3.3) Universal limits: Sine kernel (bulk) and Airy kernel (edge).

. From (17) of the last lecture, we have

$$K_N(x,y) = N \cdot e^{-\frac{x^2}{4N}} \cdot e^{-\frac{y^2}{4N}} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y} \tag{18}$$

with $q_k(x) = \frac{1}{(2\pi N)^{1/4}} \cdot \frac{1}{\sqrt{2^k k!}} \cdot P^H_k(x/\sqrt{2N})$, P^H_k the Hermite polynomials.

3.3.1) Wigner semicircle law.

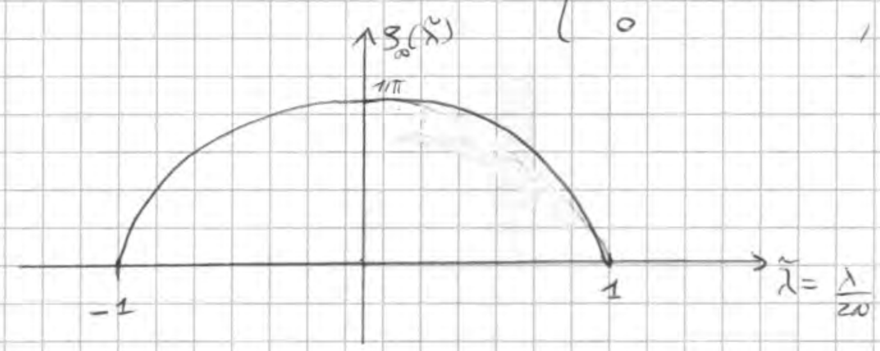
. The largest eigenvalue is close to $2N$ and the smallest eigenvalue close to $-2N$.

. Since there are exactly N eigenvalues, the density of eigenvalues between $-2N$ and $2N$ is of order one also as $N \to \infty$.

. The eigenvalue density can be written in term of the kernel:

$$\underline{\rho_N(\tilde{\lambda}) = K_N(2N\tilde{\lambda}, 2N\tilde{\lambda})} \tag{19}$$

. Computations gives: $\lim_{N\to\infty} \rho_N(\tilde{\lambda}) = \begin{cases} \frac{1}{\pi} \cdot \sqrt{1-\tilde{\lambda}^2}, & \tilde{\lambda} \in [-1,1], \\ 0, & \tilde{\lambda} \notin (-1,1). \end{cases}$



. This is called (Wigner) semicircle law.

. Next we analyze the limit kernel in the bulk and at the edge. The semicircle law can be seen from the bulk limit.

3.3.2) Limit Kernels (universal)

. There are two limit kernels which appear in a lot of models in an appropriate asymptotic limit (large time or thermodynamic limit or large N matrices). The Sine Kernel arises in the bulk of the system, while the Airy kernel at the edge.

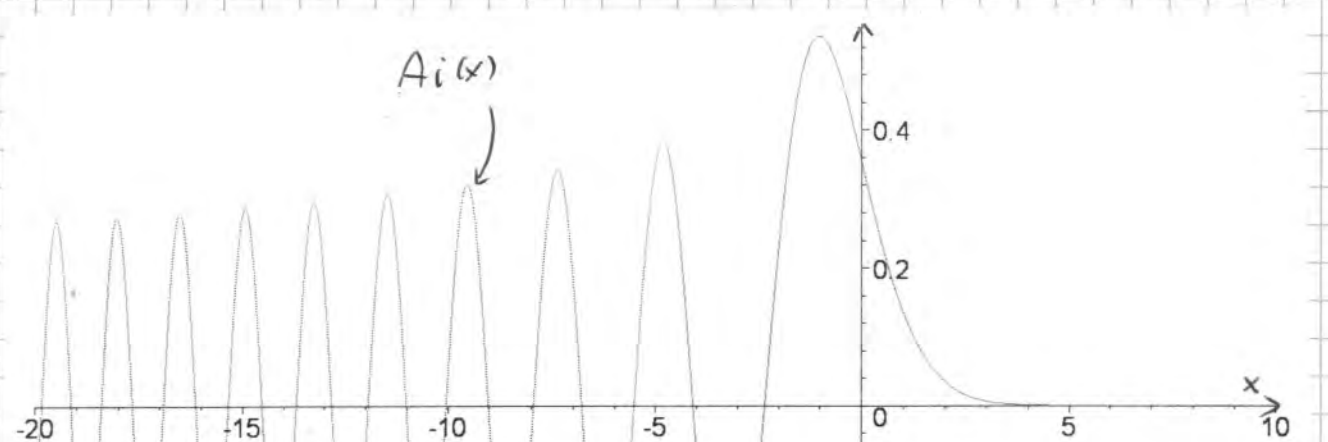
Definition: The sine kernel is defined as $S(x,y) = \dfrac{\sin(\pi(x-y))}{\pi(x-y)}$.

This describes the bulk with density of eigenvalues normalized to one.

Definition: The Airy kernel is defined as $\mathcal{A}(x,y) = \dfrac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$,

where Ai is the Airy function, i.e. solution of

$$y''(x) = x \cdot y(x) \quad , \quad y(x) \sim \frac{\exp\left(-\frac{2}{3}x^{3/2}\right)}{2\sqrt{\pi}\, x^{1/4}} \text{ as } x \to +\infty$$



3.3.3) Bulk scaling limit.

. Let us consider a $\tilde{\lambda} \in (-1,1)$ and focus around $2N \cdot \tilde{\lambda}$ as follows:

$$(20) \qquad \begin{cases} x = 2N \cdot \tilde{\lambda} + \xi_1 \\ y = 2N \cdot \tilde{\lambda} + \xi_2 \end{cases}$$

<u>Proposition:</u> $$\lim_{N \to \infty} K_N(2N\tilde{\lambda} + \xi_1, 2N\tilde{\lambda} + \xi_2) = \frac{\sin(\pi \cdot \rho(\tilde{\lambda})(\xi_1 - \xi_2))}{\pi(\xi_1 - \xi_2)} \equiv \rho(\tilde{\lambda}) \cdot S'(\rho(\tilde{\lambda})\xi_1, \rho(\tilde{\lambda})\xi_2)$$

with $\rho(\tilde{\lambda}) = \frac{1}{\pi}\sqrt{1 - \tilde{\lambda}^2}$

. <u>Sketch of the proof:</u> There are known asymptotic expansions of the Hermite polynomials [see e.g. Szegö; Abramowitz-Stegun].

These, translated into our q_n's becomes:

$$(21) \qquad \sqrt{N} \cdot q_{N-h}(x) \cdot e^{-\frac{x^2}{4N}} \underset{\substack{\uparrow \\ x = 2N\tilde{\lambda} + \xi}}{\cong} \frac{1}{\pi \cdot \sqrt{\rho(\tilde{\lambda})}} \cdot \sin(\alpha_0 N + \pi\rho(\tilde{\lambda})\xi + \tilde{\alpha} h),$$

with $\tilde{\alpha} = \arccos(\tilde{\lambda})$.

One uses (21) in (18) and gets

$$(22) \qquad K_N(2N\tilde{\lambda} + \xi_1, 2N\tilde{\lambda} + \xi_2) \cong \frac{1}{\pi^2 \rho(\tilde{\lambda})} \frac{1}{\xi_1 - \xi_2} \cdot \Big[\sin(\alpha_0 N + \pi\rho \cdot \xi_1)\sin(\alpha_0 N + \pi\rho \cdot \xi_2 + \tilde{\alpha}) - \sin(\alpha_0 N + \pi\rho\xi_1 + \tilde{\alpha})\sin(\alpha_0 N + \pi\rho\xi_2)\Big]$$

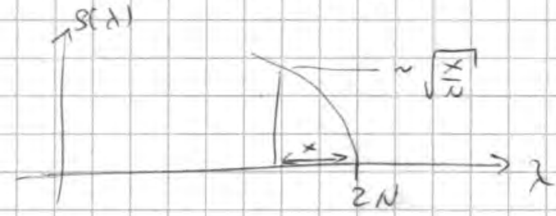
Then apply the identity: $\sin(a)\sin(b+\alpha) - \sin(a+\alpha)\sin(b) = \sin(\alpha)\sin(a-b)$

and $\sin(\arccos(x)) = \sqrt{1-x^2}$, to get the result. #

3.3.4) Edge scaling limit.

- The largest eigenvalue is around $2N$ and it fluctuates on a $N^{1/3}$ scale. The $1/3$ exponent is a consequence of the square root behavior of the density at the edge of the spectrum.

- Heuristics:



$$\#\text{ eigenvalues} \geq 2N-x \approx N\cdot\left(\frac{x}{N}\right)^{3/2} = \frac{x^{3/2}}{\sqrt{N}} \longleftrightarrow \text{over a distance } x.$$

$\Rightarrow$ # eigenvalues $\geq 2N-x$ is $\mathcal{O}(1)$, for $x \sim N^{1/3}$, i.e., the top eigenvalues fluctuates over distances $\mathcal{O}(N^{1/3})$.

- Therefore, the scaling limit is as follows:

$$(23) \qquad \begin{cases} X = 2N + \xi_1 N^{1/3} \\ Y = 2N + \xi_2 N^{1/3} \end{cases}$$

- Proposition: $\displaystyle\lim_{N\to\infty} N^{1/3} K_N(2N+\xi_1 N^{1/3}, 2N+\xi_2 N^{1/3}) = \mathcal{A}(\xi_1,\xi_2)$.

- Remark: The $N^{1/3}$ factor is because of the spacial rescaling (remember that the n-pt. correlation functions, which are densities, are given in terms of K_N).

- Sketch of the proof: One uses asymptotics of Hermite polynomials, which rewrites:
$$N^{1/3} q_{N-h}(x)\cdot e^{-\frac{x^2}{4N}} \underset{x=2N+\xi N^{1/3}}{\cong} Ai(\xi + N^{-1/3}\cdot(h-\tfrac{1}{2})).$$
By replacing it into (18) and taking the $N\to\infty$ limit we get the result. #

Remark: As stated the convergence is pointwise, but by using, for example, double integral representations for the Hermite kernel, one can get some uniform convergence on bounded sets for the bulk and for sets bounded from below for the edge.