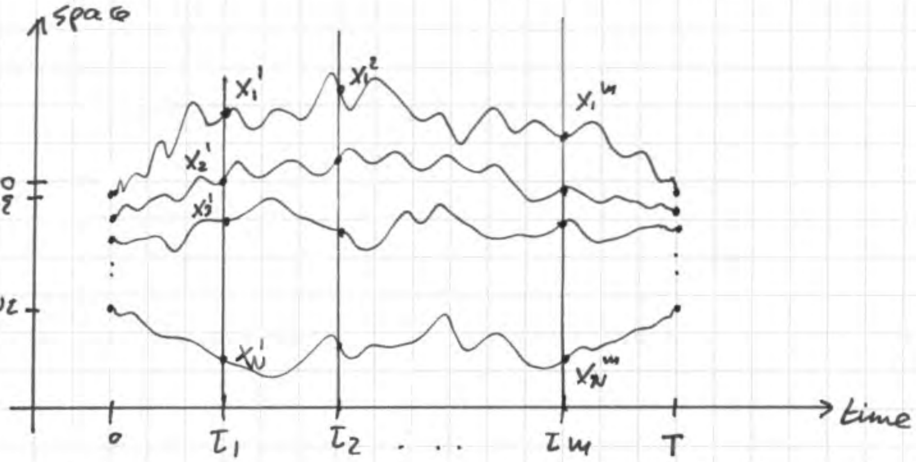
7.2) Application to non-intersecting Brownian Bridges.

- For Brownian paths, $\phi_{r,s}(x,y) = \dfrac{\exp\left(-\frac{(y-x)^2}{2(\tau_s-\tau_r)}\right)}{\sqrt{2\pi(\tau_s-\tau_r)}} \mathbb{1}_{[\tau_s>\tau_r]}$.
- By the above theorem, the point process with support on $\{x_n^k, 1\le k\le N, 1\le n\le m\}$ is determinantal in space-time provided the initial and final positions are fixed.

- We apply to N non-intersecting Brownian Bridges from time $t=0$ to time $t=T$. We do the usual construction:

  ① $x_i^0 = x_i^{m+1} = -\varepsilon\cdot i$

  ② $\lim_{\varepsilon\to 0}$.



- Let us see what happens as $\varepsilon\to 0$ to the first / last term in (3).

First term: $\det(\phi_{a_1}(x_i^0, x_j^1)) = \det\left[\frac{1}{\sqrt{2\pi\tau_1}}\cdot e^{-\frac{(x_j^1+\varepsilon i)^2}{2\tau_1}}\right]_{1\le i,j\le N}$

$$= \left(\prod_{k=1}^{N} \frac{e^{-(x_k^1)^2/2\tau_1}}{\sqrt{2\pi\tau_1}}\right)\cdot \underbrace{\det\left(e^{-\frac{i\varepsilon x_j^1}{\tau_1}}\cdot e^{-\frac{i^2\varepsilon^2}{2\tau_1}}\right)_{1\le i,j\le N}}_{\circledast}$$

Now: $\circledast = \det\left[1-\frac{i\cdot\varepsilon x_j^1}{\tau_1}+\frac{1}{2}\frac{(i\varepsilon x_j^1)^2}{\tau_1^2}+\dots+\frac{(-1)^{N-1}}{(N-1)!}\cdot\frac{(i\,\varepsilon x_j^1)^{N-1}}{\tau_1^{N-1}}+\mathcal{O}(\varepsilon^N)\right]_{1\le i,j\le N}\cdot$

$$\cdot\left(\prod_{i=1}^{N} e^{-\frac{i^2\varepsilon^2}{2\tau_1}}\right)$$

· The product $\prod_{i=1}^{N} e^{-\frac{i^2\varepsilon^2}{2\tau_1}} \longrightarrow 1$ as $\varepsilon \to 0$.

. It remains the determinant:

$$\det\left(1-\left(\frac{i\varepsilon x_j^1}{\tau_1}\right)+\dots+\left(\frac{-i\varepsilon x_j^1}{\tau_1}\right)^{N-1}\frac{1}{(N-1)!}+\mathcal{O}(\varepsilon^N)\right)_{1\le i,j\le N} \quad \text{to evaluate.}$$

. We can apply linear combinations and get [Check for example 4x4...]

$$\text{const}\times\det\left[\begin{pmatrix} 1 & +\mathcal{O}(\varepsilon^N) \\ \varepsilon x_j^1 & +\mathcal{O}(\varepsilon^N) \\ (\varepsilon x_j^1)^2 & +\mathcal{O}(\varepsilon^N) \\ \vdots & \\ (\varepsilon x_j^1)^{N-1} & +\mathcal{O}(\varepsilon^N)\end{pmatrix}_{1\le j\le N}\right] = \text{const}\cdot\varepsilon^{\frac{N(N-1)}{2}}\cdot\underbrace{\det\left[\begin{pmatrix} 1+\mathcal{O}(\varepsilon^N) \\ x_j^1+\mathcal{O}(\varepsilon^{N-1}) \\ \vdots \\ (x_j^1)^{N-1}+\mathcal{O}(\varepsilon)\end{pmatrix}_{1\le j\le N}\right]}_{**}$$

. As $\varepsilon\to 0$, $**$ $\to$ Vandermonde determinant in $(x_1^1,\dots,x_N^1)$.

Therefore, $$\lim_{\varepsilon\to 0}\frac{\det\left(\varphi_{0,1}(-i\varepsilon, x_j^1)\right)_{1\le i,j\le N}}{\varepsilon^{N(N-1)/2}} = \text{const}\cdot\Delta_N(x_1^1,\dots,x_N^1).$$

. Last term: Exactly in the same way.

. The factors $\varepsilon^{\frac{N(N-1)}{2}}$ will be compensate by the normalization constant, so, in the $\varepsilon\to 0$ limit, the measure on $\{x_k^n\}$ is given by

$$\underline{P(\{x_k^n\})} = \text{const}\times\,\Delta_N(\{x_k^1\})\cdot\Delta_N(\{x_k^{m-1}\})\cdot\prod_{k=1}^{N}\frac{e^{-(x_k^1)^2/2\tau_1}}{\sqrt{2\pi\tau_1}}\cdot\prod_{\ell=1}^{N}\frac{e^{-(x_k^{m-1})^2/2(T-\tau_{m-1})}}{\sqrt{2\pi(T-\tau_{m-1})}}\cdot$$
$$\cdot\prod_{n=1}^{m-1}\det\left(\frac{e^{-\frac{(x_j^{n+1}-x_i^n)^2}{2(\tau_{n+1}-\tau_n)}}}{\sqrt{2\pi(\tau_{n+1}-\tau_n)}}\right)_{1\le i,j\le N}.$$

$$= \text{const}\times\det\left[\frac{e^{-(x_j^1)^2/2\tau_1}}{\sqrt{2\pi\tau_1}}\cdot P_{i-1}(x_j^1)\right]_{1\le i,j\le N}\cdot\prod_{n=1}^{m-1}\det\left[\frac{e^{-\frac{(x_j^{n+1}-x_i^n)^2}{2(\tau_{n+1}-\tau_n)}}}{\sqrt{2\pi(\tau_{n+1}-\tau_n)}}\right]_{1\le i,j\le N}\cdot$$
$$\times\det\left[\frac{e^{-(x_j^{m-1})^2/2(T-\tau_{m-1})}}{\sqrt{2\pi(T-\tau_{m-1})}}\cdot\tilde{P}_{i-1}(x_j^{m-1})\right]_{1\le i,j\le N}$$

. We can still choose the p_i's and $\tilde{p}_i$'s. No surprise: they are given in terms of Hermite polynomials, defined as:

$$\int_{\mathbb{R}} dx\, e^{-x^2} H_k(x) H_\ell(x) = \sqrt{\pi}\cdot 2^k k!\cdot \delta_{k\ell}$$

. Let us define: $\Phi_i(r,y) = \dfrac{\sqrt[4]{2\pi T}}{\sqrt{i!}\, 2^{i/2}} \cdot \left(\dfrac{T-\tau r}{\tau r}\right)^{i/2} \cdot H_i\left(\dfrac{y}{\sqrt{2\tau r(T-\tau r)/T}}\right) \dfrac{e^{-\frac{y^2}{2\tau r}}}{\sqrt{2\pi\tau r}}$

and $\Psi_j(s,x) = \dfrac{\sqrt[4]{2\pi T}}{\sqrt{j!}\cdot 2^{j/2}} \cdot \left(\dfrac{\tau s}{T-\tau s}\right)^{j/2} \cdot H_j\left(\dfrac{x}{\sqrt{2\tau s(T-\tau s)/T}}\right) \dfrac{e^{-\frac{x^2}{2\tau s}}}{\sqrt{2\pi\tau s}}$

. Then, these functions satisfy (see page 4):

$$\circledast \begin{cases} \int_{\mathbb{R}} dx\, \Phi_i(r,x)\, \phi_{r,s}(x,y) = \Phi_i(s,y) \quad \text{and} \\ \int_{\mathbb{R}} dz\, \phi_{r,s}(x,z)\, \Psi_j(s,z) = \Psi_j(r,x). \end{cases}$$

. Moreover, $\{\Phi_i(1,x)\}_{i=0}^{N-1}$ generates $\{e^{-\frac{x^2}{2\tau_1}} p_{i-1}(x)\}_{i=1}^{N}$

and $\{\Psi_j(m,x)\}_{j=0}^{N-1}$ generates $\{e^{-\frac{x^2}{2(T-\tau_m)}} \cdot p_{j-1}(x)\}_{j=1}^{N}$

. Finally, with this choice, the matrix A to be inverted has entries:

$$a_{ij} \overset{\text{use } \circledast}{=} \int_{\mathbb{R}} dx\, \Phi_i(1,x)\, \Psi_j(1,x) = \int_{\mathbb{R}} dx\, H_i\left(\frac{x}{\sqrt{2\tau(T-\tau)/T}}\right) H_j\left(\frac{x}{\sqrt{2\tau(T-\tau)/T}}\right) \cdot \frac{e^{-\frac{x^2}{\tau}}}{2\pi\tau} \cdot \frac{\sqrt{2\pi T}}{\sqrt{j!\,i!}\, 2^{\frac{i+j}{2}}} \cdot \left(\frac{T-\tau}{\tau}\right)^{\frac{i-j}{2}}$$

$$\overset{\tau=\frac{T}{2}}{=} \underbrace{\int_{\mathbb{R}} dx\, H_i\left(\frac{x}{\sqrt{T/2}}\right) H_j\left(\frac{x}{\sqrt{T/2}}\right) \cdot e^{-\frac{x^2 2}{T}}}_{=\delta_{ij}\cdot\sqrt{\frac{T}{2}}\cdot\sqrt{\pi}\cdot 2^i i!} \cdot \frac{1}{\sqrt{j!\,i!}\, 2^{\frac{i+j}{2}}} \cdot \frac{\sqrt{T}\, 2}{\sqrt{2\pi}\cdot T}$$

$$= \delta_{ij}.$$

. The consistency relations $\circledast$ are implied by the following computation.

. A computation ⇒ consistency of $\Phi_i(r,x)$, $\Psi_0(r,x)$.

. The n-th Hermite polynomials, H_n, is given by

$$H_n(x) = \frac{n!}{2\pi i}\oint_{\Gamma_0} dw\, \frac{e^{2xw-w^2}}{w^{n+1}}.$$

$$\Rightarrow \boxed{\int_{\mathbb{R}} dx\, \frac{e^{-\frac{x^2}{2\tau_1}}}{\sqrt{2\pi\tau_1}}\cdot\frac{e^{-\frac{(y-x)^2}{2(\tau_2-\tau_1)}}}{\sqrt{2\pi(\tau_2-\tau_1)}}\cdot H_n\left(\frac{x}{\sqrt{2\tau_1(T-\tau_1)/T}}\right) =}$$

$$= \frac{n!}{2\pi i}\oint_{\Gamma_0} dw\, \frac{e^{-w^2}}{w^{n+1}}\cdot \underbrace{\int_{\mathbb{R}} dx\, \frac{e^{-\frac{x^2}{2\tau_1}}}{\sqrt{2\pi\tau_1}}\cdot\frac{e^{-\frac{(y-x)^2}{2(\tau_2-\tau_1)}}}{\sqrt{2\pi(\tau_2-\tau_1)}}\cdot e^{2x\cdot w\cdot\alpha}}_{=\frac{e^{-\frac{y^2}{2\tau_2}}}{\sqrt{2\pi\tau_2}}\cdot\exp\left[\frac{2yw\sqrt{\tau_1 T}}{\sqrt{2(T-\tau_1)}\,\tau_2}+\frac{w^2\cdot T(\tau_2-\tau_1)}{(T-\tau_1)\tau_2}\right]}, \quad \alpha = \frac{1}{\sqrt{\frac{2\tau_1(T-\tau_1)}{T}}}$$

$$w \doteq z\sqrt{\frac{(T-\tau_1)\tau_2}{(T-\tau_2)\tau_1}}$$

$$\doteq \frac{n!}{2\pi i}\oint_{\Gamma_0} dz\, \frac{e^{-z^2}}{z^{n+1}}\cdot e^{\frac{2yz}{\sqrt{2(T-\tau_2)\tau_2/T}}}\cdot\left(\frac{(T-\tau_2)\tau_1}{(T-\tau_1)\tau_2}\right)^{n/2}$$

$$\boxed{= \left(\frac{(T-\tau_2)\tau_1}{(T-\tau_1)\tau_2}\right)^{n/2}\cdot H_n\left(\frac{y}{\sqrt{2\tau_2(T-\tau_2)/T}}\right)}$$

. Now we have all ingredients to get the kernel for N non-intersecting Brownian Bridges:

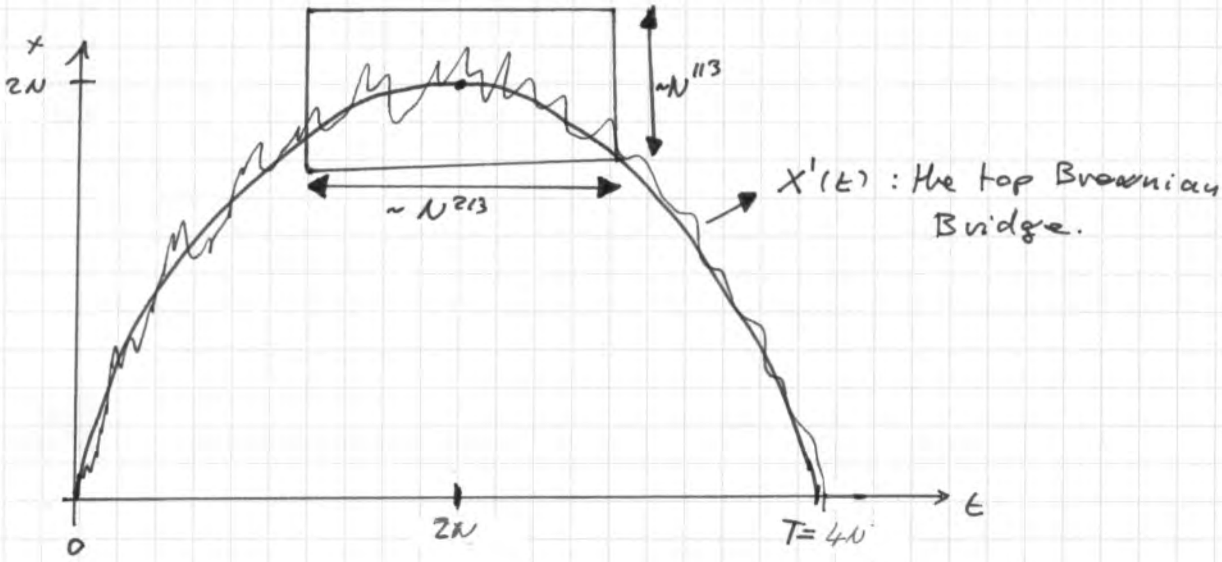
$$\boxed{\begin{gathered} K_N(r,x;s,y) = -\phi_{r,s}(x,y) + \sum_{k=0}^{N-1}\Psi_k(r,x)\cdot\Phi_k(s,y) \\ \text{with } \Phi_i(r,x),\ \Psi_j(s,y) \text{ defined at page ③ and} \\ \phi_{r,s}(x,y) = \frac{\exp\left[-\frac{(y-x)^2}{2(\tau_s-\tau_r)}\right]}{\sqrt{2\pi(\tau_s-\tau_r)}}\,\mathbb{1}_{[r<s]} \end{gathered}}$$

7.3) Edge-scaling and Airy process.

. Consider now the following rescaling:

. $T = 4N$ and focus around

. $t \cong N$: $\begin{cases} \tau_i = 2N + u_i \cdot N^{2/3} & , u_1 < u_2 < \dots < u_m \text{ fixed,} \\ x_i = 2N - \frac{u_i^2}{4} \cdot N^{1/3} + s_i \cdot N^{1/3}. \end{cases}$



<u>Remark:</u> . In the rescaling of x_i, $2N - \frac{u_i^2}{4} N^{1/3}$, is the position of the limit shape and $s_i N^{1/3}$ is the deviation from it.

. The rescaled Kernel is then:

$$K_N^{resc}(u_1, s_1; u_2, s_2) \doteq N^{1/3} \cdot K_N(\tau_1, x_1; \tau_2, x_2).$$

. Let us compute the <u>asymptotics of $\Psi_{N-k}(r, x)$</u>:

$$\cdot \; \Psi_{N-k}(r, x) = \frac{\sqrt[4]{2\pi \cdot 4N}}{\left((N-k)!\, 2^{N-k}\right)^{1/2}} \cdot \left(\frac{2N + u \cdot N^{2/3}}{2N - u \cdot N^{2/3}}\right)^{\frac{N-k}{2}} \cdot \frac{e^{-\frac{\left[2N + (s - \frac{1}{4}u^2) N^{1/3}\right]^2}{2 \cdot \left[2N + u N^{2/3}\right]}}}{\sqrt{2\pi \cdot (2N + u N^{2/3})}} \cdot$$

$$\cdot \; H_{N-k}\left[\underbrace{\frac{2N + (s - \frac{u^2}{4}) N^{1/3}}{\left(2(2N + u N^{2/3})(2N - u N^{2/3})/4N\right)^{1/2}}}_{\cong \sqrt{2N} + N^{-1/6} \cdot \frac{s}{\sqrt{2}} = \frac{2N + s N^{1/3}}{\sqrt{2N}}}\right]$$

. The asymptotics of Hermite polynomials are known, compare with 3.3.4 of this lecture series:

$$H_{N-k}\left(\frac{2N+\xi N^{1/3}}{\sqrt{2N}}\right) \cong (2\pi N)^{1/4} \cdot \sqrt{2^{N-k}\cdot (N-k)!}\cdot N^{-1/3}\cdot$$

$$\underline{\cdot\, e^{+\frac{(2N+\xi N^{1/3})^2}{4N}} \cdot Ai\left(\xi + N^{-1/3}\cdot\left(k-\tfrac{1}{2}\right)\right).}$$

Therefore: $\underline{\Phi_{N-\lambda N^{1/3}}(s,x)} \cong N^{-1/3}\cdot Ai(s+\lambda)\cdot \cdot \exp\left[+\frac{(2N+sN^{1/3})^2}{4N}\right]$

$$\cdot \exp\left[-\frac{\left(2N+sN^{1/3}-\frac{u^2}{4}N^{1/3}\right)^2}{4N+2uN^{2/3}}\right]\cdot$$

$$\cdot \exp\left[-\frac{N-\lambda N^{1/3}}{2}\cdot \ln\left(\frac{2N+uN^{2/3}}{2N-uN^{2/3}}\right)\right]$$

$$\underline{\cong N^{-1/3}\cdot Ai(s+\lambda)\cdot \exp\left[\frac{\lambda u}{2}\right]\cdot f(s,u)}$$

$$\underline{\text{with } f(s,u) = \exp\left[-\frac{u^3}{24}+\frac{su}{2}\right]}$$

. Asymptotics of $\underline{\bar{\Phi}_{N-k}(s,y)}$: Similar.

$$\cdot\ \bar{\Phi}_{N-\lambda N^{1/3}}(s,y) \cong N^{-1/3}\cdot Ai(s+\lambda)\cdot \exp\left[\frac{\lambda u}{2}\right]\cdot f(s,u)^{-1}$$

. Therefore, the main part of the kernel, i.e. $N^{1/3}\sum_{k=0}^{N-1}\Phi_k(s,x)\,\bar{\Phi}_k(r,y) =$

$$\cong \int_0^\infty d\lambda\, Ai(s_1+\lambda)Ai(s_2+\lambda)\cdot e^{-\frac{\lambda(u_2-u_1)}{2}}\cdot \frac{f(s_1,u_1)}{f(s_2,u_2)}.$$

. Asymptotics of the transition term:

$$N^{1/3}\cdot \phi_{r,s}(x,y) = \frac{\cancel{N^{1/3}}\cdot \exp\left[-\frac{\left(s_1-\frac{u_1^2}{4}-s_2+\frac{u_2^2}{4}\right)^2\cancel{N^{2/3}}}{2(u_2-u_1)\cancel{N^{2/3}}}\right]}{\sqrt{2\pi(u_2-u_1)\cancel{N^{2/3}}}}$$

$$= \frac{1}{\sqrt{2\pi(u_2-u_1)}}\cdot \exp\left[-\frac{(u_2-u_1)(u_2+u_1)^2}{32}-\frac{u_1+u_2}{4}(s_1+s_2)-\frac{(s_2-s_1)^2}{2(u_2-u_1)}\right]$$

$$= \frac{1}{\sqrt{4\pi\left(\frac{u_2-u_1}{2}\right)}} \cdot \exp\left[-\frac{(u_2-u_1)}{2}\cdot\left(\frac{u_2+u_1}{2}\right)^2\cdot\frac{1}{4} - \frac{u_2+u_1}{2}\cdot\frac{s_1-s_2}{2} - \frac{(s_2-s_1)^2}{4\cdot\frac{u_2-u_1}{2}}\right]$$

$$= \frac{f(s_1,u_1)}{f(s_2,u_2)} \cdot \frac{1}{\sqrt{4\pi\frac{u_2-u_1}{2}}} \cdot \exp\left[-\frac{(s_2-s_1)^2}{4\cdot\frac{u_2-u_1}{2}} + \frac{1}{12}\left(\frac{u_2-u_1}{2}\right)^3 - \frac{(u_2-u_1)(s_1+s_2)}{4}\right].$$

. The common factor $\frac{f(s_1,u_1)}{f(s_2,u_2)}$ cancels out in the determinant.

. Moreover, since we have a lot of factors $\frac{1}{2}$ in front of u, we redefine $u \to 2u$. We have derived the following result.

Thm.: . Let N Brownian Bridges from $t=0$ to $t=4N$ be conditioned on non-intersecting (excepts at the origin at $t=0$, $t=4N$).

. Let $\begin{cases} \tau = 2N + 2u\,N^{2/3} \\ X = 2N - u^2 N^{1/3} + s\,N^{1/3} \end{cases}$

. Then, in the $N\to\infty$ limit, we have an extended determinantal point process with kernel:

$$K_2(u,s;u',s') = -\frac{\mathbb{1}(u'>u)}{\sqrt{4\pi(u'-u)}}\cdot\exp\left[-\frac{(s'-s)^2}{4(u'-u)} + \frac{1}{12}(u'-u)^3 - \frac{(u'-u)(s+s')}{2}\right]$$
$$+ \int_0^\infty d\lambda\, Ai(s+\lambda)\,Ai(s'+\lambda)\cdot e^{\lambda(u'-u)}$$

. One can also check that $\int_{\mathbb{R}} d\lambda\, Ai(s+\lambda)Ai(s'+\lambda)\,e^{\lambda(u'-u)} =$

$$\overset{u'>u}{=} \frac{1}{\sqrt{4\pi(u'-u)}}\cdot\exp\left[-\frac{(s'-s)^2}{4(u'-u)} + \frac{1}{12}(u'-u)^3 - \frac{(u'-u)(s+s')}{2}\right].$$

- Under the assumption that not only the kernel converges but control on the decay is good enough (it can be made) the above result can be stated as follows.

Corollary: Let $X^1(t)$ be the trajectory of the top Brownian Bridge. Then, define the rescaled process

$$Y_N(u) \doteq \frac{X^1(2N+2uN^{2/3})-(2N-u^2N^{1/3})}{N^{1/3}}.$$

Then, $\lim_{N\to\infty} Y_N(u) = \mathcal{A}_2(u)$

in the sense of finite-dimensional distributions, where $\mathcal{A}_2(u)$ is called the Airy process.

- Definition: Airy process:

  - The Airy process is defined via the finite-dimensional distributions given by (set $u_1 < u_2 < \dots < u_m$)

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \mathcal{A}_2(u_k) \leq s_k\right) = \det\left(\mathbb{1} - \chi_s K_2 \chi_s\right)_{L^2(\mathbb{R}\times\{u_1,\dots,u_m\})}$$

with $\chi_s(x, u_k) = \mathbb{1}_{[x \geq s_k]}$ and K_2 is the extended Airy kernel:

$$K_2(u,s;u',s') = \begin{cases} \int_0^\infty d\lambda\, Ai(s+\lambda)Ai(s'+\lambda)e^{\lambda(u'-u)}, & \text{if } u' \leq u, \\ -\int_{-\infty}^0 d\lambda\, Ai(s+\lambda)Ai(s'+\lambda)e^{\lambda(u'-u)}, & \text{if } u' > u. \end{cases}$$