2) The classical ensembles of random matrices.

. In the 60's physicists (Dyson, Wigner, ...) considered random matrix ensembles to model statistics of heavy atom spectral measurements. The result are the classical gaussian ensembles of random matrices. The different ensembles corresponds to different intrinsic symmetries of the system (time reversal, rotation symmetry).

. First we introduce the ensembles of random matrices and derive their eigenvalues' distribution. Then we will come back and discuss more in detail the question of symmetries.

2.1) GOE, GUE, GSE.

. Consider the following classes of matrices:

1 $\beta = 1$: H are real symmetric matrices:

. $H = H^t$ i.e. $H_{ij} = H_{ji} \in \mathbb{R}$.

. There are $\frac{N(N+1)}{2}$ independent entries.

. A real symmetric matric can a priori describe a system which is time reversal and, rotation invariant or with integer magnetic momentum.

. These matrices can be diagonalized by an orthogonal transformation.

2 $\beta = 2$: H are complex hermitian matrices.

- $H = H^*$: $H_{ij} = \overline{H_{ji}} \in \mathbb{C}$, or, equivalently,

  $H = H^0 + i H^1$, H^0 real symmetric, H^1 real antisymmetric.
- There are N^2 independent entries.
- A complex hermitian matrix can apriori describe a system which is <u>not</u> time reversal (e.g., in the presence of a magnetic field).
- These matrices can be diagonalized by an <u>unitary</u> transformation.

3 $\beta = 4$: H are real quaternionic matrices.

- What are quaternionic matrices? A simple way to define them is as follows. For $\beta = 1$ each entry was a multiple of "1", for $\beta = 2$ the entries are linear combinations of "1" and "i", and for $\beta = 4$ (the present case), are simply linear combinations of "1", "i", "j", "k", with $i^2 = j^2 = k^2 = -1$, $i \cdot j \cdot k = -1$.
- One way of representing the four basis element of a quaternion number are via 2x2 matrices;

$$"1" \longleftrightarrow e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad "i" \longleftrightarrow e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$"j" \longleftrightarrow e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad "k" \longleftrightarrow e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

<u>Remark:</u> $e_k = i \cdot \sigma_k$, $k = 1, 2, 3$, σ_k the Pauli matrices.

- Then, ~~the~~ matrices which are called quaternionic real are the ones which writes: $H = H^0 e_0 + H^1 e_1 + H^2 e_2 + H^3 e_3$ with H^0 real symmetric, H^1, H^2, H^3 real antisymmetric ~~NxN~~ N matrices.
- There are $N(2N-1)$ independent entries.

. A system which a priori can be described by such matrices have time reversal symmetry but with half-integer (as spins...) magnetic momentum.

. These matrices can be diagonalized by a symplectic unitary transformation.

. Definition: The Gaussian Orthogonal Ensemble (GOE) of random matrices is the set of $N \times N$ real symmetric matrices H with probability measure, P,

$$P(H)\,dH = \frac{1}{Z_N} \cdot \exp\left(-\frac{\mathrm{Tr}(H^2)}{2N}\right) dH, \tag{1}$$

where dH is the flat reference measure:

$$dH = \prod_{1 \le i \le j \le N} dH_{ij},$$

and Z_N a constant.

. Remark: . The name "Orthogonal" comes from the fact that the measure (1) is invariant under orthogonal transformations.
. The factor $\frac{1}{2N}$ is just a normalization, useful at the end of the lectures.

. Definition: The Gaussian Unitary Ensemble (GUE) of random matrices is the set of $N \times N$ complex hermitian matrices H with measure (1), but now

$$dH = \left(\prod_{i=1}^{N} dH_{ii}\right) \cdot \prod_{1 \le i < j \le N} d\,\mathrm{Re}(H_{ij})\, d\,\mathrm{Im}(H_{ij})$$

. Definition: The Gaussian Symplectic Ensemble (GSE) of random matrices is the set of $N \times N$ quaternionic real matrices N ($\equiv$ $2N \times 2N$ matrices if one uses the e_i's) with measure (1) and $dH = \left(\prod_{1 \le i \le j \le N} dH^0_{ij}\right) \prod_{1 \le i < j \le N} \left(dH^1_{ij}\, dH^2_{ij}\, dH^3_{ij}\right)$.

<u>Remarks:</u> . "Unitary" is because of invariance under unitary transformations of the measure (1) with hermitian matrices.

. "Symplectic" is because of invariance under symplectic transformations of (1) with real quaternionic matrices.

. The original definition of the gaussian ensembles is different: $p(H)$ is defined to be invariant under the group of symmetry (orthogonal, $O(N)$, for GOE), i.e., <u>no basis is special</u>. This implies that $p(H)$ is a function of $\mathrm{Tr}(H^k)$, $k=1,\dots,N$ only, i.e., of the N eigenvalues $\lambda_1,\dots,\lambda_N$. The second requirement is that the entries of the matrices are independent random variables (up to symmetries). With these two requirement, it can be proven (see, e.g., Chapter 26 of Mehta's book) that $p(H) \propto \exp\left[-a\mathrm{Tr}(H^2)+b\mathrm{Tr}(H)+c\right]$, $a>0$, $b,c\in\mathbb{R}$.

. c enters in the normalisation, while b can be set to zero by a change in the zero of the energies.

<u>2.2) Eigenvalues' distributions</u>

. Often one is interested in the eigenvalues of the matrix, since are independent of the choice of basis.

<u>Proposition:</u> Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ be the eigenvalues of a Gaussian Ensemble of random matrix (GOE, GUE, or GSE).

Then, the joint distribution on the eigenvalues is given by

$$\underline{p(\lambda)\,d\lambda = \frac{1}{Z_N}\left(\prod_{1\leq i<j\leq N}(\lambda_i-\lambda_j)\right)^{\beta}\cdot\prod_{i=1}^{N} d\mu(\lambda_i)}, \quad d\mu(\lambda_i)=e^{-\frac{\lambda_i^2}{2N}}\,d\lambda_i$$

and Z_N a constant; $\beta=1$ for GOE, $\beta=2$ for GUE, $\beta=4$ for GSE.

<u>Remark:</u> . $\prod_{1\le i<j\le N}(\lambda_i-\lambda_j)$ is called the Vandermonde determinant, and equals, $\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N} =$

$$= \det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{bmatrix}.$$

. Let us prove this Proposition for the orthogonal case, the other two cases can be made as an exercise.

. Given H, $\exists\, g\in O(N)$ s.t. $H = g\cdot\Lambda\cdot g^{-1}$, $\Lambda = \lambda_i\cdot\delta_{ij}$, $i,j=1,\dots,N$.
Since $M=\frac{N(N+1)}{2}$ are the independent entries of H, we have $M-N$ independent variables $p_1,\dots,p_{M-N}$ in the g's.

. We have to compute the Jacobian of the change of variables $\{H_{ij}, 1\le i\le j\le N\}$ to $\{\lambda_1,\dots,\lambda_N; p_1,\dots,p_{M-N}\}$.

. An infinitesimal transformation of H gives

$$\delta H = \delta g\cdot\Lambda\cdot g^{-1} + g\cdot\delta\Lambda\cdot g^{-1} + g\cdot\Lambda\cdot\delta g^{-1}$$

and sing $\quad g g^{-1} = \mathbb{1},\ (\delta g)\cdot g^{-1} = -g\cdot\delta g^{-1}$

$$\Rightarrow \delta H = \delta g\cdot\Lambda\cdot g^{-1} + g\cdot\delta\Lambda\cdot g^{-1} - g\cdot\Lambda\cdot g^{-1}(\delta g) g^{-1}$$

$$= g\left[g^{-1}\delta g\cdot\Lambda - \Lambda\cdot g^{-1}\delta g\right]g^{-1} + g\,\delta\Lambda\cdot g^{-1}$$

$$\Rightarrow \delta H = g\cdot\delta\tilde{H}\cdot g^{-1} \quad\text{with}\quad \delta\tilde{H} = \delta\Lambda + [\underbrace{g^{-1}\delta g}_{\doteq\,\delta\Omega}, \Lambda]$$

$\doteq \delta\Omega$, the "angular" variables

. $H \xrightarrow[\text{Jacobian} \,=\, 1]{} \tilde{H} \xrightarrow[\text{Jacobian} \,=\, ?]{} (\Lambda, \Omega)$

. In components, $\delta \tilde{H}_{i,j} = \delta \Lambda_{i,j} + \sum_{k=1}^{N} \delta \Omega_{i,k} \cdot \delta_{k,j}\, \lambda_j$

$$- \sum_{k=1}^{N} \lambda_i\, \delta_{i,k} \cdot \delta \Omega_{k,j}$$

and $\delta \Lambda_{ij} = \delta \lambda_i \cdot \delta_{ij}$

$$\Rightarrow \quad \delta \tilde{H}_{i,j} = \delta \lambda_i \cdot \delta_{i,j} + \delta \Omega_{i,j} \cdot (\lambda_j - \lambda_i)$$

. Thus, the Jacobian between $\tilde{H}$ and (Λ, Ω) is

$$J = \det \left[\frac{\partial (\tilde{H}_{1,1}, \dots, \tilde{H}_{N,N};\ \tilde{H}_{1,2}, \dots, \tilde{H}_{1,N}, \dots, \tilde{H}_{N-1,N})}{\partial (\lambda_1, \dots, \lambda_N;\ \Omega_{1,2}, \dots, \Omega_{1,N}, \dots, \Omega_{N-1,N})} \right]$$

$$= \det \left(\begin{array}{ccc|ccccc} 1 & & 0 & & & 0 & & \\ & \ddots & & & & & & \\ 0 & & 1 & & & & & \\ \hline & & & \lambda_1 - \lambda_2 & & & & \\ & 0 & & & \lambda_1 - \lambda_3 & & 0 & \\ & & & & & \ddots & & \\ & & & & & \lambda_1 - \lambda_N & & \\ & & & & 0 & & \ddots & \\ & & & & & & & \lambda_{N-1} - \lambda_N \end{array} \right)$$

$$= \prod_{1 \le i < j \le N}^{N} (\lambda_i - \lambda_j) \equiv \Delta_N(\lambda).$$

. Therefore, we have $dH = \Delta_N(\lambda) \cdot \underbrace{d\lambda}_{\prod_{i=1}^{N} d\lambda_i} \cdot \underbrace{d\Omega}_{\text{Haar measure on } SO(N).}$

. On the other hand, from (1), we have

$$p(H)\, dH = \frac{1}{Z_N}\, e^{-\frac{\mathrm{Tr}(H^2)}{2N}}\, dH \quad \text{and } \mathrm{Tr}(H^2) = \sum_{k=1}^{N} \lambda_k^2$$

$$= \frac{1}{Z_N}\, d\Omega \cdot \Delta_N(\lambda) \prod_{k=1}^{N} \left(e^{-\lambda_k^2 / 2N}\, d\lambda_k \right)$$

and by integrating out the angular variables, $d\Omega$, we get the result of the Proposition.

Remark: . The proof for $\beta=2$ and $\beta=4$ are similar. The main difference is that for the non-diagonal terms, instead of 1 we have β variables $\tilde{H}_{ij}$ and $\tilde{\lambda}_{ij}$ too, since for $\beta=2$ we have Real and Imaginary parts of $\tilde{H}_{ij}$, while for $\beta=4$ the $\tilde{H}^1_{ij}, \tilde{H}^2_{ij}, \tilde{H}^3_{ij}$.
. This is the reason why one gets $(\lambda_i-\lambda_j)^\beta$ instead.

2.3) Symmetries in matrices and physical symmetries.

2.3.1) Generalities on symmetries.

. In quantum mechanics, the pure states are given by one-dimensional projections on the Hilbert space $\mathcal{H}$ (describing the space where the system "lives"), called rays: $\hat{\Psi} = \{\varphi \in \mathcal{H}, \varphi = \alpha\Psi, \alpha \in \mathbb{C}\}$.

. Take $\|\Psi\| = 1$, then a one-dimensional projector on the ray containing Ψ is $P_{\hat{\Psi}} = |\hat{\Psi}\rangle\langle\hat{\Psi}|$

. Let A be an observable, a linear operator on $L^2(\mathcal{H})$. Then its expected value on $\hat{\Psi}$ is given by $\langle A\rangle_{\hat{\Psi}} = \mathrm{Tr}(P_{\hat{\Psi}} A) = \dfrac{\langle \Psi, A\Psi\rangle}{\langle\Psi|\Psi\rangle}$,

. The transition probability of two pure states $\hat{\Psi}, \hat{\varphi}$ is given by

$$\mathrm{Tr}(P_{\hat{\varphi}} P_{\hat{\Psi}}) = \frac{|\langle\varphi,\Psi\rangle|^2}{\|\varphi\|^2 \cdot \|\Psi\|^2}.$$

Definition: A symmetry S is an application from $\mathrm{Rays}(\mathcal{H})$ into itself:

$$\begin{cases} - \text{ surjective} : \forall \hat{\varphi}, \exists \hat{\Psi} \text{ st. } \hat{\varphi} = S(\hat{\Psi}). \\ - \text{ Keeping the transition probability invariant:} \\ \quad \mathrm{Tr}(P_{\hat{\varphi}} P_{\hat{\Psi}}) = \mathrm{Tr}(P_{S(\hat{\varphi})} P_{S(\hat{\Psi})}) \end{cases}$$

Theorem (Wigner): Let S be a symmetry, then it exists an application $U : \mathcal{H} \to \mathcal{H}$ unitary or antiunitary such that $S = S_U$ is given by $S_U(\hat{\psi}) = \text{Ray of } (U\psi)$.

- Moreover, U is unique up to a phase factor, i.e., if U' is another application satisfying $S = S_{U'}$, then $U' = \tau U$, with $|\tau| = 1$.
- If $\dim(\mathcal{H}) > 1 \Rightarrow$ the unitary or antiunitary nature of U is determined by S.

2.3.2) Time reversal symmetry.

- Let T denote the time reversal operator and $|\psi\rangle$ a quantum mechanics state.
- By Wigner's Theorem, $T = K.C$ or $T = K$, K unitary, C is the complex conjugation.
- Let $|\psi(0)\rangle$ be the state at time $t=0$, then at time δt, we have (use $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$):

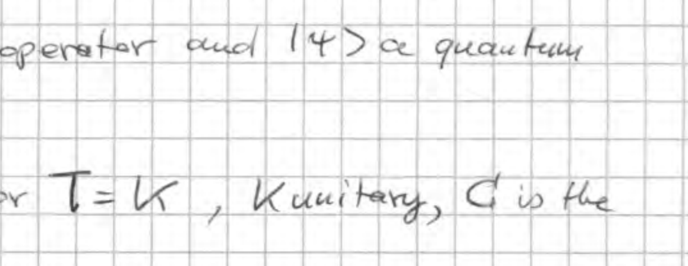
$$|\psi(\delta t)\rangle \cong \left(\mathbb{1} - \frac{iH}{\hbar}\delta t\right)|\psi(0)\rangle, \quad H = H^* \text{ the Hamiltonian}$$

- If the system is time reversal;

$$T|\psi(+\delta t)\rangle = e^{-i\frac{H}{\hbar}\delta t}\, T|\psi(0)\rangle$$

$$\Rightarrow \left(\mathbb{1} - \frac{iH}{\hbar}\delta t\right) T|\psi(0)\rangle = T\left(\mathbb{1} + \frac{iH}{\hbar}\delta t\right)|\psi(0)\rangle, \quad \forall\, |\psi(0)\rangle$$

$$\Rightarrow -iHT = T(iH)$$



- **Q: Is T unitary or antiunitary?**

- Assume T unitary $\Rightarrow T = K$, no conjugation. This means that

$$\{ H, T \} = 0 \qquad (2)$$

- A simple case, free particle, $H = \frac{P^2}{2m}$, P the momentum.

From (2) we get $T^{-1} p^2 T = -p^2$, but time reversal has to satisfy

$T p = -p \quad \Rightarrow T^{-1} p^2 T = p^2$, which is in contradiction with.

Thus T is antiunitary, so $T = K \cdot C \Rightarrow [T, H] = 0$.

- Now we have to see what possible choices of K we have.

- A change of representation is a unitary transformation on the states : $|\psi\rangle \longrightarrow U|\psi\rangle$, U unitary.

- A change of representation does not change the scalar product between two states : $\langle \varphi, \psi \rangle = \langle U\varphi, U\psi \rangle = \langle \varphi, U^* U \psi \rangle = \langle \varphi, \psi \rangle$.

- Thus also: $\langle \varphi, T\psi \rangle = \langle \varphi, U^* T U \psi \rangle$, $\forall \varphi, \psi \in L^2(\mathcal{H})$, which means that a change of representation transform

T into $U T U^*$.

Now, $T = K \cdot C$ ~~becomes~~ $U K C U^* = U \cdot K \cdot U^t C$

$C(U^* |\psi\rangle) = U^t C |\psi\rangle$ since:

$C \int U^*(x,y)\, \psi(y)\, dy = C \cdot \int \overline{U(y,x)}\, \psi(y)\, dy = \int U(y,x)\, \overline{\psi(y)}\, dy$

$= \int U^t(x,y)\, \overline{\psi(y)}\, dy$.

which means that the change of representation takes K into $U K U^t$.

- Applying twice T, the physical system has to remain unchanged, therefore $T^2 = \gamma \mathbb{1}$, $|\gamma| = 1$.

$$T^2 = K C K C = K \bar{K} C C = K \bar{K} = \gamma \cdot \mathbb{1} \text{ and } K \text{ is unitary}$$

$$\Rightarrow K K^* = \mathbb{1} \Rightarrow \underbrace{C K C}_{= \bar{K}} \cdot \underbrace{C K^*}_{= K^t C} C = \mathbb{1} \Rightarrow \underline{\bar{K} K^t = \mathbb{1}}.$$

$\Rightarrow K = K(\bar{K}K^t) = \gamma K^t = \gamma(\gamma K^t)^t = \gamma^2 K,$

Thus $\boxed{\gamma = \pm 1.}$

Case (a) $\gamma = +1$: If $K.\bar{K} = \mathbb{1}$, K unitary $\Rightarrow \exists V$ unitary s.t. $K = V.V^t$. (No proof here; check consistency: $K\bar{K} = VV^t\overline{VV^t} = VV^t\bar{V}V^* = V(\bar{V}^*\bar{V})V^* = VV^* = \mathbb{1}$).

Then, by an appropriate change of representation, $|\psi\rangle \longrightarrow V^{-1}|\psi\rangle$, one can take $K = \mathbb{1}$, i.e. $\underline{T = C}$.

- A matrix H is time invariant iff $[H,T] = 0$, i.e.,

$$T^{-1}HT = H$$

With $T = C$, we get $H = CHC = \bar{H}$.

Thus in this case the hamiltonian H can be taken as real symmetric matrix.

Case (b) $\gamma = -1$: If $K.\bar{K} = -\mathbb{1}$, then $\exists V$ s.t. $K \to VKV^t = e_2 \otimes \mathbb{1}_N$.

(No proof; consistency: $(e_2 \otimes \mathbb{1}_N)\overline{(e_2 \otimes \mathbb{1}_N)} = -\mathbb{1}_N$).

- So, in this case we can set $\underline{T = e_2 \otimes \mathbb{1} \cdot C}$

- As exercice, one can check that a matrix of the form

$$H = H^0 e_0 + H^1 e_1 + H^2 e_2 + H^3 e_3 \text{ satisfy}$$

$[H,T] = 0$ iff H^0 is real symmetric, and H^1, H^2, H^3 are real antisymmetric.

- Magnetic momentum and rotations (a remark)

- A system invariant by rotation has an Hamiltonian which commute with the generator of the rotations; the magnetic momentum J (not to be mixed up with spins; J has integer magnetic momentum).

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. Conclusion: . Time reversal symmetry implies that in an appropriate representation we have either a real symmetric matrix or a real quaternionic matrix for the Hamiltonian.

. When time reversal symmetry is not present, one has just the "usual" hermitian matrix.