6) The Tracy-Widom distribution for $\beta=2$.

. Let us remind a few results we obtained for GUE eigenvalues.

. Let H be $N\times N$ Hermitian matrices distributed according to $\exp\left(-\frac{\mathrm{Tr}(H^2)}{2N}\right)$. Then the eigenvalues of H, $\lambda_1,\dots,\lambda_N$, have joint distribution given by $\frac{1}{Z_N}\Delta_N(\lambda)^2\prod_{k=1}^{N}(e^{-\lambda_k^2/2N}\,d\lambda_k)$, with $\Delta_N(\lambda)=\det(\lambda_i^{j-1})_{1\le i,j\le N}$, the Vandermonde determinant.

. We then computed the n-point correlation functions:

$$S^{(n)}(\lambda_1,\dots,\lambda_n)=\det\left[K_N(\lambda_i,\lambda_j)\right]_{1\le i,j\le n},$$

with K_N the Hermite kernel:

$$K_N(x,y)=e^{-\frac{x^2}{4N}}\cdot e^{-\frac{y^2}{4N}}\sum_{k=0}^{N-1}q_k(x)q_k(y)$$

$$=N\cdot e^{-\frac{x^2}{4N}}e^{-\frac{y^2}{4N}}\cdot\frac{q_N(x)q_{N-1}(y)-q_{N-1}(x)q_N(y)}{x-y},$$

with $q_k(x)=\frac{1}{\sqrt[4]{2\pi N}}\cdot\frac{1}{\sqrt{2^k k!}}\cdot P_k^H(x/\sqrt{2N})$, $P_k^H(x)$ the Hermite polynomials of degree k.

. The particular structure of the correlation functions allows us to write the distribution of the largest eigenvalue as a Fredholm determinant:

$$\mathbb{P}\left(\lambda_{max}^{(N)}\le u\right)=\det\left(\mathbb{1}-P_u\cdot K_N\cdot P_u\right)_{L^2(\mathbb{R},dx)},$$

$$\text{with } P_u(x)=\begin{cases}1, & x>u,\\ 0, & x\le u.\end{cases}$$

. We also saw that as $N\to\infty$, the scaling limit relevant for the largest eigenvalue is $\frac{\lambda_{max}^{(N)}-2N}{N^{1/3}}$, and the rescaled kernel converges to the Airy kernel:

$$K_N^{resc}(x,y)\doteq N^{1/3}\cdot K_N(2N+x\cdot N^{1/3},2N+y\cdot N^{1/3})\underset{N\to\infty}{\longrightarrow}\mathcal{A}(x,y)\equiv\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$$

. Therefore,

$$\mathbb{P}\left(\lambda^{(N)}_{max} \leq 2N + s\cdot N^{1/3}\right) = \det\left(\mathbb{1} - P_{2N+sN^{1/3}} \cdot K_N \cdot P_{2N+sN^{1/3}}\right)$$

$$\underset{\substack{\text{change of}\\ \text{variables:}\\ x\to \frac{x-2N}{N^{1/3}}}}{=} \det\left(\mathbb{1} - P_s \cdot K_N^{\text{resc}} \cdot P_s\right)_{L^2(\mathbb{R}, dx)}.$$

. It is possible to prove (but we do not do here the computations), that $K_N^{\text{resc}} \to \mathcal{A}$ in trace-norm as $N\to\infty$. Thus, we have

$$\underline{F_2(s) \doteq \lim_{N\to\infty} \mathbb{P}\left(\lambda^{(N)}_{max} \leq 2N + s\cdot N^{1/3}\right) = \det\left(\mathbb{1} - P_s \cdot \mathcal{A} \cdot P_s\right)_{L^2(\mathbb{R}, dx)}}$$

. F_2 is called the Tracy-Widom distribution (for GUE eigenvalues), and it is one of the universal laws arising in a lot of different models (in the 1+1 KPZ class).

. In the following we want to discuss F_2 and relate it with solutions of Painlevé-II equation, for which high precision numerical solutions are available [Michael Prähofer's homepage].

6.1). Properties of the Airy kernel.

① $\underline{\mathcal{A}(u,v) = \int_0^\infty d\lambda \, Ai(u+\lambda) Ai(v+\lambda)}$

<u>Proof:</u> We need to show the identity:

$$\underline{Ai(u)Ai'(v) - Ai'(u)Ai(v) = (u-v)\cdot\int_0^\infty d\lambda \, Ai(u+\lambda)Ai(v+\lambda)}.$$

$$= \int_0^\infty d\lambda \,(u+\lambda) Ai(u+\lambda) Ai(v+\lambda) - \int_0^\infty d\lambda \,(v+\lambda) Ai(v+\lambda) Ai(u+\lambda)$$

$$\underset{Ai''(x) = x\cdot Ai(x)}{=} \int_0^\infty d\lambda \, Ai''(u+\lambda) Ai(v+\lambda) - \int_0^\infty d\lambda \, Ai(u+\lambda) Ai''(v+\lambda)$$

$$\underset{\int \text{by parts}}{=} Ai(v+\lambda)Ai'(u+\lambda)\Big|_0^\infty - Ai(u+\lambda)Ai'(v+\lambda)\Big|_0^\infty$$

$$- \cancel{\int_0^\infty d\lambda \, Ai'(u+\lambda) Ai'(v+\lambda)} + \cancel{\int_0^\infty d\lambda \, Ai'(u+\lambda) Ai'(v+\lambda)}$$

$$= -Ai'(u)Ai(v) + Ai(u)Ai'(v) \quad \#$$

② $\mathcal{A}^2 = \mathcal{A}$: $\int_0^\infty d\mu \int_0^\infty d\lambda \, Ai(u+\lambda) \underbrace{\int_{\mathbb{R}} dz \, Ai(\lambda+z) Ai(\mu+z)}_{= \delta(\mu-\lambda) \text{ (completeness relation)}} \cdot Ai(v+\mu) =$

$$= \int_0^\infty d\lambda \, Ai(u+\lambda) Ai(v+\lambda).$$

③ $H = -\frac{d^2}{dx^2} + x$ is the Airy operator (regarded as self-adjoint on $L^2(\mathbb{R})$).

. Generalized eigenfunctions : $\underline{f_\lambda(x) \doteq Ai(x-\lambda)} \Rightarrow \underline{H f_\lambda = \lambda \cdot f_\lambda}$. Then, the Airy Kernel is the spectral projection onto $\{H \leq 0\}$:

$$\mathcal{A}(u,v) = \int_{\mathbb{R}_-} d\mu \, Ai(u-\mu) Ai(v-\mu).$$

④ $\mathcal{A}$ is locally trace-class and $\| P_s \mathcal{A} P_s \| < 1$, $\forall s > -\infty$ (defines a det pp.).

6.2). Tracy-Widom distribution on a fixed Hilbert space.

. A convenient way for the next part, is to consider $F_2(s)$ as a Fredholm determinant on a fixed Hilbert space, $L^2(\mathbb{R}_+, dx)$, but with kernel depending on "s".

• Define the kernel $\underline{B_s(u,v) \doteq Ai(u+v+s)}$, then

$$F_2(s) = \det(\mathbb{1} - B_s^2)_{L^2(\mathbb{R}_+, dx)}$$

In fact, $B_s^2(u,v) = \int_0^\infty d\lambda \, B_s(u,\lambda) B_s(\lambda,v) = \int_0^\infty d\lambda \, Ai(u+\lambda+s) Ai(v+\lambda+s)$.

$$\Rightarrow \det(\mathbb{1} - B_s^2)_{L^2(\mathbb{R}_+, dx)} \underset{\boxed{\substack{u+s \to u \\ v+s \to v}}}{=} \det(\mathbb{1} - \mathcal{A})_{L^2((s,\infty), dx)} = F_2(s).$$

. Remark : ① B_s is not a positive operator.

② B_s is Hilbert-Schmidt on $L^2(\mathbb{R}_+, dx)$, $\forall s > -\infty$ (easy computation)

$\Rightarrow$ ③ B_s^2 is Trace-Class on $L^2(\mathbb{R}_+, dx)$, $\forall s > -\infty$.

. To check ② it is enough to consider the simple bound:

$Ai(x) \leq e^{-x}$, $x \in \mathbb{R}$.

. A few properties of the (GUE)-Tracy-Widom distribution:

mean	variance	Skewness	Kurtosis
$-1,77109$	$0,8132$	$0,224$	$0,094$

where: $\text{mean} = \mathbb{E}(X)$

$\text{variance} = \mathbb{E}(X-\mathbb{E}(X))^2$: spread of distribution

$\text{skewness} = \dfrac{\mathbb{E}(X-\mathbb{E}(X))^3}{[\mathbb{E}((X-\mathbb{E}(X))^2)]^{3/2}}$; degree of asymmetry (0 for Gaussian)

$\text{Kurtosis} = \dfrac{\mathbb{E}(X-\mathbb{E}(X))^4}{[\mathbb{E}((X-\mathbb{E}(X))^2)]^2} - 3$: degree to which a distribution is peaked/flat (0 for Gaussian).

. $\ln(F_2'(s)) \sim -\frac{4}{3} \cdot s^{3/2}$, $s \gg 1$ and $\ln(F_2'(s)) \sim -\frac{1}{12} \cdot |s|^3$, $s \ll -1$.

6.3) Tracy-Widom distribution and Painlevé-II equation.

. Theorem [Tracy-Widom]: $F_2(s) = \exp\left(-\int_s^\infty (x-s)\, q^2(x)\, dx\right)$,

where $q(x)$ is the unique solution of the Painlevé-II equation $q''(x) = s \cdot q(x) + 2 \cdot q^3(x)$ satisfying the asymptotic condition $q(s) \simeq Ai(s)$ as $s \to +\infty$.

. Proof: . Define K_s on $L^2(\mathbb{R}_+, dx)$ by $K_s = B_s^2$. We also have $\|K_s\| < 1$.

. Write $A_s(x) \doteq Ai(x+s)$, then,

$$\frac{\partial}{\partial s} K_s(x,y) = \frac{\partial}{\partial s}\left(\int_0^\infty d\lambda\, Ai(x+\lambda+s)\, Ai(y+\lambda+s)\right)$$

$$= \int_0^\infty d\lambda\, Ai'(x+\lambda+s)\, Ai(y+\lambda+s) + \int_0^\infty d\lambda\, Ai(x+\lambda+s)\, Ai'(y+\lambda+s)$$

$$= Ai(x+\lambda+s) Ai(y+\lambda+s)\Big|_0^\infty - \cancel{\int_0^\infty d\lambda\, Ai(x+\lambda+s) Ai'(y+\lambda+s)} + \cancel{\text{"}}$$

$$= -A_s(x) A_s(y).$$

In bra-ket notations: $\frac{\partial}{\partial s} K_s = -|A_s\rangle\langle A_s|$. (eq.1)

. Let $u(s) = \frac{\partial}{\partial s} \ln\left(\det(\mathbb{1} - K_s)\right)$. (eq.2)

By the identity : $\underline{\det(\mathbb{1}+A) = \exp\left(\mathrm{Tr}(\ln(\mathbb{1}+A))\right)}$ $(\|A\|<1)$

we get $\underline{u(s)} = \frac{\partial}{\partial s} \mathrm{Tr}(\ln(\mathbb{1} - K_s))$

$= -\mathrm{Tr}\left((\mathbb{1}-K_s)^{-1} \cdot \frac{\partial}{\partial s} K_s\right)$: used cyclicity of the trace

$= \mathrm{Tr}\left((\mathbb{1}-K_s)^{-1} |A_s\rangle\langle A_s|\right)$: rank-one operator

$= \underline{\langle A_s|(\mathbb{1}-K_s)^{-1} A_s\rangle}$ (eq.3)

. Another expression for $u(s)$ is

$$\underline{u(s)} = \frac{\partial}{\partial s} \sum_{n\geq 1} \frac{-1}{n} \cdot \mathrm{Tr}(K_s^n)$$

$$= -\sum_{n\geq 1} \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \mathrm{Tr}\left(K_s^{n-1} \cdot \frac{\partial}{\partial s} K_s\right) = \sum_{n\geq 1} \mathrm{Tr}\left(K_s^{n-1} |A_s\rangle\langle A_s|\right)$$

$$= \sum_{n\geq 1} \langle A_s| K_s^{n-1} |A_s\rangle$$

$|A_s\rangle = B_s|\delta_0\rangle$, $K_s = B_s^2$ $\Big\}$ $= \sum_{n\geq 1} \langle \delta_0 | K_s^n \delta_0\rangle = \underline{\langle \delta_0 | K_s(\mathbb{1}-K_s)^{-1} \delta_0\rangle}$ (eq.4)

$\underline{\text{. Define}}$:
$$\begin{cases} \cdot\ q(s) = \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s\rangle \\ \cdot\ p(s) = \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s'\rangle \\ \cdot\ v(s) = \langle A_s | (\mathbb{1}-K_s)^{-1} A_s'\rangle \end{cases}$$

<u>Lemma</u> : (a) $\frac{\partial u(s)}{\partial s} = -q^2(s)$

(b) $q^2(s) = u^2(s) - 2 \cdot v(s)$

(c) $\frac{\partial q(s)}{\partial s} = p(s) - q(s) \cdot u(s)$

(d) $\frac{\partial p(s)}{\partial s} = s \cdot q(s) - 2q(s)v(s) + p(s)u(s)$

Proof of the lemma: (a) $\frac{\partial u(s)}{\partial s} = \frac{\partial}{\partial s} \langle \delta_0 | (\mathbb{1}-K_s)^{-1} \delta_0 \rangle$

$$\overset{*}{=} \langle \delta_0 | (\mathbb{1}-K_s)^{-1} \frac{\partial K_s}{\partial s} (\mathbb{1}-K_s)^{-1} \delta_0 \rangle$$

$$\overset{(eq.1)}{=} -\left(\langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle\right)^2 = -q^2(s) .$$

(*): use the identity: $\frac{d}{d\alpha}(\mathbb{1}-K)^{-1} = (\mathbb{1}-K)^{-1} \frac{dK}{d\alpha} (\mathbb{1}-K)^{-1}$ (eq.5)
for any operator with $\|K\|<1$ depending on a parameter α.
[can be easily be proven by writing the Neumann series].

(b) We compute $\frac{\partial u(s)}{\partial s}$ using the representation (eq.3).

$$\frac{\partial u(s)}{\partial s} \underset{\substack{(eq.5)\\(eq.3)}}{=} 2 \cdot \langle A_s | (\mathbb{1}-K_s)^{-1} A_s' \rangle - \langle A_s | (\mathbb{1}-K_s)^{-1} A_s \rangle \langle A_s | (\mathbb{1}-K_s)^{-1} A_s \rangle$$

$$= \underline{2 \cdot v(s) - u(s)^2 \overset{(a)}{=} -q^2(s)}$$

(c) $$\frac{\partial q(s)}{\partial s} \underset{\substack{(eq.5)\\(eq.3)}}{=} \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s' \rangle - \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle \langle A_s | (\mathbb{1}-K_s)^{-1} A_s \rangle$$

$$= p(s) - q(s) \cdot u(s) .$$

(d) For this identity we will also need:

(eq.6): $[L, (\mathbb{1}-K)^{-1}] = (\mathbb{1}-K)^{-1} [L, K] (\mathbb{1}-K)^{-1}$

(eq.7): $[Q, K_s] = |A_s\rangle\langle A_s'| - |A_s'\rangle\langle A_s|$,
where Q is the multiplication operator of the position.

- $$\frac{\partial p(s)}{\partial s} = - \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle \langle A_s | (\mathbb{1}-K_s)^{-1} A_s' \rangle + \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s'' \rangle .$$

- Using $Ai''(x+s) = (x+s) Ai(x+s)$, we obtain

$$A_s'' = (Q+s) A_s$$

$$\Rightarrow \langle \delta_0 | (\mathbb{1} - K_s)^{-1} A''_s \rangle = s \cdot \langle \delta_0 | (\mathbb{1} - K_s)^{-1} A_s \rangle + \langle \delta_0 | (\mathbb{1} - K_s)^{-1} Q A_s \rangle$$

$$= s \cdot q(s) + \underbrace{\langle \delta_0 | Q (\mathbb{1} - K_s)^{-1} A_s \rangle}_{\equiv 0} - \langle \delta_0 | [Q, (\mathbb{1} - K_s)^{-1}] A_s \rangle$$

(eq.6) with $L=Q$ and (eq.7)

$$\Downarrow$$

$$= s \cdot q(s) - \langle \delta_0 | (\mathbb{1} - K_s)^{-1} A_s \rangle \langle A'_s | (\mathbb{1} - K_s)^{-1} A_s \rangle + \langle \delta_0 | (\mathbb{1} - K_s)^{-1} A'_s \rangle \langle A_s | (\mathbb{1} - K_s)^{-1} A_s \rangle$$

$$= s \cdot q(s) - q(s) \cdot v(s) + p(s) \cdot u(s)$$

$$\Rightarrow \frac{\partial p(s)}{\partial s} = s \cdot q(s) - 2 \cdot q(s) v(s) + p(s) u(s) .$$ # of Lemma.

. By Lemma, we have:

$$\frac{d^2 q(s)}{ds^2} = p'(s) - q'(s) u(s) - q(s) u'(s)$$

$$= s \cdot q(s) - 2 q(s) v(s) + \cancel{p(s) u(s)} - \cancel{p(s) u(s)} + q(s) u^2(s) + q^3(s)$$

$$= s \cdot q(s) + q(s) \cdot [\underbrace{q^2(s) + u^2(s) - 2 v(s)}_{= 2 \cdot q^2(s)}]$$

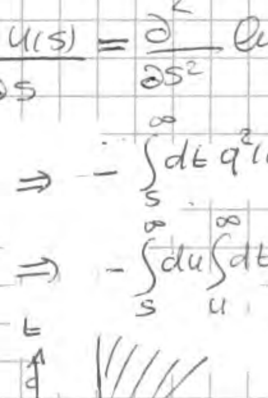
. Moreover, as $s \to \infty$, $(\mathbb{1} - K_s)^{-1} \to \mathbb{1}$, thus $q(s) \to Ai(s)$.

. To get the final formula, we need to integrate twice:

$$\cdot \frac{\partial u(s)}{\partial s} = \frac{\partial^2}{\partial s^2} \ln \left(F_2(s) \right) = - q^2(s) .$$

$$\Rightarrow - \int_s^\infty dt \, q^2(t) = \int_s^\infty dt \frac{d^2}{dt^2} (\ln F_2(t)) = \frac{d}{dt} \ln F_2(t) \Big|_s^\infty = - \frac{d}{ds} \ln F_2(s)$$

$$\Rightarrow - \int_s^\infty du \int_u^\infty dt \, q^2(t) = - \int_s^\infty du \frac{d}{du} \ln F_2(u) = - \ln F_2(u) \Big|_s^\infty = \ln F_2(s)$$



$$\hookrightarrow = - \int_s^\infty dt \, q^2(t) \cdot \int_s^t du = - \int_s^\infty dt \, (t-s) q^2(t)$$

$$\Rightarrow F_2(s) = \exp \left[- \int_s^\infty dt \, (t-s) q^2(t) \right]$$ #