Exercices: (1) Show the formula for the joint distributions of eigenvalues in the $\beta=2$ and $\beta=4$ cases.

(2) Check the operator identity $CUC=\overline{U}$ where C is the complex conjugation.

(3) Consider the "entropy" functional

$$S(p) = -\int p(H)\ \ln(p(H))\, dH$$

Show that under the condition $\mathbb{E}\left(\frac{Tr(H^2)}{2N}\right)=\frac{n}{2}$, where $n = N + \frac{\beta N(N-1)}{2}$ (degrees of freedom), the measure with

$$p(H) = \frac{1}{Z_N}\exp\left(\frac{-Tr(H^2)}{2N}\right)$$

maximize $S(p)$.

Solutions of the excercices:

1. For $\beta = 2$, the strategy is the same as for $\beta = 1$.
We have also in this case $\delta H = g \cdot \delta \tilde{H} \cdot g^{-1}$ with

$$\delta \tilde{H} = \delta \boldsymbol{\Lambda} + [g^{-1}\delta g, \Lambda].$$

In components: . $\delta \tilde{H}_{i,i} = \delta \lambda_i$ and

for $j > i$. $\delta(\mathrm{Re}\, \tilde{H}_{ij}) = (\lambda_j - \lambda_i) \cdot \delta(\mathrm{Re}\, \Omega_{ij})$

and . $\delta(\mathrm{Im}\, \tilde{H}_{ij}) = (\lambda_j - \lambda_i) \cdot \delta(\mathrm{Im}\, \Omega_{ij})$

. Thus, the Jacobian between $\tilde{H}$ and $(\boldsymbol{\Lambda}, \Omega)$ is

$$J = \det \begin{pmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \lambda_1 - \lambda_2 & & \\ & \ddots & \\ & & \lambda_{N-1} - \lambda_N \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} \lambda_1 - \lambda_2 & & \\ & \ddots & \\ & & \lambda_{N-1} - \lambda_N \end{matrix} \end{pmatrix}$$

$$= \left[\prod_{1 \le i < j \le N}^{N} (\lambda_i - \lambda_j) \right]^2 .$$

. For $\beta = 4$, it is similar. Now, we have instead of only the real and imaginary parts, we have the 4 components of the basis of the quaternionic numbers.

2. Let us see how they apply to functions Ψ:

$$(CUC\psi)(x) = C \cdot \int U(x,y)\, \overline{\psi(y)}\, dy = \int \overline{U}(x,y)\, \psi(y)\, dy$$
$$= (\overline{U}\psi)(x).$$

3 Consider the constraint

$$C' = \int p(H)\,\mathrm{Tr}(H^2)\,dH.$$

~~that~~ Let $B = -(1+\ln(A))$ and λ be the Lagrange multiplier, i.e.,

$$S(p) = -\int p(H)\ln p(H)\,dH - \lambda\cdot\left(\int p(H)\,\mathrm{Tr}(H^2)\,dH - C_1\right) + (\ln A + 1)\left(\int p(H)\,dH - 1\right)$$

. Let p_0 be the distribution which maximize $S(p)$, then at first order in δp,

$$\delta S(p_0) = S(p_0+\delta p) - S(p) = 0, \text{ i.e.,}$$

$$-\cancel{1} - \ln(p_0) - \lambda\,\mathrm{Tr}(H^2) + \cancel{1} + \ln(A) = 0,$$

which implies $p_0(H) = A\cdot\exp(-\lambda\,\mathrm{Tr}(H^2))$.

. The normalization constant fixes the value of A,

$$A^{-1} = \int e^{-\lambda\mathrm{Tr}(H^2)}\,dH \equiv a(\lambda).$$

. Thus, the second constraint writes

$$C' = \frac{1}{a(\lambda)}\int \mathrm{Tr}(H^2)\cdot e^{-\lambda\cdot\mathrm{Tr}(H^2)}\,dH = -\frac{1}{a(\lambda)}\frac{d\,a(\lambda)}{d\lambda}$$

. Let us get $a(\lambda)$ now. By the change of variable $X = \sqrt{\lambda}\,H$, we obtain

$$a(\lambda) = \lambda^{-n/2}\int e^{-\mathrm{Tr}(X^2)}\,dX,$$ where n is the dimension of the space where the integral is made.

. The integral over dX is independent of λ, thus

$\dfrac{da(\lambda)}{d\lambda} = \dfrac{-n}{2\lambda}\cdot a(\lambda)$, and finally $\lambda = \dfrac{n}{2C}$.

Since we want $\lambda = \frac{1}{2N}$, $C = N\cdot n$. This finishes the proof.