7) Dynamics on point processes: "extended point processes"

. Until now we considered only point processes "at a fixed time", e.g., eigenvalues' point process of a given random matrix ensemble. In the application to physical system that we will consider after Xmas, this corresponds to focus on a one-point random variable, like the height in a stochastic growth model at a fixed position. However, in the growth model it is natural to be interested in the spacial structure of correlations $\Rightarrow$ need to do some extension of the framework.

. In this chapter we will consider non-intersecting Brownian Motions which lead to extended point processes. Then we give a discrete version (on a graph) which is quite useful in applications. We will also discuss Dyson's Brownian Motion on matrices.

7.1) Karlin-McGregor theorem.

. The original work was in continuous time but discrete space (jump processes). Here we present something similar but for Brownian Motions.

. Consider N Brownian paths on $\mathbb{R}$. Let $\{x_i(t)\}_{i=1}^{N}$ be their positions at time t. The question is: "What is the probability that the N Brownian Motions start at $t=0$ for $y_1 > y_2 > \dots > y_N$ and reach at time t the positions $x_1 > x_2 > \dots > x_N$ and that they are non-intersecting"?

. The answer is "simple":

Theorem (K-McG): Let $y_1 > y_2 > \dots > y_N$, $x_1 > x_2 > \dots > x_N$. Then

$$(1)\quad \mathbb{P}_{\text{non-int}}\left(x_1(t)=x_1, \dots, x_N(t)=x_N \mid x_1(0)=y_1, \dots, x_1(N)=y_N \right) = \det\left[\mathbb{P}(x(t)=x_i \mid x(0)=y_j) \right]_{1\le i,j\le N},$$

where $\mathbb{P}(X(t)=x_i \mid X(0)=y_j)$ is the transition probability of a single Brownian Motion $X(t)$.

Remark: This is not the conditional proba. that they do not intersect, but the proba. that they go from y_i's to x_i and do not intersect.

. Proof: To understand the theorem one need to keep in mind the two key properties:

① Brownian Motion is a Markov process
$\Rightarrow$ "weight" is the product of "weights" on smaller intervals.

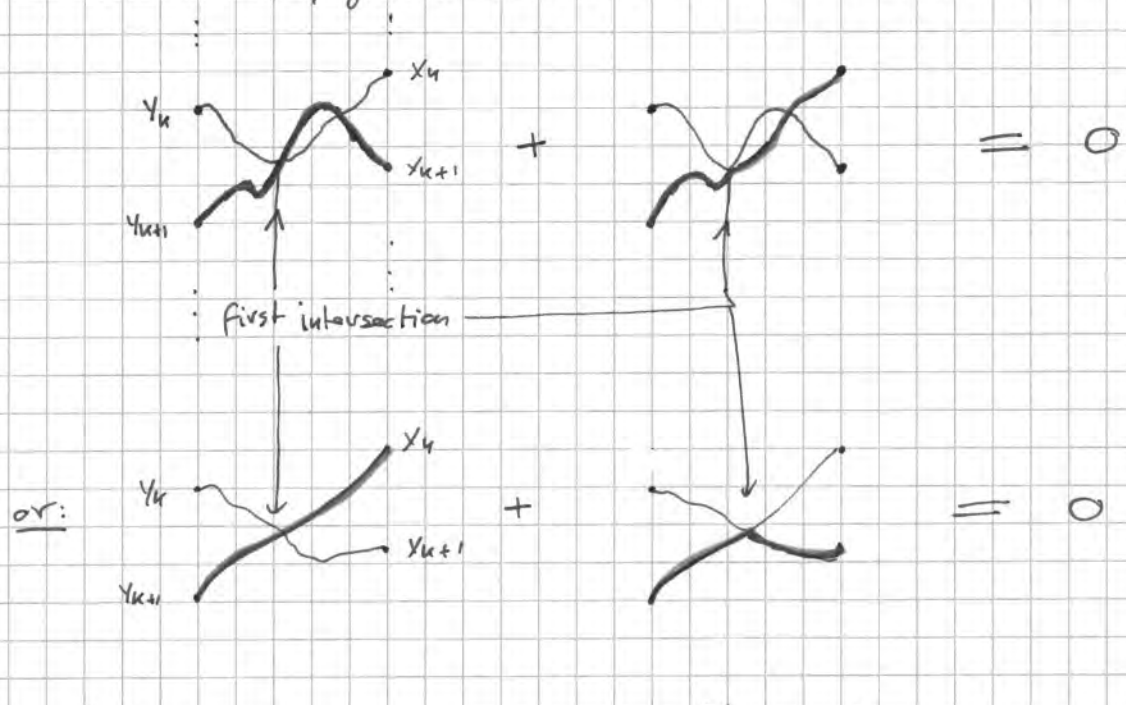
② $\exists!$ permutation $\sigma \in S_N$ s.t. it is possible to connect $\{Y_1, \dots, Y_N\}$ to $\{X_{\sigma(1)}, \dots, X_{\sigma(N)}\}$ without intersections. [$\sigma = id$ in our case].

. let us write r.h.s. of (1). Denote $p(x_i \mid y_j) = \mathbb{P}(X(t)=x_i \mid X(0)=y_j)$.

Then:

$$\det\left(p(x_i, y_j)\right)_{1 \le i,j \le N} = \sum_{\sigma \in S_N} (-1)^{|\sigma|} \cdot \prod_{k=1}^{N} p(x_k, y_{\sigma(k)}) \qquad (2)$$

. We have to see that this expression gives a measure zero to all configurations with intersections. Graphically:



. Permutations: σ $\tilde{\sigma} = \sigma$ plus one transposition

. Prefactor in (2): $(-1)^{|\sigma|}$ $(-1)^{|\tilde{\sigma}|} = (-1)^{|\sigma|+1}$.

. Weights: identical by Markov property.

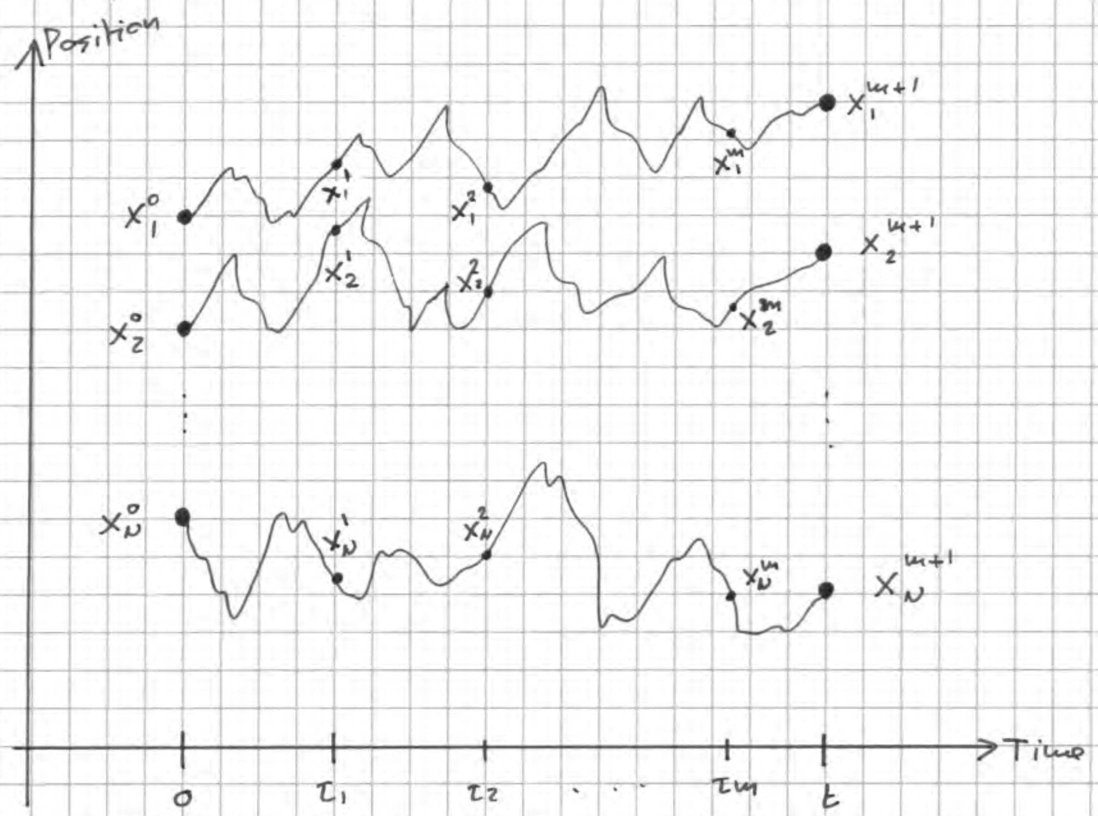
$\Rightarrow$ zero contribution.

. Therefore, all contributions coming from configurations with intersections is identically equal to zero, and

$$(2) = \mathbb{P}_{non-int}\left(\{X_i(t)=x_i\} \mid \{X_i(0)=y_i\}\right). \qquad \#$$

. Let now consider $0<\tau_1<\tau_2<\dots<\tau_m<t$. Then the measure of non-intersecting Brownian Motions at times $\tau_1, \tau_2, \dots, \tau_m$ is obtained by the previous theorem. Introduce the notations:

$$\begin{cases} \cdot\, x_k^n = \text{position of B.M. number } k \text{ at time } \tau_n \equiv X_k(\tau_n). \\ \cdot\, x_k^0 \text{ and } x_k^{m+1} \text{ be } \underline{\text{fixed}} \text{ initial and final positions.} \end{cases}$$



. Then, the measure on $\{x_k^\ell,\ 1\le k\le N,\ 1\le \ell\le m\}$ is given by:

$$(3)\ \cdot \mathbb{P}_{N,m}(\{x_k^\ell\}) = \frac{\det\left(\phi_{0,1}(x_i^0, x_j^1)\right)_{1\le i,j\le N}}{Z_{N,m}\cdot (N!)^m} \cdot \left[\prod_{k=1}^{m-1} \det\left(\phi_{k,k+1}(x_i^k, x_j^{k+1})\right)_{1\le i,j\le N}\right] \cdot \det\left(\phi_{m,m+1}(x_i^m, x_j^{m+1})\right)_{1\le i,j\le N}$$

where $\phi_{k,k+1}(x,y) =$ transition probability of one B.M. from $(x, t=\tau_k)$ to $(y, t=\tau_{k+1})$.

$\underline{\text{.Assume } Z_{N,m} \neq 0.}$

. <u>Theorem:</u> The space-time correlation functions for a measure of the form (3) is determinantal with <u>extended kernel</u>:

$$K_{N,m}(r,x;s,y) = -\phi_{r,s}(x,y) + \sum_{i,j=1}^{N} \phi_{r,m+1}(x,x_i^{m+1})[A^{-1}]_{ij}\phi_{0,s}(x_j^{0},y), \quad (4)$$

with $\phi_{r,s}(x,y) = \begin{cases} (\phi_{r,r+1} * \dots * \phi_{s-1,s})(x,y), & \text{for } r < s, \\ 0, & \text{for } r \geq s, \end{cases}$

and $A = [a_{ij}]_{1 \leq i,j \leq N}$, with $a_{ij} = \phi_{0,m+1}(x_i^{0}, x_j^{m+1})$.

. <u>Remark:</u> By applying Cauchy-Binet identity several times, one obtains $Z_{N,m} = \det(A) \neq 0$ by assumption, so A is invertible.

. The above theorem follows from the following one:

<u>Theorem:</u> Let g be a function on $\mathbb{R} \times \{1,\dots,m\}$ such that $K_{N,m} \cdot g$ is trace-class. Then,

$$\int_{\mathbb{R}^{N\times m}} \prod_{r=1}^{m} \prod_{j=1}^{N} (1 + g(r, x_j^{r})) \cdot P_{N,m}(x)\, d^{N\times m}x = \quad (5)$$

$$= \det\left(\mathbb{1} + K_{N,m} \cdot g\right)_{L^2(\mathbb{R}\times\{1,\dots,m\}, dx)}$$

where with $x = (x_1^1,\dots,x_1^m; x_2^1,\dots,x_2^m; \dots; x_N^1,\dots,x_N^m)$.

<u>Proof:</u> . L.h.s. of (5) is the expected value of $(1+g)$ with respect to the measure (3). A general expansion in terms of correlation functions leads to a series (see chapter 3) and in the particular case when the correlation functions are determinantal, this series is a Fredholm expansion of a Fredholm determinant. Thus, proving (5) will imply (4).

. Denote $w_{N,m}(x) = Z_{N,m} \cdot P_{N,m}(x)$. Then, set

$$Z_{N,m}(g) \doteq \frac{1}{(N!)^m} \int_{\mathbb{R}^{N\times m}} dx \cdot w_{N,m}(x) \cdot \prod_{r=1}^{m} \prod_{j=1}^{N} (1+g(r,x_j^r)).$$

. With this notation: $Z_{N,m} = \det(A) \equiv Z_{N,m}(0)$.

$$. Z_{N,m}(g) = \frac{1}{(N!)^m} \int_{\mathbb{R}^{N\times m}} dx \left(\prod_{r=1}^{m} \prod_{j=1}^{N} (1+g(r,x_j^r)) \right) \cdot \prod_{r=0}^{m} \det\left(\phi_{r,r+1}(x_i^r, x_j^{r+1})\right)_{1\le i,j\le N}$$

$$\underset{m \text{ times}}{\overset{\text{Cauchy-Binet}}{=\!=}} \det\left[\int_{\mathbb{R}^m} \prod_{r=1}^{m} (1+g(r,z_r)) \cdot \phi_{0,1}(x_i^0, z_1) \cdot \prod_{r=1}^{m-1} \phi_{r,r+1}(z_r, z_{r+1}) \cdot \phi_{m,m+1}(z_m, x_j^{m+1}) \cdot d^m z \right]_{1\le i,j\le N} \qquad (6)$$

. We expand: $\prod_{r=1}^{m} (1+g(r,z_r)) = 1 + \sum_{e=1}^{m} \sum_{1\le r_1 < r_2 < \dots < r_e \le m} g(r_1, z_{r_1}) \cdots g(r_e, z_{r_e})$

and plug back in (6).

. The restriction of ordering of the r_i in can be dropped since $\phi_{r,s} = 0$ for $r \geq s$.

. Also, we denote $\psi^{(0)}(r,x; s,y) = \delta_{r,s} \cdot \delta(x-y)$

and, $e\geq 1$, $\psi^{(e)}(r,x; s,y) = \sum_{u=1}^{m} \int_{\mathbb{R}} dz\ \psi(r,x; u,z) \cdot \psi^{(e-1)}(u,z; s,y)$, where

$\boxed{\psi(r,x; u,z) \doteq \phi_{r,u}(x,z) \cdot g(u,z).}$ $\boxed{\cdot\ \psi^{(e)} \text{ is the e-th convolution of } \psi.}$

. We have: $Z_{N,m}(g) = \det\left[a_{ij} + \sum_{e=1}^{m} \sum_{r_1,\dots,r_e=1}^{m} \int_{\mathbb{R}^e} dz_1 \dots dz_e\ \phi_{0,r_1}(x_i^0, z_1) \cdot g(r_1,z_1) \cdot \left(\prod_{s=1}^{e-1} \psi(r_s, z_s; r_{s+1}, z_{s+1}) \right) \cdot \phi_{r_e,m+1}(z_e, x_j^{m+1}) \right]_{1\le i,j\le N}.$

$$\left.\begin{matrix} r_1 \equiv r \\ z_1 \equiv x \\ r_e \equiv s \\ z_e \equiv y \end{matrix}\right\} \Rightarrow\ = \det\left[a_{ij} + \sum_{r,s=1}^{m} \int_{\mathbb{R}^2} dx\,dy\ \phi_{0,r}(x_i^0, x) \cdot g(r,x) \cdot \left(\sum_{e=1}^{m} \psi^{(e-1)}(r,x; s,y) \right) \cdot \phi_{s,m+1}(y, x_j^{m+1}) \right]_{1\le i,j\le N}$$

. What we need to compute is $\dfrac{Z_{N,m}(g)}{\det(A)} = \dfrac{Z_{N,m}(g)}{Z_{N,m}(0)}$.

$$\frac{Z_{N,m}(g)}{Z_{N,m}(0)} = \det\Big[\delta_{ij} + (a\cdot b)_{ij}\Big]_{1\le i,j\le N}$$

with $a(i;r,x) \doteq \sum_{k=1}^{N} [A^{-1}]_{ik}\cdot \phi_{0,r}(x_k^0,x)\cdot g(r,x)$

and $b(r,x;j) \doteq \sum_{s=1}^{N} \int_{\mathbb{R}} dy \left(\sum_{e=1}^{m} \psi^{(e-1)}(r,x;s,y)\right)\cdot \phi_{s,m+1}(y,x_j^{m+1})$.

. At this point we use the identity:

$$(7)\qquad \det(\mathbb{1} + (a\cdot b))_{\ell^2(\{1,\dots,N\})} = \det(\mathbb{1} + (b\cdot a))_{L^2(\mathbb{R}\times\{1,\dots,m\}, d\underline{x})}.$$

$$.\ (b\cdot a)(r,x;s,y) = \sum_{j=1}^{N}\sum_{e=1}^{m}\sum_{\tilde s=1}^{N}\int_{\mathbb{R}} d\tilde y\, \psi^{(e-1)}(r,x;\tilde s,\tilde y)\cdot \phi_{\tilde s,m+1}(\tilde y, x_j^{m+1}) \cdot \sum_{k=1}^{N} [A^{-1}]_{jk}\cdot \phi_{0,s}(x_k^0,y)\cdot g(s,y).$$

$\Rightarrow\ b\cdot a = \left(\sum_{e=1}^{m} \psi^{(e-1)}\right)\cdot \tilde K_{N,m}\cdot g$, where

$\tilde K_{N,m}(r,x;s,y) = K_{N,m}(r,x;s,y) + \phi_{r,s}(x,y)$.

. Finally, since $\phi_{r,s} = 0$ for $r \geq s$, we have

$$\det(\mathbb{1} - \psi) = 1 \text{ and } \underbrace{\psi^{(e)} \equiv 0}_{\text{for } e\geq m} \text{ (Nilpotent). Thus}$$

$$\det(\mathbb{1}-\psi)\det(\mathbb{1}+b\cdot a) = \det\Big(\mathbb{1} - \psi + (\mathbb{1}-\psi)\cdot\sum_{e=1}^{m}\psi^{(e-1)}\tilde K_{N,m}\, g\Big)$$

$$= \det\Big(\mathbb{1} - \psi + \underbrace{\Big(\sum_{e=1}^{m}(\mathbb{1}-\psi)\psi^{(e-1)}\Big)}_{\equiv \mathbb{1}}\tilde K_{N,m}\cdot g\Big)$$

$\equiv \mathbb{1}$. Indeed: $= (\mathbb{1}-\psi)(\mathbb{1}+\psi+\dots+\psi^{m-1})$

$= \mathbb{1} - \psi^{(m)} = \mathbb{1}$.

$$= \det(\mathbb{1} + K_{N,m}\cdot g)$$

#