Random Matrices and Related Problems

1) Introduction.

1.1) From micro to macro : universality.

. On a macroscopic scale, there are a lot of physical laws which are shared by different systems.

For example, consider the diffusion equation in the space homogeneous case:

(eq 1) $$\frac{\partial \phi}{\partial t} = D \cdot \nabla^2 \phi$$, for some function $\phi(x,t)$,

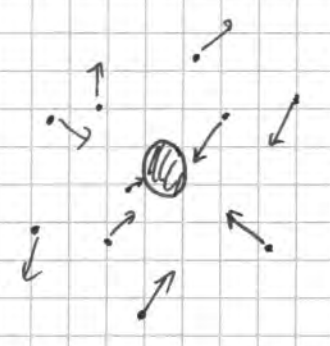
x = space ($\in \mathbb{R}^d$), t = time ($\in \mathbb{R}$),

D is called the diffusion coefficient.

This equation appears in several situations, to have two examples in mind :

(a) ϕ represent the probability density of finding a grain of pollen in suspension in water at position x and time t. The evolution of the grain of pollen being determined by the shocks with the water molecules, it looks random (→ Brownian Motion).

. The initial position of the grain of pollen is x_0, i.e., $\phi(x,0) = \delta(x-x_0)$.



(●) : grain of pollen

• : water molecules

. By changing the dimension of the grain of pollen (or replacing by something else), the same equation hold. The only difference will be the diffusion coefficient D, which is given by the following relation:

(eq 2) $$D = \frac{k_B \cdot T}{m \cdot \gamma},$$

where: . k_B is the Boltzmann constant of gas,

. T is the absolute temperature (in Kelvin),

. m is the mass of the particle suspended in the liquid,

. γ is the friction coefficient of the liquid.

(b) ϕ can also represent the temperature profile in a metal (or a solid material more generally). $\phi(x,0)$ will then be the initial temperature at position x.

. The only place where the material dependence enters in (eq1) is by the diffusion coefficient.

. In examples (a) and (b), the macroscopic equation (eq1) has only one free parameter, the diffusion constant D, which changes by changing the experimental parameters.

. On the other hand, since the Greeks (Democritus & co.) the idea that matter is built out of tiny constituents ("atoms") is present (and around one century ago checked in laboratory). The "atoms" obey their own laws of interaction. So, a natural fundamental question is to understand "how does one derive the macroscopic laws of physics from the microscopic laws of atoms?".

. The key point is that the <u>same macroscopic laws</u> should emerge <u>"no matters" of the details</u> of the atomic interaction, since on a macroscopic scale, physical systems should exhibit universality.

. In the above examples, the details of the atomic interactions emerge only in the diffusion constant D, which is material/system dependent, but <u>the form of (eq1) is the same: it is universal.</u>

Remarks: • It is the emergence of such universal behaviors for macroscopic systems, which allow the existence of the physical laws. If (eq 1) would be different (its form) for every mass, temperature, ..., then one would not have the law of diffusion.

• The two examples (a) and (b) shows that the same equation (the mathematics) can describe physical phenomena which do not have anything in common.
The same happens for random matrix related models, where the same limit laws emerge in appropriate macroscopic (or mesoscopic) scale, although the models themselves do not have any physical connection: it is the underlying mathematical structure that is shared!

1.2) Universality: a simple mathematical example.

• The simplest example of universality in mathematics is the central limit theorem (CLT); Consider independent, identically distributed random variables $\{X_n\}_{n\geq 1}$. Let $\mu = \mathbb{E}(X_i)$ the mean (supposed finite) and $\sigma^2 = \mathrm{Var}(X_i) \in (0,\infty)$ its variance (supposed finite and non-zero).
Then,

$$\text{(eq 3)} \qquad \lim_{N\to\infty} \mathbb{P}\left(\frac{\sum_{k=1}^{N} X_k - \mu\cdot N}{\sigma\cdot\sqrt{N}} \leq s \right) = \int_{-\infty}^{s} du\, \frac{e^{-u^2/2}}{\sqrt{2\pi}}.$$

• The r.h.s. of (eq 3) is universal, it does not depend on the details of the distribution of the x_i's (as soon as more than two moments are finite). The details of the distribution enter only via the centering (μ) and rescaling (σ). The macroscopic variable is $S_N = \sum_{k=1}^{N} X_k$.

. In this example there are <u>two universal quantities</u>:

(a) the <u>fluctuation exponent</u> is 1/2, i.e., $S_N - \mathbb{E}(S_N) \approx N^{1/2}$,

(b) the <u>limit law</u> : Gaussian distribution for $\frac{S_N - \mathbb{E}(S_N)}{N^{1/2}}$.

1.3) Random Matrices (R.M.)

. There are systems which behave according to rules leading for example to limit laws which are non-Gaussian. One class of such models are the ones we consider in this lecture, namely "random matrix ensembles and related problems". With "related problems" are meant models sharing the same limit laws (on a macroscopic or mesoscopic level) as random matrices.

. In the next lecture we will start defining properly some random matrix ensembles, but for the moment consider the following example.

. Let us take a matrix H, of size $N \times N$, symmetric, real, whose entries are (up to symmetry: $H_{ij} = H_{ji}$) independent random variables, with mean zero, variance $N(1+\delta_{i,j})$:

$$\begin{cases} H_{i,i} \sim \mathcal{N}(0, 2N) \\ H_{i,j} \equiv H_{j,i} \sim \mathcal{N}(0, N), \ i \neq j \end{cases}$$

. Often (but not always) the quantity of interest are the eigenvalues (the energies if the matrix represent an Hamiltonian of a physical system), and not the specific entries, which depends on the basis used to describe the system.

. As we will see, the spectrum has N eigenvalues (real) $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ which are distributed as follows:

$$P(\lambda_1, \dots, \lambda_N)\, d\lambda_1 \dots d\lambda_N = \text{const} \times \underbrace{\left[\prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)\right]}_{= V(\lambda) \text{ the Vandermonde determinant.}} \cdot \prod_{i=1}^{N} \left(e^{-\frac{\lambda_i^2}{2N}} d\lambda_i\right)$$

where $p(\lambda_1, \dots, \lambda_N)$ is the probability density of observing the eigenvalues $\lambda_1, \dots, \lambda_N$.

. One of the <u>essential features</u> of random matrices, in contrast to a Poisson point process, is that there is a <u>repulsion between eigenvalues</u> due to the Vandermonde determinant (i.e., the probability density to see eigenvalues very close goes to zero polynomially with their distance $\Rightarrow$ effective repulsion).

. What does one do with random matrices?

→ Analyze some statistical properties.

For large matrices, $N \gg 1$, one can analyze the behavior of the density of eigenvalues:

$$\rho(\lambda) \approx \begin{cases} \frac{1}{\pi} \cdot \sqrt{1 - \left(\frac{\lambda}{2N}\right)^2}, & \lambda \in (-2N, 2N) \\ 0, & \lambda \notin (-2N, 2N). \end{cases} \qquad \text{(eq 4)}$$

This is the so-called Wigner semi-circle law [See Figures 1,2].

. One can also focus on an interval and consider the statistical properties like the: → nearest-neighbor spacings

→ fluctuations of the largest (or n-th largest) eigenvalue

→ ....

. Now, suppose that a scientist makes an experiment and has as output some data (e.g., the spectrum of neutron resonance of heavy nuclei [Figure 3]).

Question: Is the system behaving like a random matrix?

. Roughly speaking, we say that a system is modeled by a random matrix theory if it behaves statistically like eigenvalues of large random matrices.

. So, but how to compare the experimental data with R.M.?

. The standard procedure is the following; to compare statistical quantities $\{a_k\}$ in the neighborhood of some point A with the eigenvalues $\{\lambda_k\}$ of some random matrices in the neighborhood of some energy E, say, in the "bulk" of the spectrum (i.e., where $g(\lambda) > 0$), one has to:

(a) Center,
(b) Rescale to the same density, say 1,

$$\Rightarrow \quad \begin{cases} a_k \longrightarrow \breve{a}_k = \gamma_a \cdot (a_k - A) \\ \lambda_k \longrightarrow \tilde{\lambda}_k = \gamma_\lambda \cdot (\lambda_k - E), \end{cases}$$

so that $\mathbb{E}\left(\#\{\breve{a}_k \text{ per unit interval}\}\right) = \mathbb{E}\left(\#\{\tilde{\lambda}_k \text{ per unit interval}\}\right) = 1$.

. At this point we have two comparable data sets.
The scientist can compare them and <u>if the fit is good</u>, then he concludes that <u>the system is well-modeled by random matrix theory.</u>

1.4] A few examples: Statistical quantity = nearest-neighbor spacing statistics.

Analytic

(a) The example of random matrix considered above is called GOE. There one looks at the eigenvalues and determine the following: for eigenvalues λ_k s.t. $\lim_{N\to\infty} \frac{\lambda_k}{N} = 0$ ($\Leftrightarrow k \approx \frac{N}{2}$), then

$$\lim_{N\to\infty} \frac{d}{du} \underbrace{\mathbb{P}(\lambda_k - \lambda_{k+1} \leq u)}_{\text{= proba. density that } (\lambda_k - \lambda_{k+1} = u).} = \mathcal{G}_{GOE}(u), \text{ a well-defined function.}$$

. This function is plotted in Figures 4.n with the legend "GOE".

Exp.

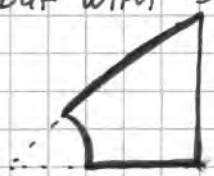
(b) Figure 4.1 and 4.2 : The histograms are experimental data for spectrum like the ones of Figure 3. Instead of eigenvalues one has the positions of the resonances.

Numerics

(c) Figure 4.3 : The function $\mathcal{G}_{GOE}(u)$ is compared to nearest neighbor spacings between energies of a free particle moving in a stadium :

(Solve $H\psi = E\psi$, $H = -\Delta$, $\psi(\text{border}) = 0$).

(d) Figure 4.4 : Like in Figure 4.3 but with Sinai's billiard instead of a stadium :



Exp.

(e) Figure 4.5 : Spacings distribution for frequencies observed in an aluminium block.

. The comparison with GOE result are not bad, considering the experimental precisions.

(f) A final example is connected with the ~~zeta~~ Riemann function,

$$\zeta(z) \doteq \sum_{n\geq 1} \frac{1}{n^z} = \prod_{p \in \text{Primes}} \left(1 - \frac{1}{p^z}\right)^{-1}, \text{ for } \mathrm{Re}\, z > 1, \text{ and}$$

defined by analytic continuation ~~for~~ $\mathrm{Re}\, z \leq 1$.

. let $z = \frac{1}{2} \pm i\gamma_n$ the nontrivial zeros of $\zeta(z)$ on $\mathrm{Re}\, z = 1/2$. The analysis of the spacing distribution of $\{\gamma_n\}_{n\geq 1}$ leads to the Figures 5.1 - 5.3 . This time the comparison is made with a slightly different random matrix ensemble, where instead of real symmetric one has complex hermitian matrices.

. When $\gamma_n \gg 1$, the agreement is astonishing good. So, the positions of the [illegible] zeros have an effective repulsion like the eigenvalues of the hermitian matrix, although there is nothing random in the ζ-Riemann function.

1.5) Example connected with the largest eigenvalue.

. The above examples concerned the statistical properties of the bulk of R.M. spectrum. However, one can also consider the statistic of the largest eigenvalue. In the example above, one finds :

for symmetric matrices $\left\{ \lim_{N\to\infty} \mathbb{P}\left(\frac{\lambda_1 - 2N}{N^{1/3}} \leq s \right) = F_1(s) \right.$, a well-defined function

for hermitian matrices $\left\{ \lim_{N\to\infty} \mathbb{P}\left(\frac{\lambda_1 - 2N}{N^{1/3}} \leq s \right) = F_2(s) \right.$, " .

The densities of $F_1(s)$ and $F_2(s)$ can be seen in Figure 6 -

. Note that the fluctuation exponent of λ_1 is 1/3 ; $F_1(s)$ and $F_2(s)$ are some universal laws different from the Gaussian one.

. Where does such a distribution occurs outside random matrix theory? Actually it appears in several models as we will see in this lecture. The first example, simple to explain, is the "longest increasing subsequence in a random permutation" problem.

. Consider a permutation $\sigma \in S_N$ (i.e., permutation of $\{1,\dots,N\}$).
Example with $N=8$, $\left(\begin{smallmatrix} i \\ \sigma_i \end{smallmatrix}\right) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 5 & 1 & 4 & 8 & 7 & 3 \end{pmatrix}$.

Let $\ell_N(\sigma)$ be the longest subsequence of $(\sigma_1,\dots,\sigma_N)$ s.t. it is increasing (i.e., the σ_i's in the sequence are increasing). In the example, $\ell_8(\sigma)=3$, and there is more than one subsequence of maximal lenght:

$(2,5,8)$; $(2,5,7)$; $(1,4,8)$; $(1,4,7)$.

. What happens to $\ell_N(\sigma)$ is σ is taken uniformly at random on S_N and $N \to \infty$?

. The answer is : $\lim_{N\to\infty} \mathbb{P}\left(\ell_N(\sigma) \le 2\sqrt{N} + s \cdot N^{1/6} \right) = F_2(s)$.

. In this case, the macroscopic scale is $\sqrt{N}$ and fluctuations are once again on a $(\sqrt{N})^{1/3}$ scale as for the largest eigenvalue of our random matrices. Moreover, the limit law F_2 arise in a model unrelated a priori to random matrices!

. This is just a simple example, but during the lecture we shall see similar results on equilibrium / non-equilibrium statistical mechanics, combinatorics, ...

1.6) Plan of the Lecture.

. Before Xmas : introduce the mathematical tools using the Gaussian Unitary Ensemble of random matrices as main model of reference.

. After Xmas : apply the learned techniques to a few models, which are non random matrix models. Between them one can choose according to the interest of the audience. A few examples are:

- . Longest increasing subsequence problem,
- . Directed percolation on $\mathbb{Z}^2$,
- . Totally asymmetric simple exclusion process,
- . Random tiling of the Aztec diamond,
- . 3D Ising corner model at low temperature,
- . 6-vertex model with domain wall boundary conditions,
- . a stochastic growth model in 1+1 dimension.

. In the next lecture and until Xmas, we start by setting the GUE ensemble, then look at correlation functions of their eigenvalues, and focus on the bulk and edge of the spectrum. At the edge the Tracy-Widom arises, which is linked with a determinantal point process. We will also see multi-matrix extensions of the model and get a limit process, which occurs also in the models listed above.

. The exercices are though to complete the lecture by deriving some property which might not be proven in details in the lecture, as well as checking some curiosities (small computations).