4) Point processes

. One way of thinking at point processes is as measurable mapping from a probability space into a space of point measures. Otherwise stated, a point process is a random point measure.

. Let us first define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

. Let Λ be the one-particle space, typically will be $\mathbb{R}^d, \mathbb{Z}^d$, or $\mathbb{R} \times \{1, \dots, n\}$ (in general: a complete separable metric space).

. Let Ω be the space of locally finite particle configurations, i.e., for each configuration $\xi = (x_i)$, $x_i \in \Lambda$, $i \in \mathbb{N}$, and for all bounded set $B \subset \Lambda$, $\xi(B) \doteq (\# x_i \in B) < \infty$.

. σ-algebra, $\mathcal{F}$: $\forall$ bounded set $B \subset \Lambda$, and for any $n \geq 0$, $C_n^B \doteq \{\xi \in \Omega, \xi(B) = n\}$ is a cylinder set.
Then, $\mathcal{F}$ is the σ-algebra generated by all cylinder sets and we denote by $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$.

. Secondly we define the space of point measures.

. Let $\mathcal{B}(\Lambda)$ be the Borel σ-algebra of Λ. A point measure on Λ is a positive measure ν on $(\Lambda, \mathcal{B}(\Lambda))$ which is a locally finite sum of Dirac measure, i.e.,

$\nu = \sum_{i \in I} \delta_{x_i}$ with $x_i \in \Lambda$, $I \subset \mathbb{N}$, and for any bounded $B \subset \Lambda$, $x_i \in B$ only for a finite number of $i \in I$.

. Denote by $M_p(\Lambda)$ the space of point measures on Λ and $\mathcal{M}_p(\Lambda)$ the σ-algebra generated by the applications $\nu \to \nu(f)$ of $M_p(\Lambda)$ to $\mathbb{N} \cup \{\infty\}$ obtained when f span $\mathcal{B}(\Lambda)$.

<u>Definition (Point process):</u> A point process η on Λ is a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(M_p(\Lambda), \mathcal{M}_p(\Lambda))$.
The probability law of this point process is the image of $\mathbb{P}$ by η.

<u>Remark:</u> For the moment we can have $x_i = x_j$ for $i \neq j$ (multiple point).

<u>Definition:</u> A <u>simple point process</u> is a point process s.t.
$\mathbb{P}(\eta(\{x\}) \leq 1) = 1$.

<u>Remark:</u> A simple point process can be identified with the support of the random point measure.

<u>Examples: ① GUE eigenvalues</u>

- For GUE eigenvalues, $\Lambda = \mathbb{R}$ and $\mathbb{P}$ is the probability coming from the eigenvalues density $\frac{1}{Z_N} \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \cdot \prod_{i=1}^{N} \exp\left(-\frac{\lambda_i^2}{2N}\right)$
- Then, point process η then is given:

(1) $$\eta = \sum_{i=1}^{N} \delta_{\lambda_i}$$

<u>② Poisson point process on $\mathbb{R}^d$ with intensity ρ.</u>

- Take $\Lambda = \mathbb{R}^d$ and $\mathbb{P}$ the probability measure s.t. $\forall B, B' \subset \Lambda$, bounded and $n, n' \geq 0$, $\mathbb{P}(C_n^B) = \frac{(\rho |B|)^n}{n!} \cdot e^{-\rho |B|}$ and if $B \cap B' = \emptyset$, $\mathbb{P}(C_n^B \cap C_{n'}^{B'}) = \mathbb{P}(C_n^B)\mathbb{P}(C_{n'}^{B'})$.
- η is a random point measure on $\mathbb{R}^d$ with intensity ρ completely decorrelated; called Poisson point process.

4.1) Correlation functions and moments

. For a point process η on Λ, the total points (counted with multiplicity) in the support of η in a set $A \subset \Lambda$ is given by

(2) $$\eta(\mathbb{1}_A), \text{ with } \mathbb{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

. For a general function f, we write $\eta(f) = \int_\Lambda d\mu(x) f(x) \eta(x)$.

. In 3.1 we already defined the correlation functions. Now we state the explicit relation to factorial moments.

Lemma: Let $A \subset \Lambda$ a subset, then

(3) $$\int_{A^n} d\mu(x_1) \ldots d\mu(x_n) \, \rho^{(n)}(x_1, \ldots, x_n) = \mathbb{E}\left(\frac{\eta(\mathbb{1}_A)!}{(\eta(\mathbb{1}_A) - n)!} \right)$$

Proof: . For $n=1$, $\eta(\mathbb{1}_A) = \sum_i \mathbb{1}_{[x_i \in A]} \Rightarrow \mathbb{E}(\eta(\mathbb{1}_A)) = \mathbb{E}(\# x_i \in A) = \int_A \rho^{(1)}(x) d\mu(x)$

. For $n=2$, notice that $\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A) - 1) = \sum_i \mathbb{1}_{[x_i \in A]} \sum_{j \neq i} \mathbb{1}_{[x_j \in A]}$.

In fact, the second sum is irrelevant if $i \notin A$, and when $i \in A$, the second sum is $\eta(\mathbb{1}_A) - 1$.

. For general n, we have: $\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A) - 1) \cdots (\eta(\mathbb{1}_A) - n + 1) =$
$$= \sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_n \neq \{i_1, \ldots, i_{n-1}\}} \left(\prod_{k=1}^n \mathbb{1}_{[x_k \in A]} \right)$$

. Thus, $$\mathbb{E}\left(\eta(\mathbb{1}_A) \cdots (\eta(\mathbb{1}_A) - n + 1) \right) = \int_{A^n} d\mu(x_1) \ldots d\mu(x_n) \cdot \rho^{(n)}(x_1, \ldots, x_n). \quad \#$$

• In particular: . $\mathbb{E}(\eta(\mathbb{1}_A)) = \int_A \rho^{(1)}(x) \, d\mu(x)$

(4)

. $$\mathrm{Var}(\eta(\mathbb{1}_A)) = \int_{A^2} \rho^{(2)}(x, y) d\mu(x) d\mu(y) + \int_A \rho^{(1)}(x) d\mu(x) - \left(\int_A \rho^{(1)}(x) d\mu(x) \right)^2$$

. By the above Lemma, we can compute factorial moments.

If we are interested in the moments, we can use the relation:

$$\mathbb{E}(X^n) = \sum_{k=1}^{n} S(n,k) \cdot \mathbb{E}\left[X(X-1)\cdots(X-k+1)\right] \qquad (5)$$

where $S(n,k)$ are the Stirling number of the second kind.

$S(n,k) = \#$ of ways to partition a set of n objects into k groups.

. They satisfy the recursion relation: $S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$ for $1 \le k < n$, and with $S(n,n) = S(n,1) = 1$.

4.2) Linear statistics

. Correlation functions are important in computing expected values of observables. In particular one can consider linear statistics, i.e., consider random variables of the form

$$\sum_i f(x_i), \qquad (6)$$

for some real function f.

. Define $u(x) = 1 - \exp(f)$. Then,

$$\underline{\mathbb{E}\left[\exp\left(\sum_i f(x_i)\right)\right]} = \mathbb{E}\left(\prod_i (1-u(x_i))\right) = \sum_{n=0}^{\infty} (-1)^n \cdot \mathbb{E}\left(\sum_{i_1<\cdots<i_n} \prod_{k=1}^{n} u(x_{i_k})\right) \qquad (7)$$

$$\underset{\text{symmetry}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathbb{E}\left(\sum_{\substack{i_1,\dots,i_n \\ \text{all different}}} \prod_{k=1}^{n} u(x_{i_k})\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \int_{\Lambda^n} d\mu(x_1)\cdots d\mu(x_n)\, \mathcal{S}^{(n)}(x_1,\dots,x_n) \cdot \prod_{k=1}^{n} u(x_k).$$

4.3) Determinantal point processes.

. The correlation functions for the GUE eigenvalues had the form $\rho^{(n)}(\lambda_1,\dots,\lambda_n) = \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le n}$. This is an example of what it is called a determinantal point process.

Definition: A point process is called determinantal if the n-point correlation functions are given by

$$\rho^{(n)}(x_1,\dots,x_n) = \det\left(K(x_i,x_j)\right)_{1\le i,j\le n},$$

where $K(x,y)$ is a kernel of an integral operator $K: L^2(\Lambda, d\mu) \to L^2(\Lambda, d\mu)$, non-negative and locally trace-class.

. Remarks: (1) Positivity is required because the n-pt. correlation functions are positive.

(2) Locally trace-class is related to the locally finite number of points ($\leftrightarrow$ point measures).

. Theorem (Soshnikov; Macchi). In the case of Hermitian K: K defines a determinantal point process iff $0 \le K \le 1$.

If the corresponding point process exists, then it is unique.

. For non-hermitian kernels, such a classification is not yet known.

Remarks: (1) The probability that the number of particle is finite or infinite is either 0 or 1, depending on whether $\mathrm{Tr}(K)$ is finite or infinite.

(2) A determinantal point process is simple.

(3) The number of particles is n with probability 1 iff K is an orthogonal projector with $\mathrm{rank}(K) = n$. (like in our GUE example).

. For a determinantal point process, (7) becomes

$$(8) \qquad \mathbb{E}\left(\prod_i \left(1-u(x_i)\right)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det\left(K(x_i,x_j)\right)_{1\le i,j\le n} \cdot \left(\prod_{i=1}^{n} u(x_i)\, d\mu(x_i)\right)$$

$$\equiv \det(\mathbb{1} - uK)_{L^2(\Lambda, d\mu)},$$

where for each $\varphi \in L^2(\Lambda, d\mu)$,

$$(9) \qquad [(uK)\varphi](x) \equiv \int_{\Lambda} u(x) K(x,y) \varphi(y)\, d\mu(y).$$

. The determinant in r.h.s. of (8), is called the Fredholm determinant of the operator uK on the space $L^2(\Lambda, d\mu)$. The series is the Fredholm series. We will discuss later Fredholm determinants.

4.3.1) An application: Hole probability

. Compute the hole probability, i.e., the probability of not having particles in a subset B of Λ:

$$(10) \qquad \mathbb{P}\left(\eta(\mathbb{1}_B)=0\right) = \mathbb{E}\left(\prod_i (1-\mathbb{1}_B(x_i))\right) = \det(\mathbb{1}-K)_{L^2(B,d\mu)}$$

. In particular, for a determinantal point process on $\mathbb{R}$ or $\mathbb{Z}$, which has a last particle, at position x_{max}, its distribution is given by

$$(11) \qquad \mathbb{P}(x_{max} \le s) = \mathbb{P}\left(\eta(\mathbb{1}_{(s,\infty)})=0\right) = \det(\mathbb{1}-K)_{L^2((s,\infty),d\mu)}$$

4.3.2) When a measure defines a determinantal point process?

. We have seen that for GUE eigenvalues, the measure

$$\frac{1}{Z_N}\left(\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N}\right)^2 \cdot \prod_{i=1}^{N} d\mu(\lambda_i), \quad d\mu(\lambda_i) = e^{-\lambda_i^2/2N} d\lambda_i$$

induces a determinantal point process. This is a particular case of the following theorem.

. <u>Theorem (Borodin; Tracy-Widom for GUE):</u> A measure of the form

(12) $$\frac{1}{Z_N} \det\left(\phi_i(x_k)\right)_{1\le i,k\le N} \cdot \det\left(\psi_i(x_k)\right)_{1\le i,k\le N} \cdot d\mu(x_1)\cdots d\mu(x_N), \quad Z_N \neq 0,$$

defines a determinantal point process with kernel

(13) $$K_N(x,y) = \sum_{i=1}^{N} \psi_i(x)\,[A^{-1}]_{ij}\,\phi_j(y),$$

where $A_{ij} = \int_\Lambda \psi_j(s)\,\phi_i(s)\,d\mu(s)$.

<u>Remark:</u> . One has an explicit formula, but in general to obtain the inverse of A explicitly for large N is not an easy task. Typically one will try first to do a change of basis such that A^{-1} is easy, e.g., if $A = \mathbb{1}$. This is what we made implicitly in the case of GUE matrices, when we introduced the orthogonal polynomials.

. <u>Proof of the theorem:</u> The basic strategy is identical to the GUE case. The difference is that now we have two different set of functions $\{\phi_i\}$ and $\{\psi_i\}$.

. <u>Notations:</u> $\langle a|b\rangle \equiv \int_\Lambda d\mu(x)\, a(x)\, b(x)$

$|b\rangle$ is a vector with components $\langle x|b\rangle = b(x)$

$\langle a|$ is a covector " " $\langle a|y\rangle = a(y)$.

. Suppose that we can find functions $\xi_k(x), \eta_k(x)$, $k=1,\dots,N$ such that

$$\mathrm{span}(\{\xi_k\}) = \mathrm{span}(\{\phi_k\}); \quad \mathrm{span}(\{\eta_k\}) = \mathrm{span}(\{\psi_k\})$$

and such that $\langle \xi_k|\eta_\ell\rangle = \delta_{k,\ell}$.

Then, $(12) = \text{const} \times \det(\xi_i(x_j)) \cdot \det(\eta_i(x_j))\, d^N\mu(x)$

$$\overset{\text{as for GUE}}{=} \text{const.} \det\Big(\underbrace{\sum_{k=1}^{N} \eta_k(x_i)\,\xi_k(x_j)}_{=K_N(x_i,x_j)}\Big)\, d^N\mu(x)$$

. The orthogonal relation $\langle \xi_k, \eta_\ell \rangle = \delta_{k,\ell}$ implies

$$\int_\Lambda d\mu(x)\, K_N(x,x) = N \quad ; \quad \int_\Lambda d\mu(z)\, K_N(x,z)\, K_N(z,y) = K_N(x,y)$$

as (8) and (9) for GUE. By the same argument as GUE we then get

$$\rho^{(n)}(x_1, \dots, x_n) = \det\big(K_N(x_i, x_j)\big)_{1\le i,j\le N}.$$

. With the bra and ket notations,

$$K_N = \sum_{k=1}^{N} |\eta_k\rangle\langle \xi_k| .$$

. Let S and T be the matrices of change of basis:

$$\phi_i = \sum_j S_{ij}\, \xi_j \quad \text{and} \quad \psi_i = \sum_j T_{ij}\, \eta_j .$$

. Thus, $K_N(x,y) = \sum_{k=1}^{N} \eta_k(x)\,\xi_k(y) = \sum_{k=1}^{N} \sum_{i,j=1}^{N} (T^{-1})_{ki}\, \psi_i(x) \cdot (S^{-1})_{kj}\, \phi_j(y)$

$$= \sum_{i,j=1}^{N} \psi_i(x)\,\phi_j(y) \cdot \underbrace{\sum_{k=1}^{N} (T^{-1})_{ki}\, (S^{-1})_{kj}}_{=\sum_k (T^t)^{-1}_{ik} (S^{-1})_{kj} = (S\cdot T^t)^{-1}_{ij}}$$

Let $S \cdot T^t = A$ and compute

$$\langle \psi_j | \phi_i \rangle = \sum_{k,\ell} T_{jk} \cdot S_{i\ell} \underbrace{\langle \eta_k | \xi_\ell \rangle}_{=\delta_{k,\ell}} = \sum_k T_{jk} \cdot S_{ik} = (S \cdot T^t)_{ij} .$$

. Thus, $A_{ij} = \langle \psi_j | \phi_i \rangle$. #

Remark: A determinantal point process with kernel $K(x,y)$ is the same as the one with kernel $\tilde{K}(x,y) = \frac{f(x)}{f(y)} K(x,y)$ for some f with $f(x) \neq 0, \forall x \in \Lambda$.

. We say that K and $\tilde{K}$ are conjugate kernels.