3) n-point correlation functions for the GUE eigenvalues

In the previous lecture we determined the joint distribution of the $N\times N$ GUE eigenvalues, $\{\lambda_1, \dots, \lambda_N\}$:

(1) $$P(\lambda_1, \dots, \lambda_N)\, d\lambda_1 \dots d\lambda_N = \frac{1}{Z_N} \left(\prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 \right) \prod_{k=1}^{N} \left(e^{-\frac{\lambda_k^2}{2N}} d\lambda_k \right)$$

In this lecture we want to obtain the expression of the n-point correlation functions, which we first have to define.

3.1) Correlation functions.

. Consider a measure on $\mathbb{R}^N$ like (1), and let us take any bounded disjoints Borel sets $A_1, \dots, A_n$ of $\mathbb{R}$. Then, let

$$M_n(A_1, \dots, A_n) \doteq \mathbb{E}\left(\prod_{i=1}^{n} (\#\text{ eigenvalues in } A_i) \right),$$

where $\mathbb{E}$ is the expectation under the measure on $\mathbb{R}^N$ (e.g., (1) in our case).

Definition: (Correlation functions) If M_n is absolutely continuous with respect to a reference measure on $\mathbb{R}^N$, μ^N, i.e.,

(2) $$M_n(A_1, \dots, A_n) = \int_{A_1\times\dots\times A_n} d\mu(x_1) \dots d\mu(x_n)\, \rho^{(n)}(x_1, \dots, x_n)$$

for all Borel sets A_i in $\mathbb{R}$,

then we call $\rho^{(n)}(x_1, \dots, x_n)$ the n-point correlation function.

. Remarks: (1) From (2) one can think of $\rho^{(n)}$ as Radon-Nikodym derivative on $\mathbb{R}^n$ of the measure M_n.

(2) $\rho^{(n)}(x_1, \dots, x_n) = \rho^{(n)}(x_{\sigma_1}, \dots, x_{\sigma_n})$, $\forall \sigma \in S_n$ (permutations of $\{1, \dots, n\}$. This symmetry is obvious since l.h.s. of (2) is independent of the order of the A_i's.

(3) We will probably come back later, when we will discuss point processes in general, to the question whether the set of all correlation functions, $\{\rho^{(n)}, n \geq 1\}$ defines uniquely a measure. One not too strong but sufficient condition will be : $\rho^{(n)}(x_1,..,x_n) \leq n^{2n} \cdot c^n$ a.s. for some $c>0$.

(4) To speak about n-point correlation functions without specifying the reference measure makes no sense, although when the reference measure is the Lebesgue measure one tends not to specify it.

In our GUE example, we might choose $d\mu_1(x) = e^{-\frac{x^2}{2N}}dx$ and get the n-point correlation functions $\rho_1^{(n)}(x_1,..,x_n)$, but also $d\mu_2(x) = dx$ and get $\rho_2^{(n)}(x_1,..,x_n)$. Then,

$$\rho_1^{(n)}(x_1,..,x_n) = \rho_2^{(n)}(x_1,..,x_n) \cdot \prod_{i=1}^{N} e^{-\frac{x_i^2}{2N}}.$$

(5) Probabilistic interpretation: In the case where a.s. one does not have double points (like in our GUE case, or more generally for simple point processes, see further lectures), then we have the following probabilistic interpretation.

Let $[x_i, x_i+\Delta x_i]$, $i=1,..,n$ be disjoints infinitesimally small sets, then we will have at most one point in each $[x_i,x_i+\Delta x_i]$, and so

$$\rho^{(n)}(x_1,..,x_n) = \lim_{\substack{\Delta x_i \to 0 \\ i=1,..,n}} \frac{\mathbb{P}(\text{one particle in each } [x_i,x_i+\Delta x_i], 1\leq i\leq n)}{\Delta x_1 \cdots \Delta x_n}$$

In particular: $\rho^{(1)}(x)$ is the density of particles (points, eigenvalues) at position x.

(6) If instead of $\mathbb{R}$ we have $\mathbb{Z}$ or any other discrete sets, then $\rho^{(n)}(x_1,\dots,x_n)$ is the probability of finding particles at $x_1,\dots,x_n$.

<u>Lemma</u>: • A particular situation is when, like in our GUE case, $P_N(x_1,\dots,x_N)$ is a <u>symmetric</u> probability measure on $\mathbb{R}^N$.

Then

$$(3) \qquad \rho^{(n)}(x_1,\dots,x_n) = \frac{N!}{(N-n)!} \cdot \int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N \, P_N(x_1,\dots,x_N)$$

<u>Proof</u>: $\rho^{(n)}(x_1,\dots,x_n)$ is the probability density of finding <u>a</u> particle at x_1, <u>a</u> particle at $x_2,\dots,$ <u>a</u> particle at x_n, but it does not keep the information of which of the N particle is at which of the x_i's.
Using the symmetry of $P_N(x_1,\dots,x_N)$, each possible choice gives a contribution equal to

$$\int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N \, P_N(x_1,\dots,x_N),$$

and there are $n! \cdot \binom{N}{n} = \frac{N!}{(N-n)!}$ possible choices. #

3.2) Application to GUE

• We will now apply (3) to our measure (1), but before we rewrite (1) in another form.

3.2.1) Orthogonal polynomials.

• Let $w(x) = e^{-\frac{x^2}{2N}}$ and define the orthogonal polynomials $\{q_k(x), k=0,\dots,N-1\}$ by the following conditions:

① $q_k(x)$ is a polynomial of degree k, with

(4) $$q_k(x) = u_k \cdot x^k + \dots \quad , \quad u_k > 0,$$

② and they satisfy the orthogonality condition:

(5) $$\int_{\mathbb{R}} dx \; w(x) \cdot q_k(x)\, q_\ell(x) = \delta_{k,\ell}.$$

3.2.2) Kernel K_N.

. Then, the measure (1) can be rewritten using the polynomials $q_k(x)$ (which will be computed at the end of this lecture), namely,

since $\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N} = \text{const} \times \det\left(q_{j-1}(\lambda_i)\right)_{1\le i,j\le N}$,

(6) $$P(\lambda_1, \dots, \lambda_N) = \frac{1}{\tilde{Z}_N} \cdot \left(\prod_{k=1}^{N} w(\lambda_k)\right) \cdot \left(\det\left(q_{i-1}(\lambda_j)\right)_{1\le i,j\le N}\right)^2.$$

$$= \frac{1}{\tilde{Z}_N} \left(\prod_{k=1}^{N} w(\lambda_k)\right) \cdot \det\left(\sum_{k=1}^{N} q_{k-1}(\lambda_i)\, q_{k-1}(\lambda_j)\right)_{1\le i,j\le N}.$$

matrix multiplication rules

The w's can also be taken into the determinant by using its multilinearity property. We then get:

(7) $$P(\lambda_1, \dots, \lambda_N) = \frac{1}{\tilde{Z}_N} \cdot \det\left[K_N(\lambda_i, \lambda_j)\right]_{1\le i,j\le N},$$

with $$K_N(x, y) = \sqrt{w(x)}\sqrt{w(y)} \cdot \sum_{k=0}^{N-1} q_k(x)\, q_k(y).$$

. Two properties of K_N: (obtained applying (5)).

(8) $$\int_{\mathbb{R}} dx \; K_N(x,x) = N$$

(9) $$\int_{\mathbb{R}} dz \; K_N(x,z)\, K_N(z,y) = K_N(x,y).$$

3.2.3) n-point correlation functions

Lemma: $$\rho^{(n)}(\lambda_1, \dots, \lambda_N) = \det\left[K_N(\lambda_i, \lambda_j)\right]_{1 \le i,j \le n} \qquad (10)$$

Proof: First we determine $\tilde{Z}_N$ in (7).

For $n = N$, $\rho^{(n)}(\lambda_1, \dots, \lambda_N) \overset{(3)}{=} P(\lambda_1, \dots, \lambda_N) \cdot N!$

and: $$\int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N \, \rho^{(n)}(\lambda_1, \dots, \lambda_N) = N!$$

$$\| \, (6)$$

$$\frac{N!}{\tilde{Z}_N} \cdot \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N \; w(\lambda_1) \dots w(\lambda_N) \cdot \det(q_{i-1}(\lambda_j))_{1 \le i,j \le N} \cdot \det(q_{i-1}(\lambda_j))_{1 \le i,j \le N}$$

$$\overset{\circledast}{=} \frac{N!}{\tilde{Z}_N}\left[N! \; \det\underbrace{\left(\int_{\mathbb{R}^N} d\lambda \, w(\lambda) \, q_{i-1}(\lambda) \, q_{j-1}(\lambda) \right)_{1 \le i,j \le N}}_{\overset{(5)}{=} \delta_{ij}} \right] = \frac{(N!)^2}{\tilde{Z}_N}.$$

Thus $\tilde{Z}_N = N!$

In $\circledast$ we used the Cauchy-Binet (or Heine) identity:

$$(11) \quad \boxed{\begin{aligned} &\det\left[\int_{\Lambda} d\lambda(x) \, \phi_i(x) \, \psi_j(x)\right]_{1 \le i,j \le N} = \\ &= \frac{1}{N!} \cdot \int_{\Lambda^N} \det(\phi_i(x_j))_{1 \le i,j \le N} \cdot \det(\psi_i(x_j))_{1 \le i,j \le N} \, d\lambda(x_1) \dots d\lambda(x_N) \end{aligned}}$$

- Now, by (3), (7) we have

$$(12) \quad \rho^{(n)}(\lambda_1, \dots, \lambda_n) = \frac{1}{(N-n)!} \cdot \int_{\mathbb{R}^{N-n}} d\lambda_{n+1} \dots d\lambda_N \; \det\left[K_N(\lambda_i, \lambda_j)\right]_{1 \le i,j \le N}$$

. We need to integrate $N-n$ times, so consider one of the integral to be done, say the one with the determinant of size $m \times m$.

Then,
$$\int_{\mathbb{R}} dx_m \det\left[K_N(x_i, x_j)\right]_{1 \le i,j \le m} = \,?$$

$$\det\left(K_N(x_i,x_j)\right)_{1\le i,j\le m} = \det \left[\begin{array}{ccc|c} K_N(x_1,x_1) & \cdots & K_N(x_1,x_{m-1}) & K_N(x_1,x_m) \\ \vdots & & \vdots & \vdots \\ K_N(x_{m-1},x_1) & \cdots & K_N(x_{m-1},x_{m-1}) & K_N(x_{m-1},x_m) \\ \hline K_N(x_m,x_1) & \cdots & K_N(x_m,x_{m-1}) & K_N(x_m,x_m) \end{array}\right]$$

$$= K_N(x_m,x_m) \cdot \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} K_N(x_k,x_m) \cdot \det \left[\begin{array}{c} K_N(x_i,x_j) \\ \hline K_N(x_m,x_j) \end{array}\right]_{\substack{1\le i,j\le m-1 \\ \text{and } i\neq k}}$$

$$\underset{\substack{\uparrow \\ \text{by linearity}}}{=} K_N(x_m,x_m) \cdot \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} \cdot \det \left[\begin{array}{c} K_N(x_i,x_j) \\ \hline K_N(x_k,x_m)\cdot K_N(x_m,x_j) \end{array}\right]_{\substack{1\le i,j\le m-1 \\ \text{and } i\neq k}}$$

. We use this decomposition and compute, using (8) and (9),

$$(13) \quad \int_{\mathbb{R}} dx_m \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m} = N \cdot \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} \cdot \det \left[\begin{array}{c} K_N(x_i,x_j) \\ \hline K_N(x_k,x_j) \end{array}\right]_{\substack{1\le i,j\le m-1 \\ \text{and } i\neq k}}$$

$$\overset{\text{reordering the } K_N(x_n, x_j)}{=} [N-(m-1)] \cdot \det [K_N(x_i, x_j)]_{1 \le i,j \le m-1}.$$

. We plug this result into (12) for $m = N, N-1, \dots, n+1$, and get

$$\rho^{(n)}(\lambda_1, \dots, \lambda_n) = \frac{1 \cdot 2 \cdots \cancel{(N-n)}}{\cancel{(N-n)!}} \cdot \det [K_N(\lambda_i, \lambda_j)]_{1 \le i,j \le n}. \quad \#$$

3.2.4) Another representation of the kernel

. Christoffel-Darboux formula: For orthogonal polynomials one has the three term relationship

$$q_n(x) = (A_n x + B_n)\, q_{n-1}(x) - C_n \cdot q_{n-2}(x), \quad n = 2, 3, \dots,$$

with $A_n > 0$, B_n, $C_n > 0$ some constants, which are given in term of the highest coefficient of $q_n(x)$, u_n, by

$$A_n = \frac{u_n}{u_{n-1}}, \quad C_n = \frac{A_n}{A_{n-1}} = \frac{u_n u_{n-2}}{u_{n-1}^2}.$$

Then, (14)

$$\boxed{\sum_{k=0}^{n-1} q_k(x) q_k(y) = \begin{cases} \dfrac{u_{n-1}}{u_n} \cdot \dfrac{q_n(x) \cdot q_{n-1}(y) - q_{n-1}(x)\, q_n(y)}{x-y}, & \text{for } x \neq y, \\[2ex] \dfrac{u_{n-1}}{u_n} \cdot \left[q_n'(x)\, q_{n-1}(x) - q_{n-1}'(x)\, q_n(x) \right], & \text{for } x = y. \end{cases}}$$

. Using the Christoffel-Darboux formula (14) we get for our GUE case, that the kernel writes

$$(15) \qquad K_N(x, y) = \sqrt{w(x) \cdot w(y)} \cdot \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}$$

3.2.5) Explicit kernel for GUE

. In the GUE case, $w(x) = \exp\left(-\frac{x^2}{2N}\right)$, the orthogonal polynomials $q_k(x)$ are given in terms of Hermite polynomials:

$$(16)\quad \begin{cases} P_k^H(x) \doteq e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, \text{ they satisfy} \\ \int_{\mathbb{R}} P_k^H(x) P_\ell^H(x) e^{-x^2} dx = \sqrt{\pi} \cdot 2^k \cdot k! \cdot \delta_{k,\ell} \\ \text{with } P_k^H(x) = 2^k \cdot x^k + \dots \end{cases}$$

. From (16) one gets, up to rescaling $x \to \frac{x}{\sqrt{2N}}$, that (exercice!)

$$(17)\quad \begin{cases} q_k(x) = \frac{1}{\sqrt[4]{2\pi N}} \cdot \frac{1}{\sqrt{2^k \cdot k!}} \cdot P_k^H(x/\sqrt{2N}), \quad \frac{u_{N+1}}{u_N} = N, \text{ thus} \\ K_N(x,y) = N \cdot e^{-\frac{x^2+y^2}{4}} \cdot \frac{q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y)}{x-y} \end{cases}$$

. Since in (17) the Hermite polynomials enters, this kernel is also called the Hermite Kernel.