Exercices: (1) Consider the GUE ensemble and let

$$E_N(L) = \#\text{ eigenvalues in } [-L, L].$$

Take the $N \to \infty$ limit first, $E(L) \doteq \lim_{N\to\infty} E_N(L)$.

Prove that: $$\lim_{L\to\infty} \frac{\mathrm{Var}(E(L))}{\ln(L)} = \frac{1}{\pi^2}$$

(2) Do the same but for a poisson point process on $\mathbb{R}^d$ with density ρ (box $[0,L]^d$).

(3) Compute the probability of not having particles in a box $[0,L]^d$ for a point process on $\mathbb{R}^d$, intensity ρ.

Solutions of exercises:

1. $$\mathrm{Var}(E_N(L)) = \int_{-L}^{L} dx\, \rho_N^{(1)}(x) + \int_{-L}^{L} dx \int_{-L}^{L} dy\, \rho_N^{(2)}(x,y) - \left(\int_{-L}^{L} dx\, \rho_N^{(1)}(x)\right)^2$$

where $\rho_N^{(1)}(x) = K_N(x,x)$, $\rho_N^{(2)}(x,y) = K_N(x,x)K_N(y,y) - K_N(x,y)K_N(y,x)$.

- When $N \to \infty$, our kernel K_N converges to the sine kernel with density $\rho = \pi^{-1}$.

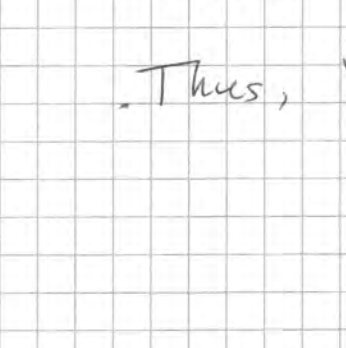
Thus, $$\mathrm{Var}(E(L)) = \int_{-L}^{L} dx\, \rho + \int_{-L}^{L} dx \int_{-L}^{L} dy \left[\rho^2 - \frac{\sin^2(\pi\rho(x-y))}{\pi^2 (x-y)^2}\right] - \left(\int_{-L}^{L} dx\, \rho\right)^2$$

$$= 2L\rho - \cancel{(2L\rho)^2} + \cancel{(2L)^2\rho^2} - \frac{1}{\pi^2} \int_{-L}^{L} dx \int_{-L}^{L} dy\, \frac{\sin^2(\pi(x-y)\rho)}{(x-y)^2}$$

Change of variable:

$$\begin{cases} \mu_1 = x-y \\ \mu_2 = x+y \end{cases}$$

$$= 2\rho L - \frac{4}{\pi^2} \int_0^{2L} \frac{d\mu_1}{2} \cdot \frac{\sin^2(\pi\rho\mu_1)}{\mu_1^2} (2L - \mu_1)$$



$$= 2\rho L - \frac{4}{\pi^2} \cdot L \underbrace{\int_0^{2L} d\mu_1 \frac{\sin^2(\pi\rho\mu_1)}{\mu_1^2}}_{\overset{\text{Maple}}{=} \frac{\pi^2\rho}{2} + O\left(\frac{1}{L}\right)} + \frac{2}{\pi^2} \underbrace{\int_0^{2L} d\mu_1 \frac{\sin^2(\pi\rho\mu_1)}{\mu_1}}_{\overset{\text{Maple}}{=} \frac{1}{2}\ln(2L\pi\rho) + O(1)}$$

$$= \frac{1}{\pi^2} \ln(2L\rho\pi) + O(1)$$

Thus, $$\lim_{L\to\infty} \frac{\mathrm{Var}(E(L))}{\ln L} = \frac{1}{\pi^2}. \quad \#$$

(2) For a point process with density ϱ on $\mathbb{R}^d$,

$$\varrho^{(n)}(x_1,\dots,x_n) = \prod_{k=1}^{n} \varrho(x_k) = \varrho^n \quad (\text{if } \varrho \text{ is constant}).$$

Thus, $\underline{\mathrm{Var}(E(L))} = \int_0^L dx_1 \dots \int_0^L dx_d\, \varrho + \int_{[0,L]^d} d\vec{x} \int_{[0,L]^d} d\vec{y}\, \varrho^2 - \left(\int_{[0,L]^d} d\vec{x}\, \varrho\right)^2$

$$= \varrho \cdot L^d + \varrho^2 \cdot L^{2d} - (\varrho L^d)^2$$

$$\underline{= \varrho \cdot L^d.}$$

$\Rightarrow$ Variance of the number of point in a Poisson point process grows like the Volume of the considered region.

(3) We have to compute a hole probability.
Let $\Lambda = [0,L]^d$, then

$$\mathbb{P}(\Lambda \text{ is empty}) = \mathbb{E}\left(\prod_i (1 - \mathbb{1}_\Lambda(x_i))\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} dx_1 \dots dx_n\, \varrho^{(n)}(x_1,\dots,x_n)$$

$$\overset{\text{Poisson point process}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varrho^n \cdot |\Lambda|^n = \exp(-\varrho|\Lambda|)$$

$$= \exp(-\varrho \cdot L^d). \quad \#$$