5.2.4) Proof of the theorems 1 - 4.

① **Proof of theorem 2:** . One starts by noticing that $|\Lambda^k(A)| = \Lambda^k(|A|)$,

since $A = U|A|$ (polar decomposition).

. Thus, $\Lambda^k(A)$ has singular values $\mu_{i_1}(A)\cdots\mu_{i_k}(A)$ with $i_1 < \cdots < i_k$. Therefore, $\|\Lambda^k(A)\|_1 \left(= \mathrm{Tr}(|A|)\right) = \sum_{i_1<\cdots<i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$.

is finite and bounded by $\sum_{i_1,\dots,i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A) = \frac{\|A\|_1^k}{k!}$.

. This bound implies that the series $\sum_{k\geq 0}^{\infty} z^k \cdot \mathrm{Tr}(\Lambda^k(A))$ is entire and the bound $\exp(|z|\cdot\|A\|_1)$ too.

. Also, $\|\Lambda^k(A)\|_1 = \sum_{i_1<\cdots<i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$ implies that

$$|\det(\mathbb{1}+zA)| \leq \sum_{k=0}^{\infty} |z|^k \cdot \underbrace{\mathrm{Tr}(|\Lambda^k(A)|)}_{=\|\Lambda^k(A)\|_1} = \prod_{k\geq 1}\left(1+|z|\cdot\mu_k(A)\right).$$

since all terms are positive

. Let us choose N s.t. $\sum_{k\geq N+1} \mu_k(A) \leq \frac{\varepsilon}{2}$, then

since $(1+x)\leq e^x$, $|\det(\mathbb{1}+zA)| \leq \underbrace{\left(\prod_{k=1}^{N}(1+|z|\mu_k(A))\right)}_{\leq C_\varepsilon \cdot e^{|z|\frac{\varepsilon}{2}}} \cdot e^{|z|\frac{\varepsilon}{2}} \leq C_\varepsilon \cdot e^{\varepsilon\cdot|z|}$. #

② **Proof of theorem 3:** Denote $F(A) = \det(\mathbb{1}+A)$. The function $g(z) = F(C+zD)$ is analytic in z since $\mathrm{Tr}\left[\Lambda^k(C+zD)\right]$ is a polynomial of degree k in z. Define $g(z) = F\left[\frac{A+B}{2} + z\cdot(A-B)\right]$. Then,

$$|F(A)-F(B)| = |g(\tfrac{1}{2}) - g(-\tfrac{1}{2})| \leq \sup_{-\frac{1}{2}\leq t\leq\frac{1}{2}} |g'(t)|$$

$$\leq R^{-1}\cdot \sup_{|z|\leq R+\frac{1}{2}} |g(z)| \quad \text{, by using a Cauchy estimate.}$$

$R^{-1} = \|A-B\|_1$

$$= \|A-B\|_1 \cdot \sup_{|z|\leq\frac{1}{2}+\|A-B\|_1^{-1}} \left|\det\left(\mathbb{1} + \frac{A+B}{2} + z\cdot(A-B)\right)\right|$$

$$\leq \|A-B\|_1 \underset{\sup_{|z|\leq\cdots}}{\cdot} \exp\left(\left\|\frac{A+B}{2} + z(A-B)\right\|_1\right)$$

$$\leq \|A-B\|_1 \cdot \exp\left(\frac{1}{2}\|A\|_1 + \frac{\|B\|_1}{2} + \frac{1}{2}\|A-B\|_1 + \cancel{\|A-B\|_1^{-1}}\cdot\cancel{\|A-B\|_1}^{\,1}\right). \ \#$$

. The <u>Cauchy</u> estimate used here is the following.

$f'(a) = \frac{1}{2\pi i}\oint dz \frac{f(z)}{z-a}$; $f(z)$ analytic $\Rightarrow$ no pole of $\frac{f(z)}{z-a}$ in any finite region except $z=a$.

$\Rightarrow |f'(a)| \leq \max_{|z|=R+a} \frac{|f(z)|}{R}$. $\boxed{\text{For } a\in[-\frac{1}{2},\frac{1}{2}],\ |f'(a)| \leq \max_{|z|\leq R+\frac{1}{2}} \frac{|f(z)|}{R}.}$

③ <u>Proof of Theorem 4:</u> Part ⓐ is proven by first showing the identity for finite rank operators (it is just the finite dimensional determinant relation $\det(AB)=\det(A)\det(B)$) and then, by continuity (Thm. 3) it follows that the same relation holds for trace-class operators.

ⓑ If $\mathbb{1}+A$ is not invertible, then A has an e.v. at -1 and by ⓒ $\det(\mathbb{1}+A)=0$. If $\mathbb{1}+A$ is invertible, let $B \doteq -A(\mathbb{1}+A)^{-1}$. Thus, $\mathbb{1}+A+B+AB=\mathbb{1}$, so that by (16), $\det(\mathbb{1}+A)\neq 0$.

ⓒ Let P_λ be the spectral projection on λ, eigenvalue of A. Then,

$(P_\lambda A)\cdot((\mathbb{1}-P_\lambda)A)=0$, thus

$$\det(\mathbb{1}+z\cdot A) = \det(\mathbb{1}+z(P_\lambda A)+z\cdot(\mathbb{1}-P_\lambda)A) \overset{ⓐ}{=}$$
$$= \det(\mathbb{1}+z\cdot P_\lambda A)\cdot\det(\mathbb{1}+z(\mathbb{1}-P_\lambda)A)$$

. Let $z_0=-\lambda^{-1} \Rightarrow \mathbb{1}+z_0(\mathbb{1}-P_\lambda)A$ is invertible since $\lambda\notin\sigma((\mathbb{1}-P_\lambda)A)$.

. By the spectral theorem for compact operators, $P_\lambda A$ is a finite rank operator with eigenvalue λ of multiplicity, say n, finite. Thus,

$$\det(\mathbb{1}+z\cdot P_\lambda A) \underset{\substack{\uparrow \\ \text{nice basis}}}{=} \det\left(\mathbb{1}+z\cdot\begin{pmatrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{pmatrix}\right) = (1+\lambda\cdot z)^n = \left(1-\frac{z}{z_0}\right)^n.$$

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④ <u>Proof of Theorem 1:</u> Let $f(z)=\det(\mathbb{1}+z\cdot A)$. By Thm. 4, it has zeros at points $z_n=-\lambda_n^{-1}$. By $\sum_{n=1}^{M}|\lambda_n|\leq\sum_{n=1}^{M}\mu_n$, for $M\to\infty$, we get $\sum_{n\geq 1}|\lambda_n| = \sum_{n\geq 1}\frac{1}{|z_n|}<\infty$, since A is trace-class.
Also, $f(0)=1$ and by Thm. 2, $|f(z)|\leq C_\varepsilon\cdot\exp(\varepsilon\cdot|z|)$. Thus we can apply Thm. 5 which gives $\det(\mathbb{1}+zA)=\prod_n(1+z\lambda_n(A))$. If one checks the first term of expansion in z, one gets: $\mathrm{Tr}(A)=\sum_n\lambda_n(A)$.

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