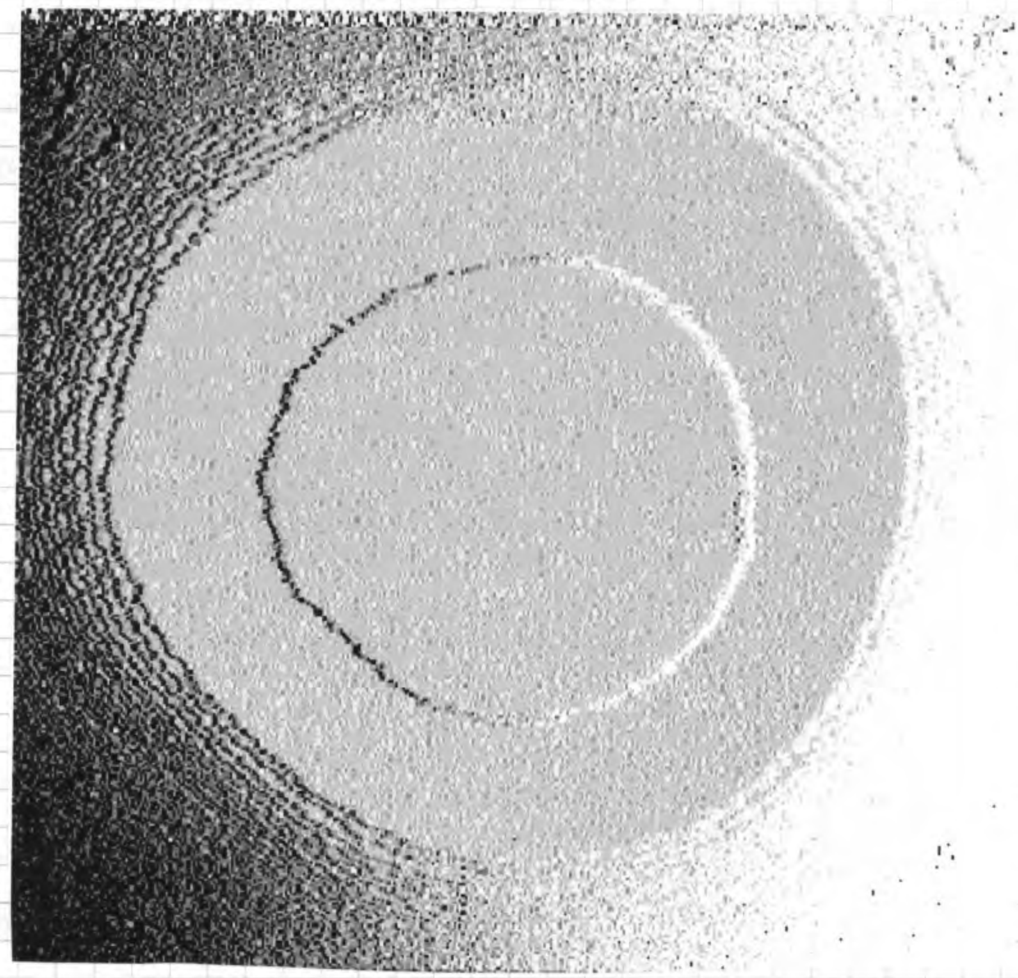
8) Application to the "3D Ising corner at $T=0$".

. The model considered is a simplified model which describes a crystal at low temperature (with short-range interactions).

. A real image of a crystal at low (i.e., much below the melting and the roughening temperatures) is taken by using an electronic microscope:



. Observations: One sees that there is a facet (with an extra island inside) and then a rounded part.

. Goal: Describe the interface between the facet and the rounded part.

. We consider the following simplified model to answer to the question.

8.1) The model.

. At each $x \in \mathbb{Z}^d$, one has an occupation variable $n_x \in \{0, 1\}$.

. Let N_a be the total number of atoms forming the crystal.

. The interaction is nearest-neighbor with strength $-J$, for some $J > 0$, namely the Hamiltonian is:

$$H(\{n\}) = -J \cdot \sum_{|x-y|=1} n_x \cdot n_y .$$

. This is a ferromagnetic Ising interaction if we replace $n_x = \frac{1+\sigma_x}{2}$, σ_x the spin at site x.

. We want to consider the equilibrium (thermal) situation, therefore the appropriate measure is the Gibbs measure with temperature T:

$$\mu(\{n\}) = e^{-H(\{n\})/k_B T} .$$

. Of course, as stated above, μ is an infinite measure (by translation-invariance). Thus what one really have to consider are configurations up to translation, i.e., consider the equivalence classes of configurations.

. To model what happens at low temperature, we look the limiting case $T = 0$. How to extend the results in a mathematically rigorous way to $T > 0$ is an open problem.

. At $T = 0$, only the configurations which minimize the energy are allowed. Here the number of atoms is fixed, thus the minimization of the energy $\equiv$ minimization of the surface area.

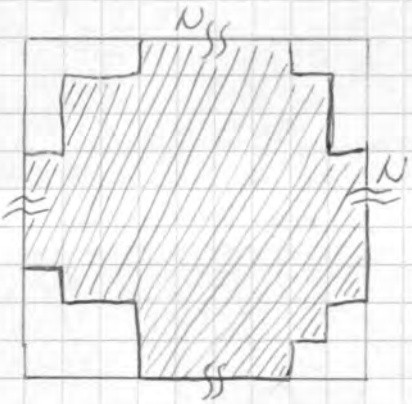
8.1.1) $d=2$ case.

- Although our goal is $d=3$, we start with $d=2$ to understand what happens.

- If $N_a = N^2$ for some $N \in \{1,2,3,\dots\}$, then there is a unique equilibrium configuration (class of).

- If, however, $N_a = N^2 - V$, with $0 < V < N-1$, then the minimal energy configurations are non-unique: we need to take away V atoms from a perfect cube by keeping the surface area unchanged.

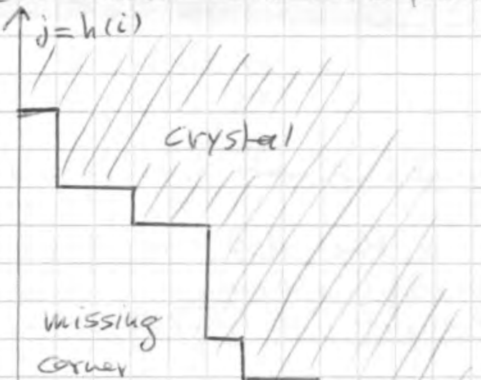
$\Downarrow$



- No overhangs!

- The further simplification can be made if $V \ll N^{1/2}$, so that we can consider the four corners independently.

Therefore, now we consider V to be the number of atoms missing from a single corner and take first $N \to \infty$ (i.e., $N_a \to \infty$).



- No overhangs

$\Downarrow$

- Crystal border described by a height function h: $h(i) \geq 0$, $i \geq 0$ and decreasing: $h(i) \geq h(i+1)$ s.t.

$$\sum_{i \geq 0} h(i) = V.$$

Question: What is the limit form and the fluctuations in the limit $V \to \infty$?

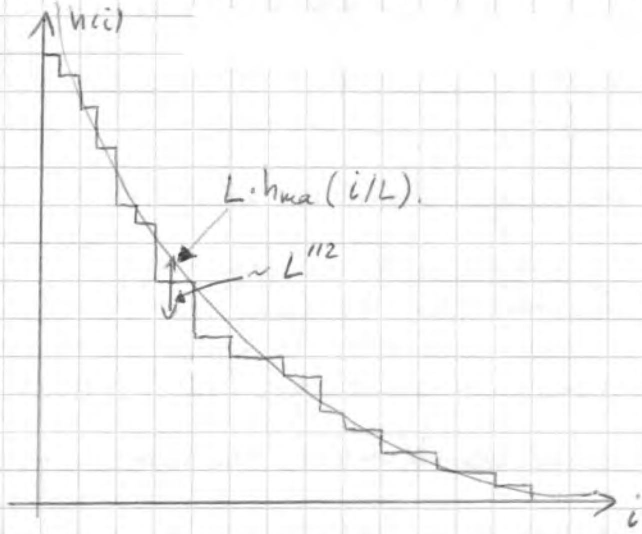
. Answer: Take $V = [L^2]$, so that L describes the linear scale in atomic units.

ⓐ Limit form: Macroscopically we have a deterministic shape: $\lim_{L\to\infty} \frac{1}{L} h([L\cdot u]) = h_{ma}(u)$, $u>0$

almost surely ($\exists$ large deviations estimates); with

$$e^{-h_{ma}(u)} + e^{-u} = 1.$$

ⓑ Fluctuations: Gaussian fluctuations on $L^{1/2}$ scale.

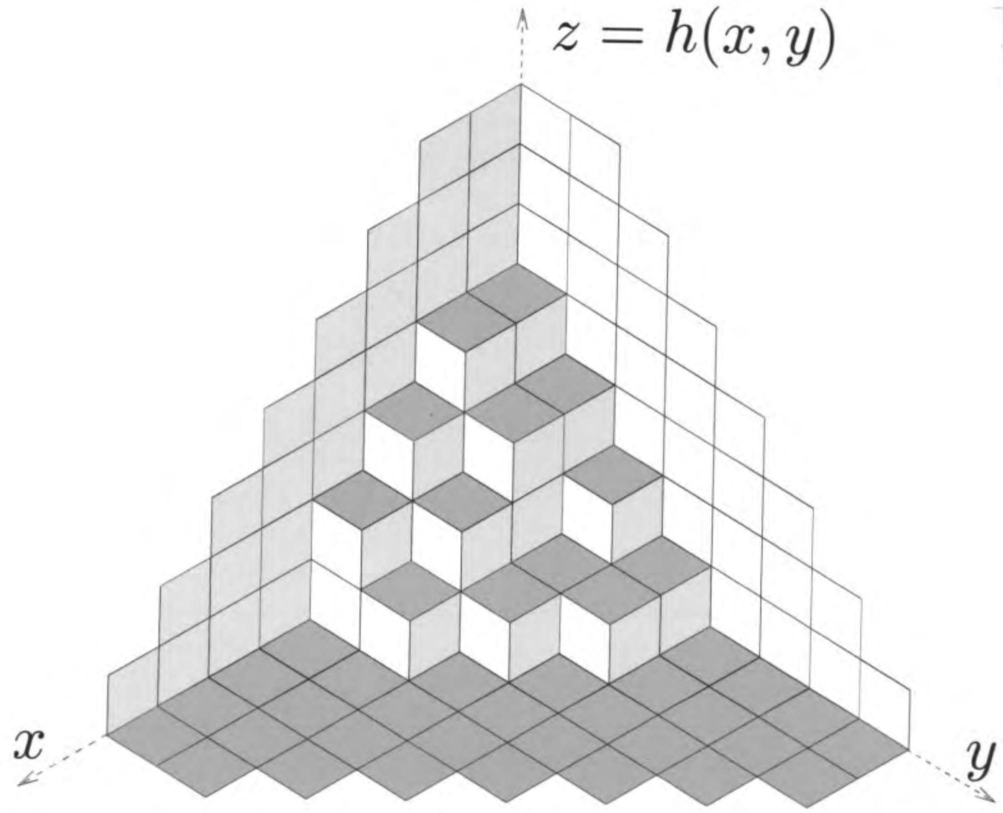


. 8.1.2) d=3 case.

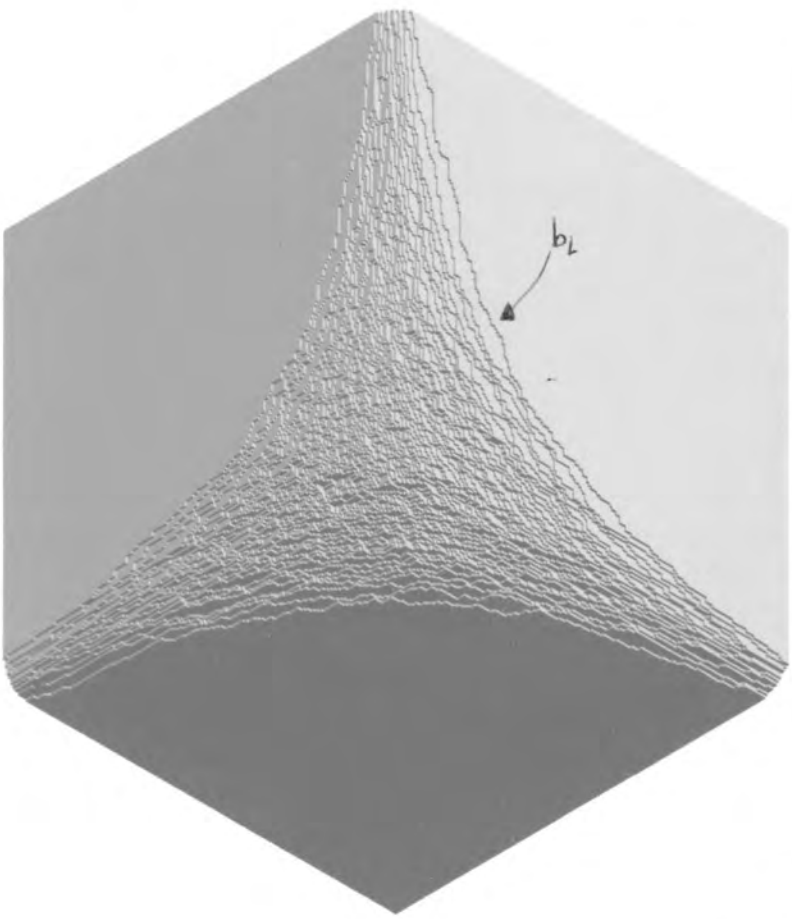
. The same arguments used in the d=2 case carry over in d=3. This time we have a height function, decreasing in both variables:

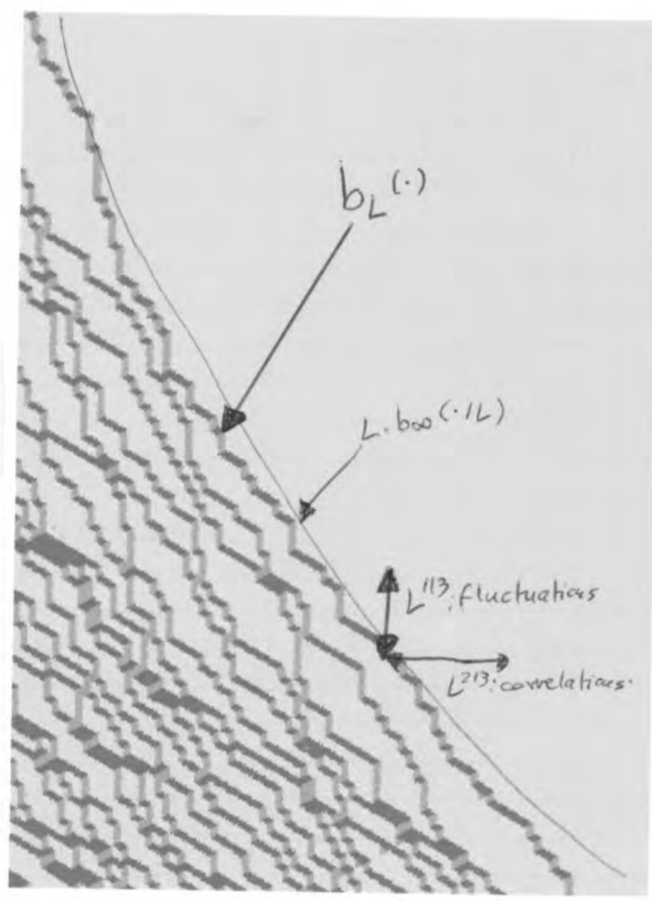
$$(x,y) \longmapsto z = h(x,y) \quad \text{with} \quad \begin{cases} h(x,y) \geq 0 & , x,y \geq 0 \\ h(x,y) \geq h(x,y+1) \\ h(x,y) \geq h(x+1,y) \end{cases}$$

with $V = \sum_{x,y\geq 0} h(x,y)$ fixed.



. Observable: Facet border $b_L(y) \equiv h(0,y)$.





(a) Limit shape: Let $V = [L^3]$. Then Cerf and Kenyon [2001] prove that

$\exists$ deterministic (non-random) limit shape h_∞:

$$\lim_{L\to\infty} \frac{1}{L} \cdot h([Lu],[Lv]) = h_\infty(u,v) \quad , \text{ a.s.}$$

In particular, the limit shape of the border is:

$$b_\infty(\tau) \doteq \lim_{L\to\infty} \frac{1}{L} b_L([\tau L]) = -2 \cdot (1 - e^{-\tau/2}).$$

$\Rightarrow$ For the fluctuations we focus, for $\tau > 0$ fixed, around

$$\begin{cases} x = 0, \\ y = \tau L, \\ z = b_\infty(\tau) \cdot L. \end{cases}$$

(b) Fluctuations: results [Ferrari, Spohn 2003]

(1) . Fluctuations live on a $L^{1/3}$ scale.

(2) . Spatial correlations live on a $L^{2/3}$ scale.

(3) . The limit process is the Airy process, $\mathcal{A}$.

. Rescaling of the edge: $b_L^{edge}(s) = \dfrac{b_L([\tau L + s L^{2/3}]) - L \cdot b_\infty(\tau + s \cdot L^{-1/3})}{L^{1/3}}$.

. Then, $\lim_{L\to\infty} b_L^{edge}(s) = \mathcal{A}(s \cdot \frac{\varkappa}{2}) \cdot \varkappa^{-1}$, $\varkappa = b''_\infty(\tau)$, in the sense of finite-dimensional distributions, i.e.,

for any $m \in \mathbb{N}$, $s_i, a_i \in \mathbb{R}$,

$$\lim_{L\to\infty} \mathbb{P}\left(\bigcap_{i=1}^m \{ b_L^{edge}(s_i) \leq a_i \} \right) = \mathbb{P}\left(\bigcap_{i=1}^m \{ \mathcal{A}(s_i \tfrac{\varkappa}{2}) \leq \tfrac{a_i}{\varkappa} \} \right).$$

8.2) How to get the $d=3$ results.

8.2.1) Canonical → Gran canonical ensemble.

- The fix missing volume V condition (canonical ensemble) is as usual difficult to implement since it is a global condition.

- Since we are interested in the $V \to \infty$ limit, it is more convenient to use the gran-canonical ensemble, where V instead of being fixed, is a random variable geometrically distributed with mean $\bar{V} \propto L^3$:

$$\underline{V(h) = [L^3]} \quad \xrightarrow[\text{by}]{\text{replaced}} \quad \underline{\exp\left(-\frac{V(h)}{L}\right) \doteq q^{V(h)}} \text{ with } q = e^{-\frac{\lambda}{L}}$$

with λ chosen s.t. $\bar{V} \equiv \mathbb{E}(V(h)) = L^3$.

Q: Is this step justified?

A: It depends on the observable one is interested in. For example, if one is interested in the fluctuations of the volume, it is obviously not true that the two descriptions agree in the $L \to \infty$ (thermodynamic) limit.

- We, however, consider an observable (the facet border) which fluctuates as $L^{1/3}$ in the canonical ensemble and in the gran-canonical one will need to average it over volumes around $\bar{V} \sim L^3$ and fluctuation on the scale $\bar{V}^{1/2} \sim L^{3/2}$.

- So, let X_L our observable for fixed L, s.t. $\frac{X_L - \alpha \cdot L}{L^{1/3}} \xrightarrow{L\to\infty} \xi$.

- Then, in the gran-canonical ensemble, we have $X_L^{g.c.} = \mathbb{E}_L^{g.c.}(X_L)$.

So, since $X_L \cong \alpha \cdot L + L^{1/3} \cdot \xi = \alpha \cdot \bar{V}^{1/3} + \xi \cdot \bar{V}^{1/9}$,

$$X_L^{g.c.} \cong \alpha \cdot \bar{V}^{1/3} \cdot (1 + \bar{V}^{-1/2})^{1/3} + \xi \cdot \bar{V}^{1/9} (1 + \bar{V}^{-1/2})^{1/9}$$

$$\cong \alpha \cdot \bar{V}^{1/3} + \frac{1}{3} \cdot \alpha \cdot \bar{V}^{-1/6} + \xi \cdot \bar{V}^{1/9} = \alpha L + \xi L^{1/3} + \mathcal{O}(L^{-1/2}).$$

Therefore, we will also have $\frac{X_L^{g.c.} - \alpha L}{L^{1/3}} \xrightarrow{L\to\infty} \xi$.

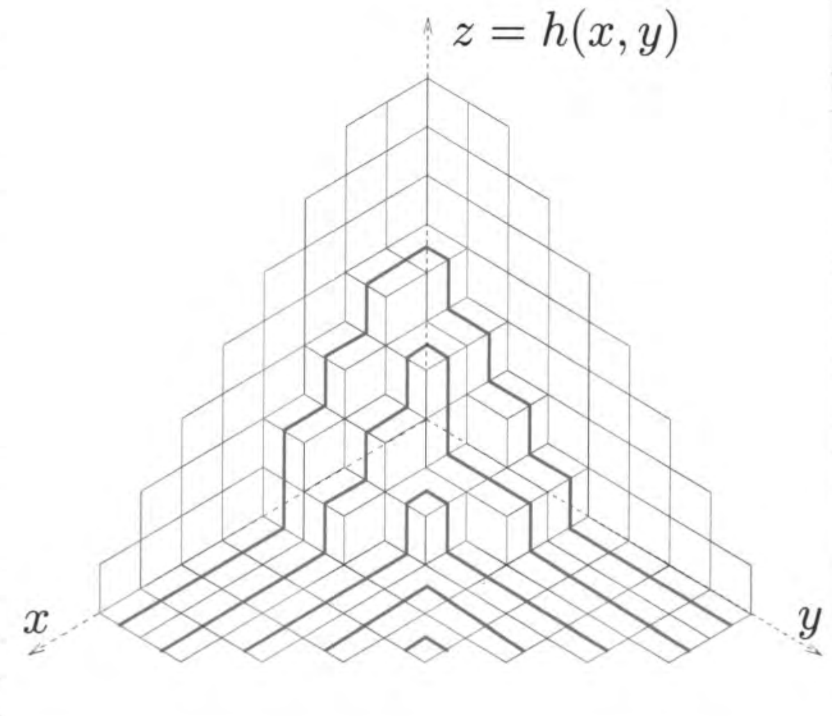
- Therefore, the fluctuations of the volume do not have any effect in the $L \to \infty$ limit to the fluctuations of our observable, and the replacement of the canonical ensemble by the grand-canonical one is fine.

Rem.: If we would have an observable scaling as $Y = \bar{V} + \xi \cdot \bar{V}^{1/3}$, then the two ensembles would not be aequivalent, since in the grand-canonical ensemble one would have $Y^{g.c.} = \bar{V} + \xi \cdot \bar{V}^{1/3} + \underbrace{\mathcal{O}(\bar{V}^{1/2})}_{\text{Volume fluctuations} \gg \bar{V}^{1/3}}$.

8.2.2) Mapping to non-intersecting line ensembles.

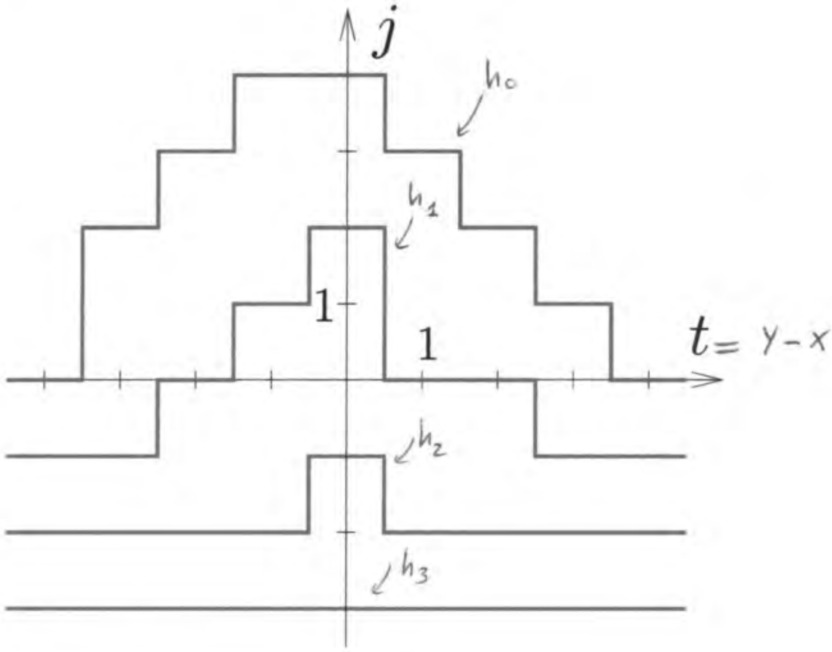
- Consider now the grand-canonical ensemble, i.e., give an extra weight $q^{V(h)}$ to the zero-temperature configurations.

- We first consider the gradient lines as in figure below:



- The next step is just a geometric transformation changing the directions of the x- and y-axis to $(-1,0)$ and $(1,0)$ respectively:

. We get, in the case of the above example:



. The <u>top line</u> is the border of the facet.

. We introduce names of the lines: $h_\ell(t)$, $\underline{t\in\mathbb{Z}}$, $\ell\geq 0$. They satisfy the non-intersecting conditions:

$$ⓐ \begin{cases} h_\ell \text{ is increasing for } t\leq 0, \text{ decreasing for } t\geq 0; \\ h_\ell(t)\leq h_\ell(t+1),\quad t<0,\quad h_\ell(t)\geq h_\ell(t+1),\ t\geq 0, \end{cases}$$

have limits: ⓑ $\lim_{t\to\pm\infty} h_\ell(t) = -\ell$,

and are non-∩: ⓒ $h_{\ell+1}(t) < h_\ell(t-1)$, $t\leq 0$; $h_{\ell+1}(t) < h_\ell(t+1)$, $t\geq 0$.

<u>. Measure on lines: Local!</u>

. In the above picture, we already extended the height functions $h_\ell(t)$ from $t\in\mathbb{Z}$ to $t\in\mathbb{R}$, with jumps at $\mathbb{Z}+1/2$.

. For a given line h_ℓ, let $t_{\ell,1}<\dots<t_{\ell,k(\ell)}<0$ be the left jump times of height $s_{\ell,1},\dots,s_{\ell,k(\ell)}$, and $0<t_{\ell,k(\ell)+1}<\dots<t_{\ell,k(\ell)+n(\ell)}$ be the right jump times of height $-s_{\ell,k(\ell)+1},\dots,-s_{\ell,k(\ell)+n(\ell)}$. Then,

$$V(h) = \sum_{\ell\geq 0}\sum_{j=1}^{k(\ell)+n(\ell)} s_{\ell,j}\cdot|t_{\ell,j}|$$

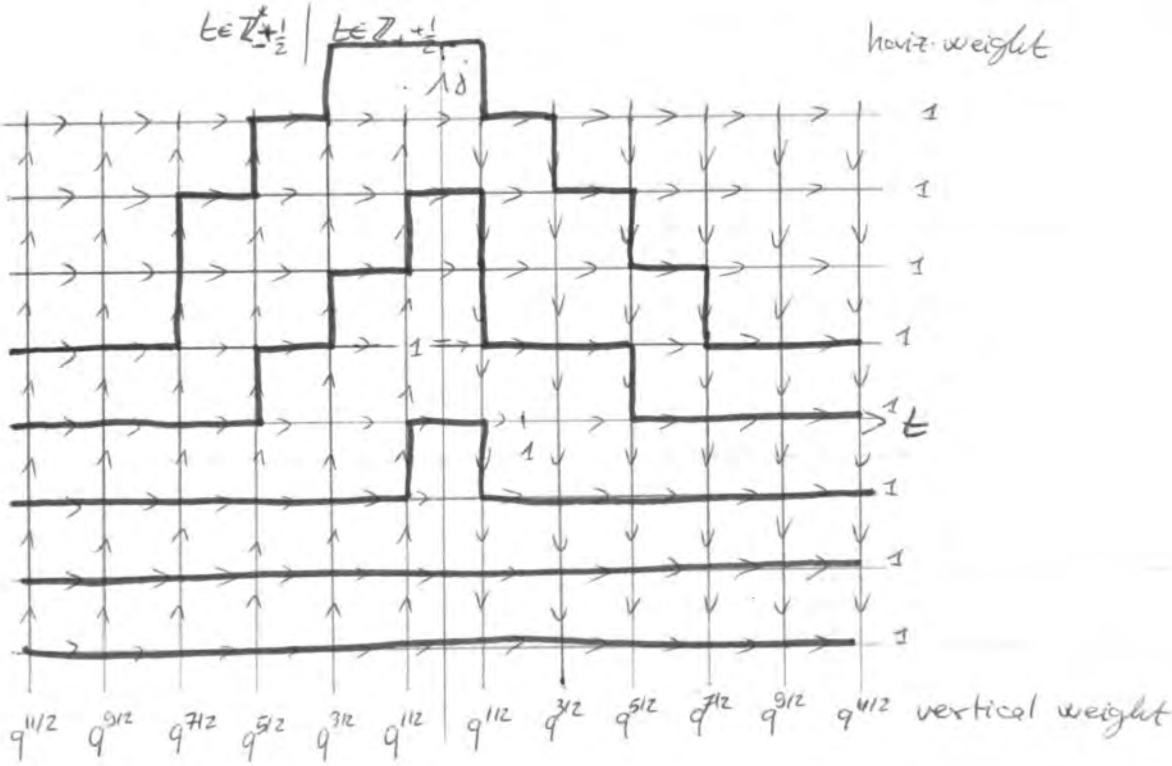
. This means that the gran-canonical weight $q^{V(h)}$ can be encoded in the jumps of the line ensembles: in this description it is a local weight!

. 8.2.3) Non-intersecting line ensemble and LGV graph.

. In our original work ("Step fluctuations for a faceted crystal" in J.S.P.) at this point we used the fermionic approach with notations usual for physicists but not well known by mathematicians. Here I present it in another way, by associating an LGV graph and then applying the theorem we learned.

. LGV graph: $\rightarrow$: weight 1

$\underbrace{\uparrow}_{t\in\mathbb{Z}_-+\frac{1}{2}}$, $\underbrace{\downarrow}_{t\in\mathbb{Z}_++\frac{1}{2}}$: weight $\underline{\underline{q_t = q^{|t|}}}$, with $|t|$ = distance to the $t=0$ axis.



. This is the LGV graph with the weight indicated at the boundaries. It is inhomogeneous in the t-variable.

. We plotted the lines of the above example too.

- To apply the LGV theorem, we need to start with some fixed initial and final positions and with a finite number of particles, N.

- Starting points : $\{ Y_\ell = (-M, -\ell) \}_{\ell \geq 0}^{N-1}$,
- Final points : $\{ X_\ell = (M, -\ell) \}_{\ell \geq 0}^{N-1}$.

- Let us now consider m values of t in $\mathbb{Z}$, say $-M < t_1 < t_2 < \dots < t_m < M$.
- The associate point process (extended)

$$\eta(t, j) = \begin{cases} 1, & \text{a line crosses } (t, j), \\ 0, & \text{no lines cross } (t, j), \end{cases}$$

is, by the LGV theorem, determinantal [see Lectures, sect 7.5].

- In fact, the measures on $\mathbb{Z} \times (t_1, \dots, t_m)$ writes

$$(*) \quad \det\Big(T((-M, -i) \to (t_1, h_j(t_1))) \Big)_{0 \leq i,j \leq N-1} \cdot \prod_{k=1}^{m-1} \det\Big(T((t_k, h_i(t_k)) \to (t_{k+1}, h_j(t_{k+1}))) \Big)_{0 \leq i,j \leq N-1} \cdot \det\Big(T((t_m, h_i(t_m)) \to (M, -j)) \Big)_{0 \leq i,j \leq N-1} .$$

where T is the transition probability of a single line on the LGV graph.

- Then, the measure $(*)$ is of the form so that Theorem in Sect. 7.1, page 4, can be applied.

- Remark : - Below we will write explicitly the transition probability. Since $q < 1$, the $M \to \infty$ limit is straightforward. A little bit more care will be needed to control the $N \to \infty$ limit. We will explain what happens without entering in the technical details of the $N \to \infty$ limit (which can be found in the references we will give).

Integral representation for the transition probability.

. It is convenient to use the Fourier representation or its contour integral analogue since on the vertical axis we have translation-invariance.

Case $t<0$: $T((t,x_1)\to(t+1,x_2)) = q_{t+\frac12}^{x_2-x_1}\cdot \mathbb{1}_{[x_2\geq x_1]})$ can be rewritten as

$$= \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{x_2-x_1+1}}\cdot\frac{1}{(1-q_{t+\frac12}\cdot w)},$$

where Γ_0 is any simple loop encircling $w=0$ and no other poles of the integrand and anticlockwise oriented.

. To verify it, just use

$$\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x+1}}\cdot\frac{1}{1-qw} \underset{|w|<\frac1q}{=} \sum_{n\geq 0}\underbrace{\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x+1}}\cdot q^n w^n}_{=\delta_{n,x}\cdot q^x}.$$

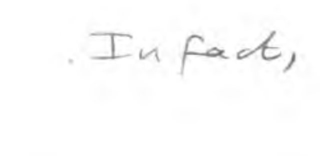
. Two-steps for $t<-1$:

$$T((t,x_1)\to(t+2,x_2)) = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_2-x_1+1}}\cdot\frac{1}{(1-q_{t+\frac12}w)(1-q_{t+\frac32}w)}.$$

. In fact, $T((t,x_1)\to(t+1,x_2)) = \sum_{y\geq x_1} T((t,x_1)\to(t+1,y))\cdot T((t+1,y)\to(t+2,x_2)) =$

$$=\sum_{y\geq x_1}\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}dz\oint_{\Gamma_0}dw\,\frac{1}{w^{x_2-y+1}}\cdot\frac{1}{(1-q_{t+\frac32}w)}\cdot\frac{1}{z^{y-x_1+1}}\cdot\frac{1}{(1-q_{t+\frac12}z)}$$

$$\overset{|z|>|w|}{=} \underbrace{\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}dz\oint_{\Gamma_0}dw}_{|w|<|z|}\,\frac{1}{(1-q_{t+\frac32}w)}\cdot\frac{1}{(1-q_{t+\frac12}z)}\cdot\frac{1}{w^{x_2-x_1+1}}\cdot\underbrace{\sum_{y\geq x_1}\left(\frac{w}{z}\right)^{y-x_1}\cdot\frac{1}{z}}_{=\frac{1}{z-w}}$$



$$\underset{\substack{\text{pole at } z=0 \\ \text{vanished, but now} \\ \text{simple pole at } z=w \Rightarrow \text{residue}}}{=\!=\!=} \frac{1}{2\pi i}\oint_{\Gamma_0}dw\,\frac{1}{(1-q_{t+\frac32}w)}\cdot\frac{1}{(1-q_{t+\frac12}w)}\cdot\frac{1}{w^{x_2-x_1+1}}\cdot \quad \#$$

. General case is analogue, as well the case $t\geq 0$ by symmetry.

. For $-\infty\leq t_1 < t_2 \leq 0$:

$$T\big((t_1,x_1)\to(t_2,x_2)\big)=\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_2-x_1+1}}\prod_{t=t_1}^{t_2-1}\frac{1}{(1-q_{t+\frac12}w)}$$

. For $0\leq t_1 < t_2 \leq \infty$:

$$T\big((t_1,x_1)\to(t_2,x_2)\big)=\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_1-x_2+1}}\prod_{t=t_1}^{t_2-1}\frac{1}{(1-q_{t+\frac12}w)}$$

. From these formulas is easy to see that $M\to\infty$ limit is immediate.

. In the case of N particles, define $A_N=[A_N]_{0\leq i,j\leq N-1}$ given by

$$[A_N]_{i,j}=T\big((-\infty,i)\to(\infty,j)\big)=\sum_{z\in\mathbb{Z}}T\big((-\infty,i)\to(0,z)\big)\cdot T\big((0,z)\to(\infty,j)\big).$$

Then, by the theorem of Sect. 7.1, we get that the kernel of our determinantal point process is given by:

$$K_N\big((t_1,x_1);(t_2,x_2)\big)=-T\big((t_1,x_1)\to(t_2,x_2)\big)\cdot\mathbb{1}_{[t_1<t_2]}+\sum_{i,j=-N+1}^{0}T\big((t_2,x_2)\to(\infty,i)\big)\cdot[A_N^{-1}]_{i,j}\cdot T\big((-\infty,j)\to(t_1,x_1)\big)$$

. The $N\to\infty$ limit: A_N is a $N\times N$ matrix, a minor of a Toeplitz matrix, i.e., $[A_N]_{i,j}$ depends only on $j-i$. In our case is even symmetric. The inverse of infinite-dimensional* Toeplitz matrices are "easy" to compute. Moreover, the transitions T in the kernel K_N depending on i or j goes to zero exponentially fast as $i,j\to-\infty$. This is the

* $i,j\leq 0$ instead of $i,j\in[-N+1,0]$.

consequence that the probability that the line l_e is not straight goes to ~~zero~~ exponentially fast (in ℓ) as $\ell\to\infty$.

. In our case, one can also get that the entries of A_N go rapidly (exponentially) to zero, in the distance to the diagonal: $|[A_N]_{i,j}| \leq C_1 \cdot e^{-C_2 \cdot |j-i|}$ for some $C_1, C_2 > 0$.

- The matrix A_N is then "almost" like a band matrix and the $N\to\infty$ limit is unproblematic. The details of the justification are like in Section 2 of "arXiv math/0006097" by Eric Rains ("Correlation functions for symmetrized increasing subsequences").

. Denote by A_∞ the half-infinite matrix, limit of A_N as $N\to\infty$.

. Inverse of the matrix for $N=\infty$.

. Denote by $(T_+)_{x,y} = T((-\infty,x)\to(0,y))$ and $(T_-)_{x,y} = T((0,x)\to(0,y))$

Then, let A be the infinite matrix, $A=\{A_{xy}\}_{x,y\in\mathbb{Z}}$, which is the product of T_+ and T_-: $\quad A = T_+\cdot T_-$.

- First remark that $T_+\cdot T_- = T_-\cdot T_+$ since $A=A^t$ is a Toeplitz matrix.

. Moreover, $[A_\infty]_{x,y} = [A]_{x,y}$, $x,y\leq 0$.

. The matrices $T_\pm$ are triangular, $T_+ = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$, $T_- = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$

. Denote by $P_+ =$ projection on $\{1,2,\ldots\}$,
$P_- =$ " " $\{\ldots,-1,0\}$ and decompose $\ell^2 = P_-\ell^2 \oplus P_+\ell^2$.

Then, the bloc representations of $T_\pm$ are: $T_+ = \begin{pmatrix} a' & c' \\ 0 & b' \end{pmatrix}$, $T_- = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$,

so that $A = \begin{pmatrix} aa' & * \\ * & * \end{pmatrix}$. <u>We need to compute $(aa')^{-1} = (a')^{-1}\cdot(a)^{-1}$.</u>

<u>. a) Inverse of a:</u> $\quad a_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{x-y+1}\,g(w)}$, $\quad g(w) = \prod_{t=0}^{\infty}(1-q_{t+\frac12}\, w)$

Then, $\quad a^{-1}_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dz}{z^{x-y+1}}\cdot g(z)$.

. In fact: $\sum_{y\le 0} a_{x,y}\, a^{-1}_{y,x'} = \sum_{y\le x} a_{x,y}\, a^{-1}_{y,x'}$

$$= \sum_{y\le x} \frac{1}{(2\pi i)^2}\oint_{\Gamma_0} \frac{dw}{w^{x-y+1}} \oint_{\Gamma_0}\frac{dz}{z^{y-x'+1}}\cdot \frac{g(z)}{g(w)}$$

$$= \frac{1}{(2\pi i)^2} \underbrace{\oint_{\Gamma_0}\frac{dw}{g(w)\,w}\oint_{\Gamma_0}\frac{dz\; g(z)}{z^{x-x'+1}}}_{|z|<|w|} \cdot \underbrace{\sum_{y\le x}\frac{z^{x-y}}{w^{x-y}}}_{=\frac{w}{w-z}}$$

$$= \underbrace{\underbrace{\oint dz \oint dw}_{|z|>|w|}}_{=0,\ \text{since no poles at } w=0} + \int (\text{Residue at } w=z)\, dz = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{x-x'+1}} = \delta_{x,x'}.$$

. Similarly, one verifies $\sum_{y\le 0} a^{-1}_{x,y}\cdot a_{y,x'} = \delta_{x,x'}$.

<u>b) Inverse of a':</u> $\quad a'_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{y-x+1}}\cdot\frac{1}{\tilde g(w)}$, with $\tilde g(w)=\prod_{t=-\infty}^{-1}(1-q_{t+\frac12}w)$

and $(a')^{-1}_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{y-x+1}}\cdot\tilde g(z)$.

<u>. Finite size Kernel.</u>

. Therefore, the Kernel of our determinantal point process is

$$K((t_1,x_1);(t_2,x_2)) = -T((t_1,x_1)\to(t_2,x_2))\,\mathbb{1}_{[t_1<t_2]}$$
$$+\sum_{x,y\le 0} T((t_2,x_2)\to(\infty,x))\sum_{\ell\le 0}(a')^{-1}_{x,\ell}(a^{-1})_{\ell,y}\cdot T((-\infty,y)\to(t_1,x_1)).$$

↳ replaced by $u\in\mathbb{Z}$, since $x\le 0$ and $(a')^{-1}_{x,u}=0$ if $u>x$.

. To our purpose we need only $t_1,t_2\ge 0$. In this case,

$$T((t_2,x_2)\to(\infty,x)) = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_2-x+1}}\cdot\prod_{t=t_2}^{\infty}\frac{1}{(1-q_{t+\frac12}w)}$$

and $$T((-\infty,y)\to(t_1,x_1)) = \sum_{v\in\mathbb{Z}} (a')_{y,v}\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{v-x_1+1}}\cdot\prod_{t=0}^{t_1-1}\frac{1}{(1-q_{t+\frac12}z)}\right)$$

$v \doteq y+x_1-\tilde v$

$$= \sum_{\tilde v\in\mathbb{Z}}\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_1-\tilde v+1}}\,\frac{1}{\tilde g(w)}\right)\cdot\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{y-\tilde v+1}}\cdot\prod_{t=0}^{t_1-1}\frac{1}{(1-q_{t+\frac12}z)}\right)$$

($\tilde v \le y$ is enough)

. Putting all together we get:

$$K((t_1,x_1),(t_2,x_2)) = -T((t_1,x_1)\to(t_2,x_2))\,\mathbb{1}_{[t_1<t_2]} + \overline{K}((t_1,x_1),(t_2,x_2)),$$

where
$$\overline{K}((t_1,x_1),(t_2,x_2)) = \sum_{\ell\le 0}\underbrace{\left(\sum_{x\le 0}\frac{1}{2\pi i}\oint_{\Gamma_c}\frac{dw}{w^{x_2-x+1}}\cdot\prod_{t=t_2}^{\infty}\frac{1}{(1-q_{t+\frac12}w)}\cdot\frac{1}{2\pi i}\oint_{\Gamma_c}\frac{dz}{z^{\ell-x+1}}\cdot\prod_{t=-\infty}^{-1}(1-q_{t+\frac12}z)\right)}_{\alpha}$$
$$\cdot\underbrace{\left(\sum_{y\le 0}\sum_{v(\le y)}\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_1-v+1}}\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\cdot\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{y-v+1}}\prod_{t=0}^{t_1-1}\frac{1}{(1-q_{t+\frac12}z)}\cdot\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{d\tilde z}{\tilde z^{\ell-y+1}}\prod_{t=0}^{\infty}(1-q_{t+\frac12}\tilde z)\right)}_{\beta}$$

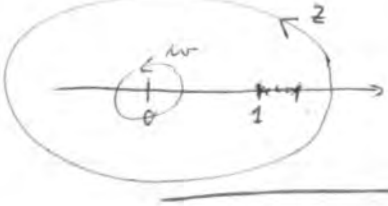
. First we do a simplification of β: Since $\ell\le 0$, we can replace the double sum by $\sum_{v\le 0}\sum_{y\ge v}$ (the restriction $y\le 0$ is not needed, the last term being exactly zero). Then we perform the sum over $y\ge v$ of the last two terms and obtain finally:

$$\boxed{\beta} = \sum_{v\le 0}\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_1-v+1}}\cdot\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\right)\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{\ell-v+1}}\cdot\prod_{t=t_1}^{\infty}(1-q_{t+\frac12}z)\right).$$

$$= \sum_{v\le \ell}\left(\quad '' \quad\right)\cdot\left(\quad '' \quad\right)$$

$$u \doteq \ell - v$$

$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_c}dw\oint_{\Gamma_0}dz}_{|z|>\frac{1}{|w|}}\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\cdot\prod_{t=t_1}^{\infty}(1-q_{t+\frac12}z)\cdot\underbrace{\sum_{u\ge 0}\frac{(wz)^{\ell}}{(wz)^{u}\cdot w^{x_1+1}\,z^{\ell+1}}}_{=\frac{1}{w^{x_1-\ell+1}(z-\frac{1}{w})}}$$



$$= \underbrace{\oint_{\Gamma_c}dw\oint_{\Gamma_0}dz}_{|z|<\frac{1}{|w|}} + \boxed{\frac{1}{2\pi i}\oint_{\Gamma_c}dw\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\prod_{t=t_1}^{\infty}\left(1-q_{t+\frac12}\cdot\frac{1}{w}\right)\cdot\frac{1}{w^{x_1-\ell+1}}}$$

$\longrightarrow 0$, since no pole at $z=0$ anymore

Residue at $z=\frac{1}{w}$

. Similarly,

$$(\alpha) = \frac{1}{2\pi i}\oint_{\Gamma_0} dw \prod_{t=t_2}^{\infty} \frac{1}{(1-q_{t+\frac{1}{2}} w)} \cdot \prod_{t=-\infty}^{-1} \left(1 - q_{t+\frac{1}{2}} \frac{1}{w}\right) \cdot \frac{1}{w^{x_2-e+1}}$$

. It is also possible to check that $\sum_{e\in\mathbb{Z}} (\alpha)(\beta) = T((t_1,x_1)\to(t_2,x_2))$.

. Therefore, the final formula for the Kernel writes:

$$K((t_1,x_1),(t_2,x_2)) = \begin{cases} \displaystyle\sum_{e\le 0} \left[\frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{\prod_{t=-\infty}^{-1}(1-q_{t+\frac{1}{2}}\frac{1}{w})}{\prod_{t=t_2}^{\infty}(1-q_{t+\frac{1}{2}}\cdot w)} \cdot \frac{1}{w^{x_2-e+1}}\right] \cdot \left[\frac{1}{2\pi i}\oint_{\Gamma_0} dz \frac{\prod_{t=t_1}^{\infty}(1-q_{t+\frac{1}{2}}\frac{1}{z})}{\prod_{t=-\infty}^{-1}(1-q_{t+\frac{1}{2}} z)} \cdot \frac{1}{z^{x_1-e+1}}\right], & \text{for } t_1 \geq t_2, \\ \displaystyle -\sum_{e>0} [\;''\;]\cdot[\;''\;] & \text{for } t_1 < t_2. \end{cases}$$

. Scaling limit

What remains to be done is the asymptotic analysis in the scaling limit:

$$\begin{cases} t_i = [\tau L + s_i L^{2/3}] \\ x_i = [b_\infty(\tau) L + b'_\infty(\tau) \cdot s_i L^{2/3} + \frac{1}{2} b''_\infty(\tau) \cdot s_i^2 L^{1/3} + r_i L^{1/3}], \end{cases}$$

i.e., the rescaled point process is

$$\eta_L^{edge}(r,s) \doteq L^{1/3} \cdot \eta_L(x,t).$$

Similarly, the rescaled kernel is:

$$K_L^{edge}((r_1,s_1),(r_2,s_2)) \underset{\substack{\text{up to}\\ \text{a conjugation}}}{\cong} L^{1/3} \cdot K((t_1,x_1),(t_2,x_2)).$$

. In "Step fluctuations for a faceted crystal", J. Stat. phys. 113. (2003), 1-46, we prove that $\lim_{L\to\infty} K_L^{edge} = K_{Airy}$. The convergence is such that, it follows the finite-dimensional distribution of the rescaled process to the Airy process.