Functional analysis appendix.

Here we collect some analysis background. The Hilbert spaces are complex (and separable) with $\langle \psi, a\varphi_1 + b\varphi_2 \rangle = a\langle \psi, \varphi_1\rangle + b\langle \psi, \varphi_2\rangle$

$\langle a\psi_1 + b\psi_2, \varphi\rangle = \bar{a}\langle \psi_1, \varphi\rangle + \bar{b}\langle \psi_2, \varphi\rangle$

- An operator A is positive iff $\langle \phi, A\phi\rangle \geq 0$, $\forall \phi$ (Notation: $A \geq 0$).
- If $A \geq 0$, then $A = A^*$.
- If $A \geq 0$, then $\exists!\ B \geq 0$ s.t. $B^2 = A$. Then we write $B = \sqrt{A}$.
- For any bounded operator A, $A^*A \geq 0$ and one define $|A| \doteq \sqrt{A^*A}$.
- Polar decomposition: Given an operator A, there exists an unique U s.t.
  - ⓐ $A = U \cdot |A|$
  - ⓑ $\|U\psi\| = 0$ for $\psi \in \mathrm{Ker}\,A$
  - ⓒ $\|U\psi\| = \|\psi\|$ for $\psi \in (\mathrm{Ker}\,A)^\perp$
- Norm: $\|A\| = \sup_{\|\varphi\|_2 = 1} \|A\varphi\|_2 \approx$ the largest e.v..

- A bounded operator A is called finite rank if $\dim(\mathrm{Im}(A)) < \infty$, and $\dim(\mathrm{Im}(A))$ is the rank of A.
- A bounded operator is compact iff it is a limit of a finite rank operator (limit $\equiv$ norm limit).

- Spectral theorem: Let A be a compact operator on a Hilbert space $\mathcal{H}$. Then:
  - ⓐ $\sigma(A)$, the spectrum of A, is a discrete set with at most zero as accumulation point,
  - ⓑ all $\lambda \in \sigma(A)\setminus\{0\}$ is an eigenvalue with finite (algebraic ~~geometric~~) multipicity
  - ⓒ $\forall \lambda \in \sigma(A)\setminus\{0\}$, $\exists$ finite rank projection P_λ s.t.
    $AP_\lambda = P_\lambda A$, $\sigma(A|_{P_\lambda \mathcal{H}}) = \{\lambda\}$, $\sigma(A|_{(\mathbb{1}-P_\lambda)\mathcal{H}}) = \sigma(A)\setminus\{\lambda\}$.
    $\mathrm{Dim}(P_\lambda)$ is the (geometric ~~algebraic~~) multiplicity of λ.

- Theorem (o.n. basis): Let A be a self-adjoint compact operator on $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis of eigenvectors for A.

  (Hilbert space)

<u>Remark</u>: Just to remind the difference between <u>algebraic</u> and <u>geometric</u> multiplicity, consider a finite rank operator A.

. Take its Jordan bloc decomposition J_A, then the algebraic multiplicity is the order of the zero of the characteristic polynomial $\det(\lambda \mathbb{1} - A) = 0$, and is equal to the number of times that "λ" appears on the diagonal of J_A.

. The geometric multiplicity of "λ" is the number of blocs with eigenvalue "λ" $\equiv \operatorname{Ker}(A - \lambda \mathbb{1})$.

. A is diagonalizable <u>iff</u> $\text{geom}(\lambda) = \text{algebr}(\lambda)$, $\forall \lambda$.

. <u>Theorem (canonical expansions)</u>:

Let A be a compact operator. Then A has a norm-convergent expansion

$$A = \sum_{n \geq 1} \mu_n(A) |\psi_n\rangle\langle\phi_n| ,$$

each $\mu_n(A) > 0$, $\mu_1(A) \geq \mu_2(A) \geq \dots$, and the $\{\phi_n\}_n$, $\{\psi_n\}_n$ are orthonormal sets (not necessarily complete, i.e., not necessarily bases).

. Moreover, $\mu_n(A)$ are uniquely determined and ϕ_n, ψ_n are essentially determined [i.e., up to a change of basis in the subspaces of given μ_n's].

. <u>Notation</u>: <u>$\mu_n(A)$</u> are called <u>singular values of A</u>.

. The $\mu_n(A)$ are the non-zero eigenvalues of $|A|$.

. <u>Weyl's inequality</u>: For any compact operator A and for all n,

$$\sum_{k=1}^{n} |\lambda_k(A)| \leq \sum_{k=1}^{n} \mu_n(A),$$

where $\lambda_n(A)$ are ordered satisfying $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \dots$ (counted up to their algebraic multiplicity).

Proof of the canonical expansion

Using the polar decomposition, we have $|A| = U^* A$. Since $|A|$ is compact and self-adjoint, by the o.n. basis theorem, we have

$$|A| = \sum_{n\geq 1} \mu_n(A) |\phi_n\rangle\langle\phi_n|, \qquad (\mu_n(A) > 0)$$

where $\mu_n(A)$ are the nonzero e.v. of $|A|$ and ϕ_n the associated $\vec{e.v.}$. Since U is an isometry on $\mathrm{Im}(|A|)$, $\psi_n \doteq U\phi_n$ are orthonormal.

The uniqueness of $\mu_n(A)$ is due to the fact that $\mu_n(A)^2$ are the non-zero e.v. of A^*A and A^*A is positive. $\{\phi_n\}$ are the $\vec{e.v.}$ of A^*A and $\{\psi_n\}$ are the $\vec{e.v.}$ of AA^*. Those vectors are determined uniquely, up to basis transformation in the subspaces of given μ_n's. #

Trace-class operators.

Theorem A: Let $\mathcal{H}$ be a separable Hilbert space and $\{\varphi_n\}_{n\geq 1}$ an orthonormal basis of $\mathcal{H}$. Then, for any positive operator $A \in \mathcal{L}(\mathcal{H})$,

(1) $$\mathrm{Tr}(A) \doteq \sum_{n\geq 1} \langle \varphi_n, A\varphi_n\rangle,$$

is <u>independent of basis</u>. $\mathrm{Tr}(A)$ is called the <u>trace of A</u>.

Definition: An - operator A is called <u>trace-class</u> iff $\mathrm{Tr}(|A|) < \infty$. The family of all trace-class operators is denoted by $\mathfrak{J}_1$.

Theorem B: $\mathfrak{J}_1$ is a $*$-ideal in $\mathcal{L}(\mathcal{H})$, i.e.,

(a) $\mathfrak{J}_1$ is a vector space,

(b) If $A \in \mathfrak{J}_1$ and $B \in \mathcal{L}(\mathcal{H})$, then $AB \in \mathfrak{J}_1$ and $B A \in \mathfrak{J}_1$.

(c) If $A \in \mathfrak{J}_1$, then $A^* \in \mathfrak{J}_1$

Remarks: • $\|\cdot\|_1$ norm: let $\|\cdot\|_1$ be defined on $\mathcal{J}_1$ by $\|A\|_1 \doteq \mathrm{Tr}(|A|)$. Then $\mathcal{J}_1$ is a Banach space with norm $\|\cdot\|_1$ and $\|A\| \leq \|A\|_1$.

• $A \mapsto \mathrm{Tr}(A)$ is a linear bounded functional on $\mathcal{J}_1$ and for $A \in \mathcal{J}_1$, $B \in \mathcal{L}(\mathcal{H})$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

• Every $A \in \mathcal{J}_1$ is compact.

Theorem C: let K be a continuous, positive function on $M \times M$. If $\int_M K(x,x)\,d\mu(x) < \infty$, then $\exists!$ operator $A \in \mathcal{J}_1$ s.t. $(A\phi)(x) = \int K(x,y)\phi(y)\,d\mu(y)$ and $\|A\|_1 = \int K(x,x)\,d\mu(x)$.

. A simple case when an operator A is trace-class, is when it is a product of two Hilbert-Schmidt operators.

. Hilbert-Schmidt operators.

Definition: An operator $A \in \mathcal{L}(\mathcal{H})$ is called Hilbert-Schmidt iff $\mathrm{Tr}(A^*A) < \infty$. The family of Hilbert-Schmidt operators is denoted by $\mathcal{J}_2$.

Theorem D: $\mathcal{J}_2$ is a $*$-ideal.

Theorem E: let $\mathcal{H} = L^2(M, d\mu)$. If $A \in \mathcal{J}_2$, then there exists a unique function $K \in L^2(M \times M, d\mu \otimes d\mu)$ with

$$(A\phi)(x) = \int K(x,y)\phi(y)\,d\mu(y). \tag{2}$$

. Conversely, any $K \in L^2(M\times M, d\mu \otimes d\mu)$ defines an operator A by (2) which is Hilbert-Schmidt and

$$\|A\|_2 = \|K\|_2 \doteq \left(\int |K(x,y)|^2 d\mu(x)\,d\mu(y) \right)^{1/2}. \tag{3}$$

Remarks:
- $\mathcal{J}_2$ with inner product $\langle \cdot, \cdot \rangle_2$ is an Hilbert Space.
- Every $A \in \mathcal{J}_2$ is compact.
- $A \in \mathcal{J}_1$ iff $A = B \cdot C$ for some $B, C \in \mathcal{J}_2$.

Theorem F: Let B a bounded operator on $\mathcal{H}$ and $\{\phi_n\}$, $\{\psi_n\}$ be orthonormal basis. Then,

$$(4) \qquad \sum_{n=1}^{\infty} \|B\phi_n\|^2 \quad \text{and} \quad \sum_{m,n=1}^{\infty} |\langle \psi_m, A\phi_n \rangle|^2$$

are independent of the basis and are equal. (4) are finite iff $B \in \mathcal{J}_2$ and in that case, $= (4) = \|B\|_2^2$