-2t^2}$$

$$. \; \nu_t\big|_{2N\,\text{jumps}} = e^{-2t^2} \cdot dY_1^+ \dots dY_N^+ \, dY_1^- \dots dY_N^-$$



$\Rightarrow$ On the line time $= t$, by the geometric factor $\sqrt{2}$ (dilatation) we have a Lebesgue measure time $\sqrt{\frac{\rho}{2}} = 1$ with $\rho = 2$.

. The simple measure on the step positions is due to the particular geometry, since at every point the length of the backwards light cone intersection with the forward light cone from $(0,0)$ is constant $= t\sqrt{2}$, so is the step intensity!



$a + b = t\sqrt{2}$.