Exercices: 1) Prove properties (8) and (9) on the kernel $K_N$.

2) Prove the Cauchy-Binet formula (11).

3) Prove the Christoffel-Darboux formula (14).

Solutions to the exercices:

1. $$K_N(x,y) = \sqrt{w(x)}\sqrt{w(y)} \sum_{k=0}^{N-1} q_k(x) q_k(y)$$

$$\Rightarrow \int_{\mathbb{R}} dx\, K_N(x,y) = \sum_{k=0}^{N-1} \int_{\mathbb{R}} dx\, w(x)\, q_k(x)^2 \underset{(5)}{=} \sum_{k=0}^{N-1} 1 = N$$

Thus (8) is proven.

$$\int_{\mathbb{R}} dz\, K_N(x,z) K_N(z,y) = \sum_{k,l=0}^{N-1} \sqrt{w(x)}\sqrt{w(y)}\, q_k(x) q_l(y) \cdot \underbrace{\int_{\mathbb{R}} dz\, w(z)\, q_k(z) q_l(z)}_{=\delta_{k,l} \text{ by } (5)}$$

$$= \sum_{k=0}^{N-1} \sqrt{w(x)}\sqrt{w(y)}\, q_k(x) q_k(y) = K_N(x,y),$$

which is property (9).

2. We start with l.h.s. of (11):

$$\det\left[\int_{\Lambda} d\lambda(x)\, \Phi_i(x) \Psi_j(x)\right]_{1\le i,j\le N} \overset{\text{linearity}}{=} \int_{\Lambda^N} d\lambda(x_1)\ldots d\lambda(x_N) \det\left[\Phi_i(x_i)\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\text{linearity}}{=} \int_{\Lambda^N} d\lambda(x_1)\ldots d\lambda(x_N) \left[\prod_{i=1}^N \Phi_i(x_i)\right] \cdot \det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\forall \text{ permutation } \sigma\in S_N}{=} \int_{\Lambda^N} d\lambda(x_1)\ldots d\lambda(x_N) \left[\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\right] \cdot \det\left[\Psi_j(x_{\sigma(i)})\right]_{1\le i,j\le N}$$

$$\overset{\text{antisymmetry of determinant}}{=} \int_{\Lambda^N} d\lambda(x_1)\ldots d\lambda(x_N)\, \mathrm{sgn}(\sigma) \cdot \left[\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\right] \cdot \det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\text{integral indep. of } \sigma}{=} \frac{1}{N!} \int_{\Lambda^N} d\lambda(x_1)\ldots d\lambda(x_N) \underbrace{\sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \cdot \left[\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\right]}_{\overset{\text{def of the determinant}}{=} \det\left[\Phi_i(x_j)\right]_{1\le i,j\le N}} \cdot \det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

#

3 . First we prove the three term relation.

- $\dfrac{q_N(x)}{u_N} - \dfrac{x\cdot q_{N-1}(x)}{u_{N-1}}$ is a polynomial of degree $N-1$,

thus $\dfrac{q_N(x)}{u_N} = \dfrac{x\cdot q_{N-1}(x)}{u_{N-1}} + \sum_{k=0}^{N-1} \alpha_k \cdot q_k(x)$,

with $\alpha_k = \left\langle \dfrac{q_N}{u_N} - \dfrac{x\cdot q_{N-1}}{u_{N-1}}, q_k \right\rangle$, where the scalar product is defined as: $\langle a,b\rangle = \int_{\mathbb{R}} dx\, w(x)\, a(x)\, b(x)$.

- For $k=0,\dots,N-3$, $\alpha_k = 0$. In fact, we use

$\langle x\cdot a, b\rangle = \langle a, xb\rangle$, to see that

$$\alpha_k = \frac{1}{u_N}\underbrace{\langle q_N, q_k\rangle}_{=0} - \frac{1}{u_{N-1}}\cdot\langle q_{N-1}, \underbrace{x\cdot q_k}_{\substack{\text{polynomial,}\\ \text{degree } k+1}}\rangle \quad = 0 \text{ for } k+1<N-1$$

- For $k=N-2$:

$$\alpha_{N-2} = -\frac{1}{u_{N-1}}\langle q_{N-1}, x\cdot q_{N-2}\rangle$$

and we can write $x\,q_{N-2} = u_{N-2}\, x^{N-1} + \mathrm{poly}_{N-2}(x)$

$$= \frac{u_{N-2}}{u_{N-1}}\cdot q_{N-1}(x) + \mathrm{poly}'_{N-2}(x)$$

$$= -\frac{1}{u_{N-1}}\cdot\frac{u_{N-2}}{u_{N-1}}.$$

- From this, by setting $B_N = \alpha_{N-1}\cdot u_N$, $A_N = \dfrac{u_N}{u_{N-1}}$, $C_N = \dfrac{u_N u_{N-2}}{(u_{N-1})^2}$,

we get: $\underline{q_N(x) = (A_N x + B_N)\cdot q_{N-1}(x) - C_N\cdot q_{N-2}(x).}$ (*)

- The second step is to use (*) to show

$$q_{N+1}(x)q_N(y) - q_N(x)q_{N+1}(y) = (x-y)\, q_N(x) q_N(y)\cdot A_{N+1} + C_{N+1}\cdot\left[q_N(x)q_{N-1}(y) - q_{N-1}(x)q_N(y)\right]$$

- $q_{N+1}(x)\, q_N(y) - q_N(x)\, q_{N+1}(y) =$

$$= \left[(A_{N+1}\, x + \cancel{B_{N+1}})\, q_N(x) - C_{N+1}\, q_{N-1}(x)\right] \cdot q_N(y)$$

$$- q_N(x) \cdot \left[(A_{N+1}\, y + \cancel{B_{N+1}})\, q_N(y) - C_{N+1} \cdot q_{N-1}(y)\right]$$

$$= A_{N+1} \cdot (x-y) \cdot q_N(x)\, q_N(y) + C_{N+1} \cdot \left[q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)\right]$$

- Divide this identity by $\underline{(x-y)\, A_{N+1}}$, and get:

$$\frac{u_N}{u_{N+1}} \cdot \frac{q_{N+1}(x)\, q_N(y) - q_N(x)\, q_{N+1}(y)}{x-y} = q_N(x)\, q_N(y) + \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}$$

$\Rightarrow$ <u>For $k \geq 1$:</u> $$q_k(x)\, q_k(y) = \frac{u_k}{u_{k+1}} \, \frac{q_{k+1}(x)\, q_k(y) - q_k(x)\, q_{k+1}(y)}{x-y} - \frac{u_{k-1}}{u_k} \cdot \frac{q_k(x)\, q_{k-1}(y) - q_{k-1}(x)\, q_k(y)}{x-y}$$

$$\Rightarrow \sum_{k=0}^{N-1} q_k(x)\, q_k(y) = \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}.$$

<u>we use:</u> $k=0$: $q_0(x)\, q_0(y) = u_0^2 = \frac{u_0}{u_1} \cdot \frac{q_1(x)\, q_0(y) - q_0(x)\, q_1(y)}{x-y}$ since $q_0(x) = u_0$, $q_1(x) = u_1 x + \alpha$.

- For $x = y$, one does just take the limit of the previous formula as $x \to y$.

#