Some properties of the Airy Process, $\mathcal{A}_2$.

. From the formula of the extended Kernel, it is immediate that the Airy Process is stationary.

. The one-point distribution is given by the $\beta=2$ Tracy–Widom distribution:

$$\mathbb{P}(\mathcal{A}_2(0) \leq s) = F_2(s).$$

. Covariance: $\quad \mathrm{Cov}(\mathcal{A}_2(0), \mathcal{A}_2(u)) = \begin{cases} \mathrm{Var}(\mathcal{A}_2(0)) - u + \mathcal{O}(u^2), & u \ll 1, \\ \frac{1}{u^2} + \mathcal{O}(u^{-4}), & u \gg 1. \end{cases}$

. The Airy process is not a Markov process.

7.4) Dyson's Brownian Motion.

. Consider matrices H in one of the GOE/GUE/GSE ensembles. Then, the independent parameter are:

$$\begin{cases} H_{ii}, & 1 \leq i \leq N \\ H_{ij}^{(\lambda)}, & 1 \leq i < j \leq N, \ 0 \leq \lambda \leq \beta - 1 \end{cases}$$

$\Rightarrow p = N + \frac{\beta}{2} N(N-1)$ independent real "entries", which we denote by H_μ, $\mu = 1, \dots, p$.

. Dyson's Brownian Motion on matrices consists in independent Ornstein-Uhlenbeck processes on H_μ (i.e., Brownian Motions in a quadratic potential) as follows:

$$(1) \quad \begin{cases} dH_\mu = -\alpha \cdot H_\mu dt + \sigma_\mu dB_\mu, \quad \sigma_\mu = \begin{cases} 1, & \mu = (i,i) \\ \frac{1}{2}, & \text{otherwise} \end{cases} \\ \text{where } dB_\mu \text{ are independent standard B.M.} \end{cases}$$

. Denote by $P(H_1, \ldots, H_p; t)$ the probability density at time t. Then, one can check that P satisfies the Smoluchowski equation:

$$(2) \qquad \frac{\partial P}{\partial t} = \sum_{\mu=1}^{P} \left[\frac{1}{2} \sigma_\mu \cdot \frac{\partial^2}{\partial H_\mu^2} P + \alpha \cdot \frac{\partial}{\partial H_\mu} (H_\mu P) \right]$$

. Moreover, the solution of (2) with initial condition $H(0)$ at $t=0$ is given by

$$(3) \quad \begin{cases} P(H,t) = \dfrac{\text{const}}{(1-q_t^2)^{P/2}} \cdot \exp\left[- \dfrac{\alpha \operatorname{Tr}\left((H - q_t H(0))^2 \right)}{(1-q_t^2)} \right] \\ \text{where} \quad q_t = \exp(-\alpha \cdot t) \end{cases}$$

. In particular, $P^{stat}(H) = \text{const.} \exp\left[-\alpha \operatorname{Tr}(H^2) \right]$ is the stationary solution of (3) [take $t \to \infty$ in (3)].

. <u>Question:</u> What happens to the eigenvalues when the matrices evolves according to (1)?

. <u>Answer:</u> The N eigenvalues, $\lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_N(t)$ satisfy the system of SDE:

$$(4) \quad \begin{cases} d\lambda_i = \left[-\alpha \lambda_i + \dfrac{\beta}{2} \displaystyle\sum_{j \neq i} \dfrac{1}{\lambda_j - \lambda_i} \right] dt + db_i, \quad i = 1, \ldots, N \\ \text{where } db_i \text{ are independent standard B.M.} \end{cases}$$

. Let us now determine (4). First of all, notice that the evolution (1) does not depends on the choice of basis. In fact, the solution (3) involves only a trace, which is representation-independent.

. Therefore we choose a representation such that at time t, $H(t)$ is diagonal and see what happens at $t+dt$.

. We have: $H(t)|\psi_i\rangle = \lambda_i |\psi_i\rangle$ with the eigenvalues a.s. all different. The problem can be formulate as follows:

at time $t+dt$, we have an Hamiltonian $H(t+dt) = H(t) + \delta H$ with $\delta H = \mathcal{O}(\sqrt{dt}) + \mathcal{O}(dt)$, in any case with δH small.

$\Rightarrow$ Goal : find the perturbation of the eigenvalues.

. Result: (see below)
$$\lambda_i(t+dt) \overset{\circledast}{=} \lambda_i(t) + \delta H_{ii}^{(0)} + \sum_{j\neq i}\sum_{\lambda=0}^{\beta-1} \frac{(\delta H_{ij}^{(\lambda)})^2}{\lambda_i - \lambda_j} + \mathcal{O}((\delta H)^3)$$

Since $\delta H_{ii}^{(0)} = -\alpha\cdot\lambda_i(t)dt + dB_{ii}$

and $(\delta H_{ij}^{(\lambda)})^2 = \frac{dt}{2}$ $\left(=\frac{1}{2}(dB_{ij}^{(\lambda)})^2\right)$.

. Therefore, $d\lambda_i = -\alpha\lambda_i dt + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}dt + dB_{ii}$.

. Now we just have to justify $\circledast$:

Lemma: Let H a $N\times N$ matrix, V another $N\times N$ matrix. Suppose $H|\psi_i\rangle = \lambda_i|\psi_i\rangle$ with λ_i all distincts $\Rightarrow \{|\psi_i\rangle\}_{i=1}^N$ form an o.n. basis.
Then $(H+\varepsilon V)|\psi_i'\rangle = \lambda_i'|\psi_i'\rangle$, with
$$\lambda_i' = \lambda_i + \varepsilon\langle\psi_i, V\psi_i\rangle + \varepsilon^2\cdot\sum_{j\neq i}\frac{|\langle\psi_i, V\psi_j\rangle|^2}{\lambda_i - \lambda_j} + \mathcal{O}(\varepsilon^3)$$

Proof.: We use $\{|\psi_i\rangle, i=1,\dots,N\}$ as basis and write
$$\begin{cases} |\psi_i'\rangle = |\psi_i\rangle + \varepsilon\sum_j A_i^{\,j}|\psi_j\rangle + \varepsilon^2\sum_j B_i^{\,j}|\psi_j\rangle + \mathcal{O}(\varepsilon^3), \text{ and} \\ \lambda_i' = \lambda_i + \varepsilon\cdot a_i + \varepsilon^2\cdot b_i + \mathcal{O}(\varepsilon^3)\end{cases}$$

$\Rightarrow (H+\varepsilon V)|\psi_i'\rangle = \lambda_i'|\psi_i'\rangle$ becomes, order by order;

. $\mathcal{O}(\varepsilon^0)$: $H|\psi_i\rangle = \lambda_i|\psi_i\rangle$: v.

. $\mathcal{O}(\varepsilon^1)$: $V\cdot|\psi_i\rangle = a_i|\psi_i\rangle + (\lambda_i - H)\sum_j A_i^{\,j}|\psi_j\rangle$,

multiplied by $\langle\psi_k| \Rightarrow \langle\psi_k, V\psi_i\rangle = \delta_{k,i}\,a_i + (\lambda_i - \lambda_k)\ A_i^{\,k}$

In particular, for $k=i \Rightarrow \underline{\langle \psi_i, V\psi_i\rangle = a_i}$

and for $k\neq i$: $\langle \psi_k, V\psi_i\rangle = (\lambda_i-\lambda_k)A_i^k$

. $\underline{\mathcal{O}(\varepsilon^2)}$: $V\sum_j A_i^j|\psi_j\rangle = b_i|\psi_i\rangle + a_i\sum_j A_i^j|\psi_j\rangle + (\lambda_i - H)\sum_j B_i^j|\psi_j\rangle$

multiplied by $\langle\psi_k|$

$$\Rightarrow \sum_j A_i^j\langle\psi_k, V\psi_j\rangle = b_i\cdot\delta_{k,i} + a_i\cdot A_i^k + (\lambda_i-\lambda_k)B_i^k$$

$$\Rightarrow \text{For } k=i: \quad b_i = \sum_j A_i^j\langle\psi_i, V\psi_j\rangle - a_i\cdot A_i^i$$

$$= \sum_{j\neq i} A_i^j\langle\psi_i, V\psi_j\rangle + \cancel{A_i^i\langle\psi_i, V\psi_i\rangle - \langle\psi_i, V\psi_i\rangle A_i^i}$$

$$= \sum_{j\neq i}\frac{\langle\psi_i, V\psi_j\rangle\cdot\langle\psi_j, V\psi_i\rangle}{\lambda_i-\lambda_j}. \quad \#$$

7.4.1) Multimatrix model and Airy process (for Hermitian case).

. Now we consider Dyson's Brownian Motion for $\beta=2$ and $\alpha = \frac{\beta}{4N} = \frac{1}{2N}$. Let $H(t=0)$ be distributed according to the stationary measure $e^{-\frac{\mathrm{Tr}(H_0^2)}{2N}}dH_0$. Consider m times $0<t_1<t_2<\dots<t_m$. Then, by (3), the measure on matrices at these times is given by

$$(5) \quad \frac{1}{Z_{N,m}}\cdot e^{-\frac{\mathrm{Tr}(H_0^2)}{2N}}\cdot\prod_{j=0}^{m-1} e^{-\frac{\mathrm{Tr}(H_{j+1}-q_jH_j)^2}{2N(1-q_j^2)}}\, dH_0\cdots dH_m,$$

where $q_j \doteq \exp[-(t_{j+1}-t_j)/2N]$ and H_j is the matrix at time t_j.

. (5) can be rewritten as

$$(6) \quad \frac{1}{Z_{N,m}}\cdot e^{-\frac{\mathrm{Tr}(H_0^2)}{2N(1-q_0^2)}}\prod_{j=1}^{m-1} e^{-\frac{\mathrm{Tr}(H_j^2)}{2N}\left[\frac{1}{1-q_{j-1}^2}+\frac{q_j^2}{1-q_j^2}\right]}\cdot e^{-\frac{\mathrm{Tr}(H_m^2)}{2N(1-q_{m-1}^2)}}$$

$$\cdot\prod_{j=0}^{m-1} e^{\frac{q_j}{N(1-q_j^2)}\mathrm{Tr}(H_j\cdot H_{j+1})}\, dH_0\dots dH_m.$$

. Denote by $\alpha_k \doteq \frac{1}{2N(1-q_k^2)}$, $\gamma_k \doteq \frac{1-q_{k-1}^2 q_k^2}{2N(1-q_{k-1}^2)(1-q_k^2)}$, and

$$\beta_k \doteq \frac{q_k}{N(1-q_k^2)}.$$

. Then, (6) writes

$$(7) \quad \frac{1}{Z_{N,m}} \cdot e^{-\alpha_0 \cdot \mathrm{Tr}(H_0^2)} \cdot \left(\prod_{k=1}^{m-1} e^{-\gamma_k \cdot \mathrm{Tr}(H_k^2)} \right) \cdot e^{-\alpha_{m-1} \cdot \mathrm{Tr}(H_m^2)}$$

$$\cdot \prod_{k=0}^{m-1} e^{\beta_k \cdot \mathrm{Tr}(H_k \cdot H_{k+1})} \, dH_0 \dots dH_m .$$

. As in the case of a single GUE matrix, if we are interested only in the eigenvalues, we have to integrate out the angular variables. We have seen that

$$dH_k = \Delta_N^2(\lambda_k) \cdot d\lambda_k \cdot dU_k \quad ,$$

where $\lambda_k = (\lambda_{k,1}, \dots, \lambda_{k,N})$, are the eigenvalues of H_k and $H_k = U_k \Lambda_k U_k^{-1}$ with $U_k \in U(N)$ and $\Lambda_k = \begin{pmatrix} \lambda_{k,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{k,N} \end{pmatrix}$.

. Replacing $H_k = U_k \Lambda_k U_k^{-1}$ in (7), the only terms in which the U_k do not disappear is the last product:

$$(8) \quad \prod_{k=0}^{m-1} e^{\beta_k \cdot \mathrm{Tr}(U_k \Lambda_k U_k^{-1} U_{k+1} \Lambda_{k+1} U_{k+1}^{-1})} .$$

Defining $V_k \doteq U_k^{-1} U_{k+1}$, we get $(8) = \prod_{k=0}^{m-1} e^{\beta_k \mathrm{Tr}(\Lambda_k V_k \Lambda_{k+1} V_k^{-1})}$.

The problem is to integrate over the unitary group $U(N)$ the expressions.

<u>Lemma:</u> [Harish-Chandra / Itzykson-Zuber formula]:

$$(9) \quad \int_{U(N)} dU \exp\left[\beta \cdot \mathrm{Tr}(\Lambda_1 U \Lambda_2 U^{-1})\right] = \frac{1}{\beta^{N(N-1)/2}} \left(\prod_{p=1}^{N-1} p! \right) \cdot \frac{\det\left[e^{\beta \lambda_{1,i} \cdot \lambda_{2,j}}\right]_{1 \le i,j \le N}}{\Delta_N(\lambda_1) \Delta_N(\lambda_2)}$$

if $\Lambda_1 = \begin{pmatrix} \lambda_{1,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{1,N} \end{pmatrix}$ and $\Lambda_2 = \begin{pmatrix} \lambda_{2,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2,N} \end{pmatrix}$.

. We are not going to prove this Lemma here. Look e.g. J. Math. Phys., 21 (1980), 411–421.

. Applying this Lemma, we get the following expression for the joint distribution of eigenvalues:

$$(10)\quad \begin{cases} P(\{\lambda_{k,i}\}_{k,i}) = \dfrac{1}{\tilde{Z}_{N,m}} \left(\prod_{i=1}^{N} e^{-\alpha_0 \cdot \lambda_{i,0}^2}\right) \cdot \left(\prod_{k=1}^{m-1} \prod_{i=1}^{N} e^{-\gamma_k \cdot \lambda_{k,i}^2}\right) \cdot \left(\prod_{i=1}^{N} e^{-\alpha_{m-1} \cdot \lambda_{m-1,i}^2}\right) \cdot \\ \cdot \Delta_N(\lambda_1) \cdot \left(\prod_{k=0}^{m-1} \det\left(e^{\beta_k \cdot \lambda_{k,i} \cdot \lambda_{k+1,j}}\right)_{1 \le i,j \le N}\right) \cdot \Delta_N(\lambda_m)\, d\lambda_0 \dots d\lambda_m \end{cases}$$

since the Vandermonde determinants coming from (9) are all canceled by the ones coming from the $dH_k = \Delta_N^2(\lambda_k) d\lambda dU_k$ except for one at the beginning and at the end.

. The measure (10) is of the form needed to apply theorem at page ④ of chapter 7.1). We do not do the computations here (analogue to the ones of the N non-intersecting Brownian Bridges). The final result is that the eigenvalues forms a space-time extended determinantal point process with Kernel

$$(11)\quad K_N(t_1, x_1; t_2, x_2) = \begin{cases} \sum_{k=0}^{N-1} e^{-\frac{k(t_2-t_1)}{2N}} \cdot P_k(x) P_k(y) \cdot e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1 \le t_2, \\ -\sum_{k=N}^{\infty} e^{-\frac{k(t_2-t_1)}{2N}} \cdot P_k(x) P_k(y) \cdot e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1 > t_2, \end{cases}$$

where $P_k(x) = \dfrac{1}{\sqrt[4]{2\pi N}\sqrt{2^k k!}} \cdot H_k\left(\dfrac{x}{\sqrt{2N}}\right)$, with H_k the standard Hermite polynomials.

. The kernel (11) is called <u>extended Hermite kernel</u>.

. Edge scaling: The edge scaling is the following. Let

$\lambda_i(t)$ be the i-th largest eigenvalue of the stationary solution of (4) for $\beta = 2$. Then, define

$$(12) \qquad \lambda_i^{resc}(s) \doteq \frac{\lambda_i(2s.N^{2/3}) - 2N}{N^{1/3}} .$$

. Then one can prove that: $\lim_{N\to\infty} \lambda_1^{resc}(s) = \mathcal{A}_2(s)$, the Airy process.

7.5) The LGV Theorem ("discrete version" of Karlin-Mc Gregor).

. In a lot of applications, it is useful the following result on non-intersecting paths on directed graphs.

. Consider a graph (V,E) of vertices V and edges E:

. The edges are directed.

. A path π is a sequence of consecutive vertices joined by directed edges.

. Let $\mathcal{P}(u,v)$ denote the set of all paths starting from $u \in V$ and ending at $v \in V$.

. Two paths π and π' intersects if they have a common vertex.

. To every edge assign a weight $w(e)$, $e \in E$. Then, the weight of a path π is given by

$$w(\pi) \doteq \prod_{e \in \pi \cap E} w(e), \quad \text{and}$$

define the total weight of paths from u to v is defined by

$$h(u,v) \doteq \sum_{\pi \in \mathcal{P}(u,v)} w(\pi) .$$

. The final <u>important condition</u> on the graph is the following: Given initial points $(u_1, \dots, u_m)$ and final points $(v_1, \dots, v_m)$, there exists at most <u>a unique</u> permutation $\sigma \in S_m$ s.t. we can connect u_i to $v_{\sigma(i)}$, $i = 1, \dots, m$, by a set of non-intersecting paths.

. We say that $(u_1, \dots, u_m)$ and $(v_1, \dots, v_m)$ are <u>compatible</u> if there exist a way of connecting them. Then we choose the numbering of the v_i's s.t. the above permutation is the identity.

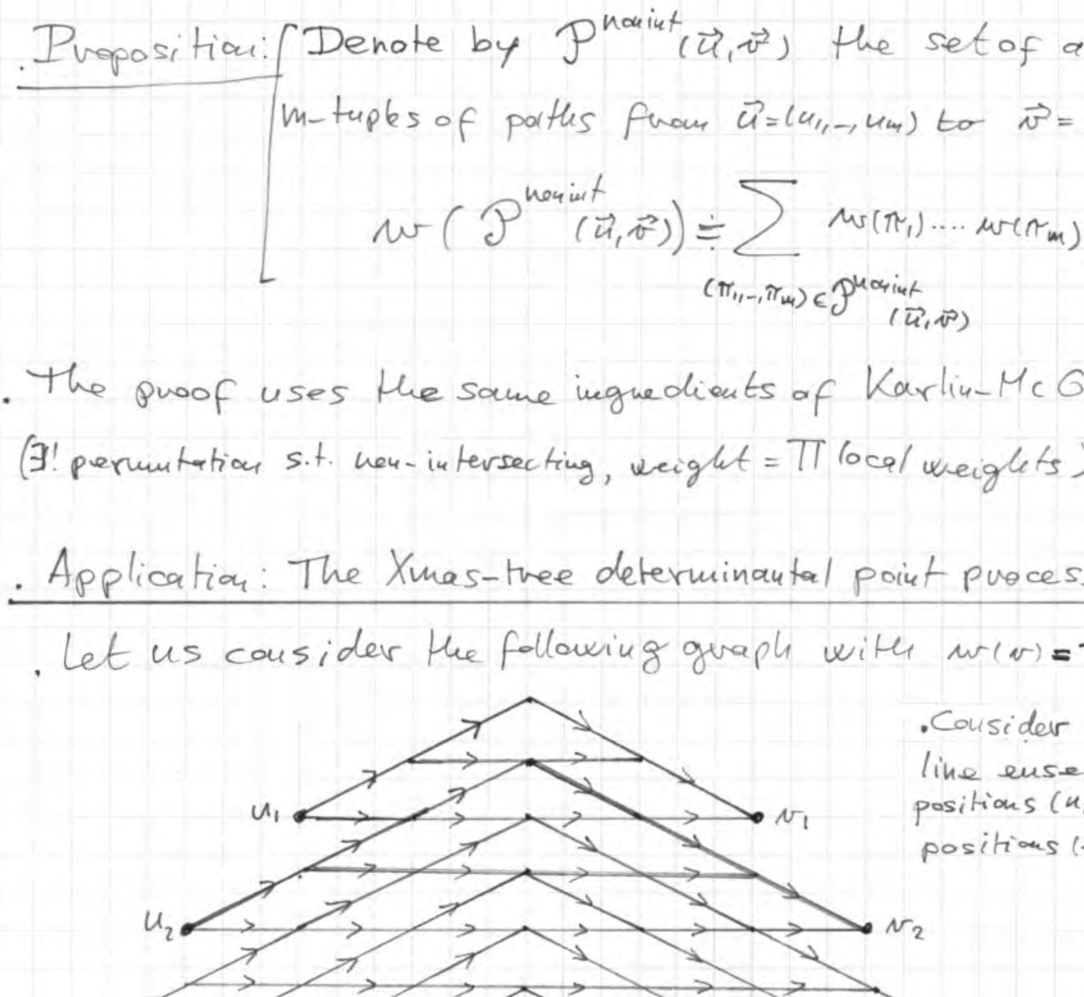
<u>Proposition:</u> Denote by $\mathcal{P}^{\text{nonint}}(\vec{u}, \vec{v})$ the set of all non-intersecting m-tuples of paths from $\vec{u} = (u_1, \dots, u_m)$ to $\vec{v} = (v_1, \dots, v_m)$. Then,

$$w\left(\mathcal{P}^{\text{nonint}}(\vec{u}, \vec{v})\right) \doteq \sum_{(\pi_1, \dots, \pi_m) \in \mathcal{P}^{\text{nonint}}(\vec{u}, \vec{v})} w(\pi_1) \cdots w(\pi_m) = \det\left(h(u_i, v_j)\right)_{1 \le i, j \le m}.$$

. The proof uses the same ingredients of Karlin-McGregor theorem ($\exists!$ permutation s.t. non-intersecting, weight $= \prod$ local weights).

<u>. Application: The Xmas-tree determinantal point process.</u>

. Let us consider the following graph with $w(v) = 1$, $\forall$ vertex.



. Consider the non-intersecting line ensembles with initial positions (u_1, u_2, u_3, u_4) and final positions (v_1, v_2, v_3, v_4).

. Show that the point process at $t = 0$ is determinantal and compute the correlation kernel. Have a nice Xmas!