5) Fredholm determinant

. In the last lecture we saw that the "hole probability" is given by a series. We called it "Fredholm expansion" of an object called "Fredholm determinant".

(1) We had: $$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} dx_1 \dots dx_n \det(K(x_i,x_j))_{1\leq i,j\leq n} =: \det(\mathbb{1}-K)_{L^2(\Lambda,dx)}$$

. There are essentially two point of views:

(a) Take the series in (1) as definition of r.h.s. As soon as the series / integrals are absolutely summable / integrable, everything is well defined.

(b) One can think at (1) as the Fredholm determinant of an operator, with eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq 0$. Then, the Fredholm determinant will be equal to $\prod_{n\geq 0}(1-\lambda_n)$.

Now we describe the two approaches:

5.1) Kernel approach.

. The idea is the following. Assume that we can bound
$|K(x,y)| \leq f(x)\cdot f(y) \cdot C$, with C a finite constant and
that $\int_\Lambda f(x)^2 dx < \infty$.

. Then $$\left|\det(\mathbb{1}-K)_{L^2(\Lambda,dx)}\right| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} dx_1 \dots dx_n \, C^n \prod_{i=1}^{n} f(x_i)^2 \cdot \det\left[\frac{K(x_i,x_j)}{C \cdot f(x_i) f(x_j)}\right]_{1\leq i,j\leq n}$$

. The entries of the $n\times n$ determinant have absolute value less or equal to 1, and by Hadamard's bound, it can be then bounded by $n^{n/2}$.

. From this will follows that $|\det(\mathbb{1}-K)| < \infty$.

. More generally, let us do the above statement precise.

. Consider (M,μ) a measure space and $A(x)$ a positive, continuous function on M satisfying $\frac{1}{A(x)} \in L^2(M,\mu)$. [$\frac{1}{A(x)}$ plays the role of $f(x)$].

. Then one defines thin and thick sets:

→ A measurable set $S \subset M \times M$ is thin if $\forall x_0, y_0 \in M$,

$$\mu(\{x\in M \mid (x,y_0)\in S\})=0,\ \mu(\{y\in M \mid (x_0,y)\in S\})=0,\ \mu(\{x\in M \mid (x,x)\in S\})=0.$$

→ A thick subset of $M\times M$ is a subset of $M\times M$ which is not thin.

Definition: A function $K(x,y)$ on $M\times M$ is a kernel if:

(a) $K(x,y)$ is measurable,

(b) for some thick open subset $U \subset M\times M$, $K(x,y)$ is continuous on U,

(c) $\|K\|_A \doteq \sup\limits_{(x,y)\in M\times M} A(x)A(y)|K(x,y)| < \infty$.

. The class of kernels forms a vector space with the norm $\|\cdot\|_A$.

Define, for any kernel K and $n>0$:

$$\Delta_n(K) \doteq \int_{M^n} d\mu(x_1)\cdots d\mu(x_n)\, \det\left[K(x_i,x_j)\right]_{1\le i,j\le n}, \quad n\geq 1 \tag{2}$$

and $\Delta_0(K) \doteq 1$.

. Hadamard bound: Let T be a $n\times n$ matrix with entries satisfying $|T_{ij}|\le 1$. Then, $|\det(T)| \le n^{n/2}$.

: Lemma: $|\Delta_n(K)| \le C^n \cdot (\|K\|_A)^n \cdot n^{n/2}$ for $C = \|A^{-1}\|_2$.

Proof: $|\Delta_n(K)| \le \int_{M^n} d^n\mu(x)\, \|K\|_A^n \left(\prod_{k=1}^n \frac{1}{A(x_k)^2}\right) \left|\det(\tilde{K}(x_i,x_j))\right|$, $\tilde{K}(x,y) = \frac{K(x,y)A(x)A(y)}{\|K\|_A}$

$$\underset{\text{Hadamard}}{\le} \|K\|_A^n \cdot n^{n/2} \left(\int_M d\mu(x) A(x)^{-2}\right)^n . \#$$

- Using this Lemma, we then define the Fredholm determinant attached to the kernel K by

(3) $$\Delta(K) \doteq \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \Delta_n(K).$$

- Remark: $|\Delta(K)| \leq \sum_{n\geq 0} \|K\|_A^n \cdot \|A^{-1}\|_2^n \cdot \frac{n^{n/2}}{n!} < \infty$, because

$n! \cong n^n e^{-n}$ for large n, thus,

$|\Delta(K)| \leq \text{const.} \sum_{n\geq 1} \frac{\tilde{C}^n}{\sqrt{n!}} < \infty.$

5.2) Operator approach.

- The operator approach is more focused around the spectral properties of the operators. Here we assume some knowledge on functional analysis, see an appendix to this lecture.

- The idea can be understood starting with finite-rank operators, whose closure gives the compact operators. Consider a finite-rank operator K with $\operatorname{rank}(K) = n < \infty$ ($\approx$ a $n\times n$ matrix).
- Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of K.

Then, $\det(\mathbb{1} + K) = \prod_{k=1}^{n} (1+\lambda_k)$.

(4) $$= 1 + \sum_i \lambda_i + \sum_{i<j} \lambda_i \lambda_j + \sum_{i_1<i_2<i_3} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} + \dots + \lambda_1 \cdots \lambda_n.$$

- "Obviously", $|\det(\mathbb{1}+K)| \leq \sum_{k=0}^{n} \left(\sum_{i=1}^{n} |\lambda_i|\right)^k \frac{1}{k!} \leq \exp(\operatorname{Tr}(|K|))$

use: $\sum_i |\lambda_i| \leq \sum_i \mu_i$, where μ_i are the e.v. of $|K|$.

. The question is to give a sense of

$$\det(\mathbb{1}+K) = \prod_{k\geq 1} (1+\lambda_k) \tag{5}$$

also for more general compact operators.

. The previous bound indicates that for trace-class operators it should not be a problem. Indeed, the Fredholm determinant will be well defined for trace-class operators.

. <u>Remark:</u> It is possible to define (5) for operators which are only Hilbert-Schmidt in some cases, but it will not be made here [see, e.g., Simon's book "Trace Ideals and Their Applications].

. Before going into the construction, let us give an example where the operator approach can be useful.

. Consider our example of GUE eigenvalues. Let $\lambda_{max}^{(N)}$ be the largest eigenvalue for $N\times N$ matrices. Then,

$$\mathbb{P}_N\left(\lambda_{max}^{(N)} \leq 2N + s\, N^{1/3}\right) = \det(\mathbb{1}-K_N)_{L^2((s,\infty),dx)}$$

for a kernel K_N.

<u>Question:</u> Does $\lim_{N\to\infty} \det(\mathbb{1}-K_N) = \det(\mathbb{1}-K_\infty)$, where

$$K_\infty = \lim_{N\to\infty} K_N \ ?$$

. <u>Answer:</u> To answer to this question we can apply the bound:

$$\left| \det(\mathbb{1}-K_N) - \det(\mathbb{1}-K_\infty)\right| \leq \| K_N - K_\infty\|_1 \cdot e^{1+\|K_N\|_1+\|K_\infty\|_1} \tag{6}$$

. Thus we need to prove that K_N, K_∞ are trace-class <u>and</u> that $K_N \to K_\infty$ in <u>trace-class norm</u>.

5.2.1) (Antisymmetric) tensor product.

- First we construct the tensor product.

- Let $\mathcal{H}$ be an Hilbert space, then the tensor product of $\mathcal{H}$, n times, denoted by $\otimes^n \mathcal{H}$ is the vector space of multilinear functionals on $\mathcal{H}$:

  (a) for given $\varphi_1, \dots, \varphi_n \in \mathcal{H}$, $\varphi_1 \otimes \dots \otimes \varphi_n \in \otimes^n \mathcal{H}$ satisfies
  $$(\varphi_1 \otimes \dots \otimes \varphi_n)(\eta_1, \dots, \eta_n) = \prod_{k=1}^{n} (\varphi_k, \eta_k), \quad \forall (\eta_1, \dots, \eta_n) \in \mathcal{H} \times \dots \times \mathcal{H}$$

  (b) inner product:
  $$(\varphi_1 \otimes \dots \otimes \varphi_n, \psi_1 \otimes \dots \otimes \psi_n) = \prod_{k=1}^{n} (\varphi_k, \psi_k)$$

  (c) for any operator $A \in \mathcal{L}(\mathcal{H})$, $\exists$ an operator $\Gamma_n(A) \in \mathcal{L}(\otimes^n \mathcal{H})$ with $\Gamma_n(A)(\varphi_1 \otimes \dots \otimes \varphi_n) = A\varphi_1 \otimes \dots \otimes A\varphi_n$.

- Γ_n satisfies $\Gamma_n(A \cdot B) = \Gamma_n(A) \cdot \Gamma_n(B)$.

- Basis: If $\{\phi_k\}_k$ is an orthonormal basis of $\mathcal{H}$, then $\{\phi_{k_1} \otimes \dots \otimes \phi_{k_n}\}_{k_1, \dots, k_n}$ is an orthonormal basis of $\otimes^n \mathcal{H}$.

- The space we are actually looking for is a subspace of $\otimes^n \mathcal{H}$, namely its antisymmetric subspace, $\Lambda^n \mathcal{H}$.

- Let S_n denote the group of permutation of $\{1, \dots, n\}$. Then, for given $\psi_1, \dots, \psi_n \in \mathcal{H}$,

$$\psi_1 \wedge \dots \wedge \psi_n \doteq \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^{|\sigma|} \cdot \psi_{\sigma(1)} \otimes \dots \otimes \psi_{\sigma(n)} \tag{7}$$

belongs to $\Lambda^n \mathcal{H}$.

- Indeed, $\Lambda^n \mathcal{H}$ is spanned by the vectors $\psi_1 \wedge \dots \wedge \psi_n$, when $\psi_1, \dots, \psi_n$ span over $\mathcal{H}$.

- For $n=0$, we define $\Lambda^0 \mathcal{H} = \mathbb{C}$.

Lemma 1: $(\phi_1 \wedge \dots \wedge \phi_n, \psi_1 \wedge \dots \wedge \psi_n) = \det\left((\phi_i, \psi_j)\right)_{1 \leq i,j \leq n}$

Proof: $(\phi_1 \wedge \dots \wedge \phi_n, \psi_1 \wedge \dots \wedge \psi_n) = \frac{1}{n!} \sum_{\sigma, \sigma' \in S_n} (-1)^{|\sigma|} (-1)^{|\sigma'|} \cdot$

$$\cdot \left(\phi_{\sigma(1)} \otimes \dots \otimes \phi_{\sigma(n)}, \psi_{\sigma'(1)} \otimes \dots \otimes \psi_{\sigma'(n)}\right)$$

$$= \frac{1}{n!} \sum_{\sigma, \sigma' \in S_n} (-1)^{|\sigma|} (-1)^{|\sigma'|} \prod_{k=1}^{n} \big(\underbrace{\phi_{\sigma(k)}}_{=j}, \psi_{\sigma'(k)}\big)$$

let $\pi = \sigma' \circ \sigma^{-1}$

$$= \frac{1}{n!} \sum_{\sigma, \pi \in S_n} (-1)^{|\pi|} \prod_{j=1}^{n} (\phi_j, \psi_{\pi(j)}) = \det\left((\phi_i, \psi_j)\right)_{1 \leq i,j \leq n} \quad \#$$

Lemma 2: Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the eigenvalues of A on $\mathcal{H}$, $\dim \mathcal{H} = N < \infty$. Then, $\Lambda^n(A) \equiv \underbrace{A \otimes \dots \otimes A}_{n \text{ times}}$ applied on $\Lambda^n \mathcal{H}$, satisfy

(8) $$\underline{\operatorname{Tr}\left[\Lambda^n(A)\right]} = \sum_{i_1 < \dots < i_n} \underline{\lambda_{i_1} \cdot \dots \cdot \lambda_{i_n}}.$$

In particular, $\operatorname{Tr}(\Lambda^N(A)) = \Lambda^N(A) = \det(A)$; $\Lambda^N \mathcal{H} = \mathbb{C}$.

Proof: Let $\{e_1, \dots, e_N\}$ be a basis of $\mathcal{H}$, e_i eigenvector of A with eigenvalue λ_i. Then, $\{e_{i_1} \wedge \dots \wedge e_{i_n}\}_{i_1 < \dots < i_n}$ is a basis of $\Lambda^n \mathcal{H}$ and $\operatorname{Tr}(\Lambda^n(A)) = \sum_{i_1 < \dots < i_n} (e_{i_1} \wedge \dots \wedge e_{i_n}, A e_{i_1} \wedge \dots \wedge A e_{i_n}) = \text{r.h.s. } (8)$ #

By Lemma 2, comparing with (4), we see the identity

(9) $$\sum_{n=0}^{N} \operatorname{Tr}(\Lambda^n(A)) = \det(\mathbb{1} + A) = \prod_{k=1}^{N} (1 + \lambda_k(A)).$$

The next step is to see that (9) holds also when $N = \infty$, for trace-class operators.

5.2.2) Traces and determinants

. Looking at (9), the first actually non-trivial point is to prove that, if $A \in \mathcal{J}_1$ and $\{\lambda_n(A)\}$ are its eigenvalues, then

(10) $$\sum_n \lambda_n(A) = \mathrm{Tr}(A).$$ [Lidskii's equality]

· The proof of (10) gives at the same time the justification for (9).

. The final theorem is namely:

Theorem 1: For any $A \in \mathcal{J}_1$ (trace-class),

(11) $$\det(\mathbb{1} + z \cdot A) = \prod_n \left(1 + z \cdot \lambda_n(A)\right).$$

. In particular (10) holds.

· The proof of Theorem 1 goes along the following steps.

Theorem 2: If A is trace class on an Hilbert space $\mathcal{H}$, then $\Lambda^k(A)$ is a trace class operator on $\Lambda^k \mathcal{H}$ with

(12) $$\|\Lambda^k(A)\|_1 \leq \frac{\|A\|_1^k}{k!}.$$

· In particular, the series

(13) $$\det(\mathbb{1} + zA) \doteq \sum_{k=0}^{\infty} z^k \, \mathrm{Tr}(\Lambda^k(A)) \qquad (z \in \mathbb{C})$$

defines an entire function satisfying

(14) $$|\det(\mathbb{1} + zA)| \leq \exp(|z| \cdot \|A\|_1).$$

. Moreover, $\forall \varepsilon > 0$, $\exists\, C_\varepsilon < \infty$ s.t. $|\det(\mathbb{1} + zA)| \leq C_\varepsilon \cdot \exp(\varepsilon \cdot |z|)$.

. This theorem is important because tell us that $\mathrm{Tr}(\Lambda^k(A))$ in (9) are well defined. The second step is to prove that (13) is a continuous function, as stated in the following thm. It provides at the same time the criteria cited at page ④.

Theorem 3: $A \mapsto \det(\mathbb{1} + A)$ is a continuous function on $\mathcal{J}_1$. In particular,

(15) $$|\det(\mathbb{1} + A) - \det(\mathbb{1} + B)| \leq \|A - B\|_1 \cdot \exp\left[1 + \|A\|_1 + \|B\|_1\right].$$

. The next theorem is important since it tell us that the series (13) has the right zeros with the right multiplicity.

<u>Theorem 4:</u>
ⓐ For any $A, B \in \mathfrak{I}_1$,

$$\det(\mathbb{1} + A + B + AB) = \det(\mathbb{1}+A) \cdot \det(\mathbb{1}+B) \tag{16}$$

ⓑ For $A \in \mathfrak{I}_1$, $\det(\mathbb{1}+A) \neq 0$ <u>iff</u> $\mathbb{1}+A$ is invertible.

ⓒ $\forall A \in \mathfrak{I}_1$ and $z_0 = -\lambda^{-1}$ with λ an eigenvalue of algebraic multiplicity n, $\det(\mathbb{1} + zA)$ has a zero of order n at $z = z_0$.

. From this theorem, we know that $\det(\mathbb{1}+zA)$ has to be proportional to $\prod_n (1 + z\lambda_n(A))$. One has still to see that it is this expression.

<u>Theorem 5:</u> Let $f(z)$ be an entire function with zeros at $z_1, z_2, \ldots$ (repeated by their multiplicity). Suppose $f(0) = 1$ and $\sum_{n\geq 1} \frac{1}{|z_n|} < \infty$, and that $\forall \varepsilon > 0$, $|f(z)| \leq C_\varepsilon \cdot \exp(\varepsilon \cdot |z|)$.

Then

$$f(z) = \prod_{n \geq 1} \left(1 - \frac{z}{z_n}\right). \tag{17}$$

. The proofs of Theorems 1–4 can be found in [Simon: "Traces Ideals and their Applications" chapter 3.]

. Maybe they will be added for next lecture.

5.2.3) Explicit formulas.

· Suppose that

$$(18) \qquad (Af)(x) = \int_a^b K(x,y) f(y)\, dy$$

on $L^2(a,b)$, with $-\infty < a < b < \infty$ and with K continuous on $[a,b]\times[a,b]$.

· Theorem: Let $A \in \mathcal{J}_1$ as (18). Then

$$(19) \qquad \mathrm{Tr}(A) = \int_a^b K(x,x)\, dx.$$

· Theorem: Let $A \in \mathcal{J}_1$ as (18). Then

$$(20) \qquad \det(\mathbb{1} + A) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n} \quad \text{with } \alpha_n = \int_{[a,b]^n} dx_1 \dots dx_n \det\left[K(x_i,x_j)\right]_{1\le i,j\le n}.$$

· This last expression tell us that the Fredholm determinant defined by (13) can be written in series, which corresponds to the one given in (1).

· Final remark: Sometimes it is either difficult to check or not true that the operator considered is trace-class. Alternative definition for operators which are only Hilbert-Schmidt exist and have the series expansion (20). We do not enter in details here, since we do not plan to use it in this Lecture.