Random matrices and related problems

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Description: In this lecture we will explore some aspects of random matrix theory. We will then see how the underlying mathematical structure can be applied to various models in mathematics and physics. There models span from directed percolation, random domino tiling, equilibrium and non-equilibrium statistical mechanics.

Schedule: Monday 10-12; MA 645.

References used in the preparation of the lecture notes:

- Lecture notes on the same framework [16, 6]
- My PhD thesis [2]
- The standard book on Random Matrices [8]
- Booklet on random matrices [7]
- Universality in Mathematics and Physics [1]
- Point processes [9]
- Determinantal point processes [14, 15, 6]
- Airy processes [10, 5] (papers) and my review [3]
- Tracy-Widom distribution and Painlevée II [17]
- Functional analysis [13, 11, 12]
- Application to the 3D-Ising corner [4]
- Application to the PNG droplet [10]

References

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Random Matrices and Related Problems

1) Introduction.

1.1) From micro to macro : universality.

. On a macroscopic scale, there are a lot of physical laws which are shared by different systems.
For example, consider the diffusion equation in the space homogeneous case:

(eq 1) $$\frac{\partial \phi}{\partial t} = D \cdot \nabla^2 \phi$$, for some function $\phi(x,t)$,
x = space ($\in \mathbb{R}^d$), t = time ($\in \mathbb{R}$),
D is called the diffusion coefficient.

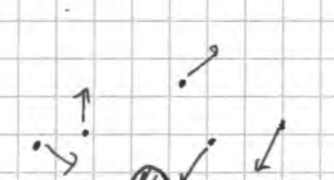
This equation appears in several situations, to have two examples in mind :

(a) ϕ represent the probability density of finding a grain of pollen in suspension in water at position x and time t. The evolution of the grain of pollen being determined by the shocks with the water molecules, it looks random (→ Brownian Motion).

. The initial position of the grain of pollen is x_0, i.e., $\phi(x,0) = \delta(x-x_0)$.

. By changing the dimension of the grain of pollen (or replacing by something else), the same equation hold. The only difference will be the diffusion coefficient D, which is given by the following relation:

(eq2) $$D = \frac{k_B \cdot T}{m \cdot \gamma},$$



(III) : grain of pollen
• : water molecules

where: . k_B is the Boltzmann constant of gas,

. T is the absolute temperature (in Kelvin),

. m is the mass of the particle suspended in the liquid,

. γ is the friction coefficient of the liquid.

(b) ϕ can also represent the temperature profile in a metal (or a solid material more generally). $\phi(x,0)$ will then be the initial temperature at position x.

. The only place where the material dependence enters in (eq1) is by the diffusion coefficient.

. In examples (a) and (b), the macroscopic equation (eq1) has only one free parameter, the diffusion constant D, which changes by changing the experimental parameters.

. On the other hand, since the Greeks (Democritus & co.) the idea that matter is built out of tiny constituents ("atoms") is present (and around one century ago checked in laboratory). The "atoms" obey their own laws of interaction. So, a natural fundamental question is to understand "how does one derive the macroscopic laws of physics from the microscopic laws of atoms?".

. The key point is that the same macroscopic laws should emerge "no matters" of the details of the atomic interaction, since on a macroscopic scale, physical systems should exhibit universality.

. In the above examples, the details of the atomic interactions emerge only in the diffusion constant D, which is material/system dependent, but the form of (eq1) is the same: it is universal.

Remarks: . It is the emergence of such universal behaviors for macroscopic systems, which allow the existence of the physical laws. If (eq 1) would be different (its form) for every mass, temperature, ..., then one would not have the law of diffusion.

. The two examples (a) and (b) shows that the same equation (the mathematics) can describe physical phenomena which do not have anything in common.
The same happens for random matrix related models, where the same limit laws emerge in appropriate macroscopic (or mesoscopic) scale, although the models themselves do not have any physical connection: it is the underlying mathematical structure that is shared!

1.2) Universality: a simple mathematical example.

. The simplest example of universality in mathematics is the central limit theorem (CLT); Consider independent, identically distributed random variables $\{X_n\}_{n\geq 1}$. Let $\mu = \mathbb{E}(X_i)$ the mean (supposed finite) and $\sigma^2 = Var(X_i) \in (0,\infty)$ its variance (supposed finite and non-zero).
Then,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\sum_{k=1}^{N} X_k - \mu\cdot N}{\sigma\cdot\sqrt{N}} \leq s \right) = \int_{-\infty}^{s} du \, \frac{e^{-u^2/2}}{\sqrt{2\pi}} . \qquad \text{(eq 3)}$$

. The r.h.s. of (eq 3) is universal, it does not depend on the details of the distribution of the x_i's (as soon as more than two moments are finite). The details of the distribution enter only via the centering (μ) and rescaling (σ). The macroscopic variable is $S_N = \sum_{k=1}^{N} X_k$.

. In this example there are two universal quantities:

(a) the fluctuation exponent is 1/2, i.e., $S_N - \mathbb{E}(S_N) \approx N^{1/2}$,

(b) the limit law: Gaussian distribution for $\dfrac{S_N - \mathbb{E}(S_N)}{N^{1/2}}$.

1.3) Random Matrices (R.M.)

. There are systems which behave according to rules leading for example to limit laws which are non-Gaussian. One class of such models are the ones we consider in this lecture, namely "random matrix ensembles and related problems". With "related problems" are meant models sharing the same limit laws (on a macroscopic or mesoscopic level) as random matrices.

. In the next lecture we will start defining properly some random matrix ensembles, but for the moment consider the following example.

. Let us take a matrix H, of size $N \times N$, symmetric, real, whose entries are (up to symmetry: $H_{ij} = H_{ji}$) independent random variables, with mean zero, variance $N(1+\delta_{i,j})$:

$$\begin{cases} H_{i,i} \sim \mathcal{N}(0, 2N) \\ H_{i,j} \equiv H_{j,i} \sim \mathcal{N}(0, N), \; i \neq j \end{cases}$$

. Often (but not always) the quantity of interest are the eigenvalues (the energies if the matrix represent an Hamiltonian of a physical system), and not the specific entries, which depends on the basis used to describe the system.

. As we will see, the spectrum has N eigenvalues (real) $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ which are distributed as follows:

$$P(\lambda_1, \dots, \lambda_N)\, d\lambda_1 \dots d\lambda_N = \text{const} \times \underbrace{\left[\prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)\right]}_{= V(\lambda) \text{ the Vandermonde determinant.}} \cdot \prod_{i=1}^{N} \left(e^{-\frac{\lambda_i^2}{2N}} d\lambda_i\right)$$

where $p(\lambda_1, \dots, \lambda_N)$ is the probability density of observing the eigenvalues $\lambda_1, \dots, \lambda_N$.

. One of the essential features of random matrices, in contrast to a Poisson point process, is that there is a repulsion between eigenvalues due to the Vandermonde determinant (i.e., the probability density to see eigenvalues very close goes to zero polynomially with their distance $\Rightarrow$ effective repulsion).

. What does one do with random matrices?

→ Analyze some statistical properties.

For large matrices, $N \gg 1$, one can analyze the behavior of the density of eigenvalues:

(eq 4)
$$\rho(\lambda) \approx \begin{cases} \frac{1}{\pi} \cdot \sqrt{1 - \left(\frac{\lambda}{2N}\right)^2}, & \lambda \in (-2N, 2N) \\ 0, & \lambda \notin (-2N, 2N). \end{cases}$$

This is the so-called Wigner semi-circle law [See Figures 1,2].

. One can also focus on an interval and consider the statistical properties like the:
→ nearest-neighbor spacings
→ fluctuations of the largest (or n-th largest) eigenvalue
→ ....

. Now, suppose that a scientist makes an experiment and has as output some data (e.g., the spectrum of neutron resonance of heavy nuclei [Figure 3]).

<u>Question:</u> Is the system behaving like a random matrix?

. Roughly speaking, we say that a system is modeled by a random matrix theory if it behaves statistically like eigenvalues of large random matrices.

. So, but how to compare the experimental data with R.M.?

. The standard procedure is the following; to compare statistical quantities $\{a_k\}$ in the neighborhood of some point A with the eigenvalues $\{\lambda_k\}$ of some random matrices in the neighborhood of some energy E, say, in the "bulk" of the spectrum (i.e., where $\rho(\lambda) > 0$), one has to:

(a) Center,

(b) Rescale to the same density, say 1,

$$\Rightarrow \quad \begin{cases} a_k \longrightarrow \breve{a}_k = \gamma_a \cdot (a_k - A) \\ \lambda_k \longrightarrow \tilde{\lambda}_k = \gamma_\lambda \cdot (\lambda_k - E), \end{cases}$$

So that $\mathbb{E}\left(\#\{\breve{a}_k \text{ per unit interval}\}\right) = \mathbb{E}\left(\#\{\tilde{\lambda}_k \text{ per unit interval}\}\right) = 1$.

. At this point we have two comparable data sets.

The scientist can compare them and <u>if the fit is good</u>, then he concludes that <u>the system is well-modeled by random matrix theory.</u>

1.4) A few examples: Statistical quantity = nearest-neighbor spacing statistics.

Analytic

(a) The example of random matrix considered above is called GOE. There one looks at the eigenvalues and determine the following: for eigenvalues λ_k s.t. $\lim_{N\to\infty} \frac{\lambda_k}{N} = 0$ $(\Leftrightarrow k \simeq \frac{N}{2})$, then

$$\lim_{N\to\infty} \underbrace{\frac{d}{du}\mathbb{P}\left(\lambda_k - \lambda_{k+1} \leq u\right)}_{\substack{= \text{ proba. density that} \\ (\lambda_k - \lambda_{k+1} = u).}} = \mathcal{Y}_{GOE}(u), \text{ a well-defined function.}$$

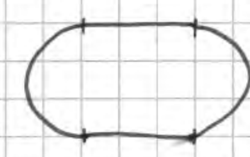
. This function is plotted in Figures 4.n with the legend "GOE".

Exp.

(b) Figure 4.1 and 4.2: The histograms are experimental data for spectrum like the ones of Figure 3. Instead of eigenvalues one has the positions of the resonances.

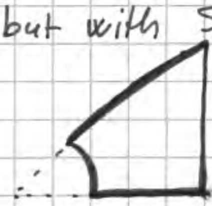
Numerics

(c) Figure 4.3: The function $\mathcal{Y}_{GOE}(u)$ is compared to nearest neighbor spacings between energies of a free particle moving in a stadium:



(Solve $H\psi = E\psi$, $H = -\Delta$, $\psi(\text{border}) = 0$).

(d) Figure 4.4: Like in Figure 4.3 but with Sinai's billiard instead of a stadium:



Exp.

(e) Figure 4.5: Spacings distribution for frequencies observed in an aluminium block.

. The comparison with GOE result are not bad, considering the experimental precisions.

(f) A final example is connected with the zeta Riemann function,

$$\zeta(z) \doteq \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \in \text{Primes}} \left(1 - \frac{1}{p^z}\right)^{-1}, \text{ for } \operatorname{Re} z > 1, \text{ and}$$

defined by analytic continuation for $\operatorname{Re} z \leq 1$.

. Let $z = \frac{1}{2} \pm i\gamma_n$ the nontrivial zeros of $\zeta(z)$ on $\operatorname{Re} z = 1/2$. The analysis of the spacing distribution of $\{\gamma_n\}_{n \geq 1}$ leads to the Figures 5.1 - 5.3. This time the comparison is made with a slightly different random matrix ensemble, where instead of real symmetric one has complex hermitian matrices.

. When $\gamma_n \gg 1$, the agreement is astonishing good. So, the positions of the zeros have an effective repulsion like the eigenvalues of the hermitian matrix, although there is nothing random in the ζ-Riemann function.

1.5) Example connected with the largest eigenvalue.

. The above examples concerned the statistical properties of the bulk of R.M. spectrum. However, one can also consider the statistic of the largest eigenvalue. In the example above, one finds:

for symmetric matrices $\left\{ \lim_{N\to\infty} \mathbb{P}\left(\frac{\lambda_1 - 2N}{N^{1/3}} \leq s\right) = F_1(s) \right.$, a well-defined function

for hermitian matrices $\left\{ \lim_{N\to\infty} \mathbb{P}\left(\frac{\lambda_1 - 2N}{N^{1/3}} \leq s\right) = F_2(s) \right.$, "

The densities of $F_1(s)$ and $F_2(s)$ can be seen in Figure 6.

. Note that the fluctuation exponent of λ_1 is 1/3; $F_1(s)$ and $F_2(s)$ are some universal laws different from the Gaussian one.

. Where does such a distribution occurs outside random matrix theory? Actually it appears in several models as we will see in this lecture. The first example, simple to explain, is the "longest increasing subsequence in a random permutation" problem.

. Consider a permutation $\sigma \in S_N$ (i.e., permutation of $\{1, \dots, N\}$).

Example with $N = 8$, $\begin{pmatrix} i \\ \sigma_i \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 5 & 1 & 4 & 8 & 7 & 3 \end{pmatrix}$.

Let $\ell_N(\sigma)$ be the longest subsequence of $(\sigma_1, \dots, \sigma_N)$ s.t. it is increasing (i.e., the σ_i's in the sequence are increasing). In the example, $\ell_8(\sigma) = 3$, and there is more than one subsequence of maximal lenght:

$(2,5,8)$; $(2,5,7)$; $(1,4,8)$; $(1,4,7)$.

. What happens to $\ell_N(\sigma)$ is σ is taken uniformly at random on S_N and $N \to \infty$?

. The answer is: $\lim\limits_{N \to \infty} \mathbb{P}\left(\ell_N(\sigma) \leq 2\sqrt{N} + s \cdot N^{1/6} \right) = F_2(s)$.

. In this case, the macroscopic scale is $\sqrt{N}$ and fluctuations are once again on a $(\sqrt{N})^{1/3}$ scale as for the largest eigenvalue of our random matrices. Moreover, the limit law F_2 arise in a model unrelated a priori to random matrices!

. This is just a simple example, but during the lecture we shall see similar results on equilibrium / non-equilibrium statistical mechanics, combinatorics, ...

1.6) Plan of the Lecture.

. Before Xmas : introduce the mathematical tools using the Gaussian Unitary Ensemble of random matrices as main model of reference.

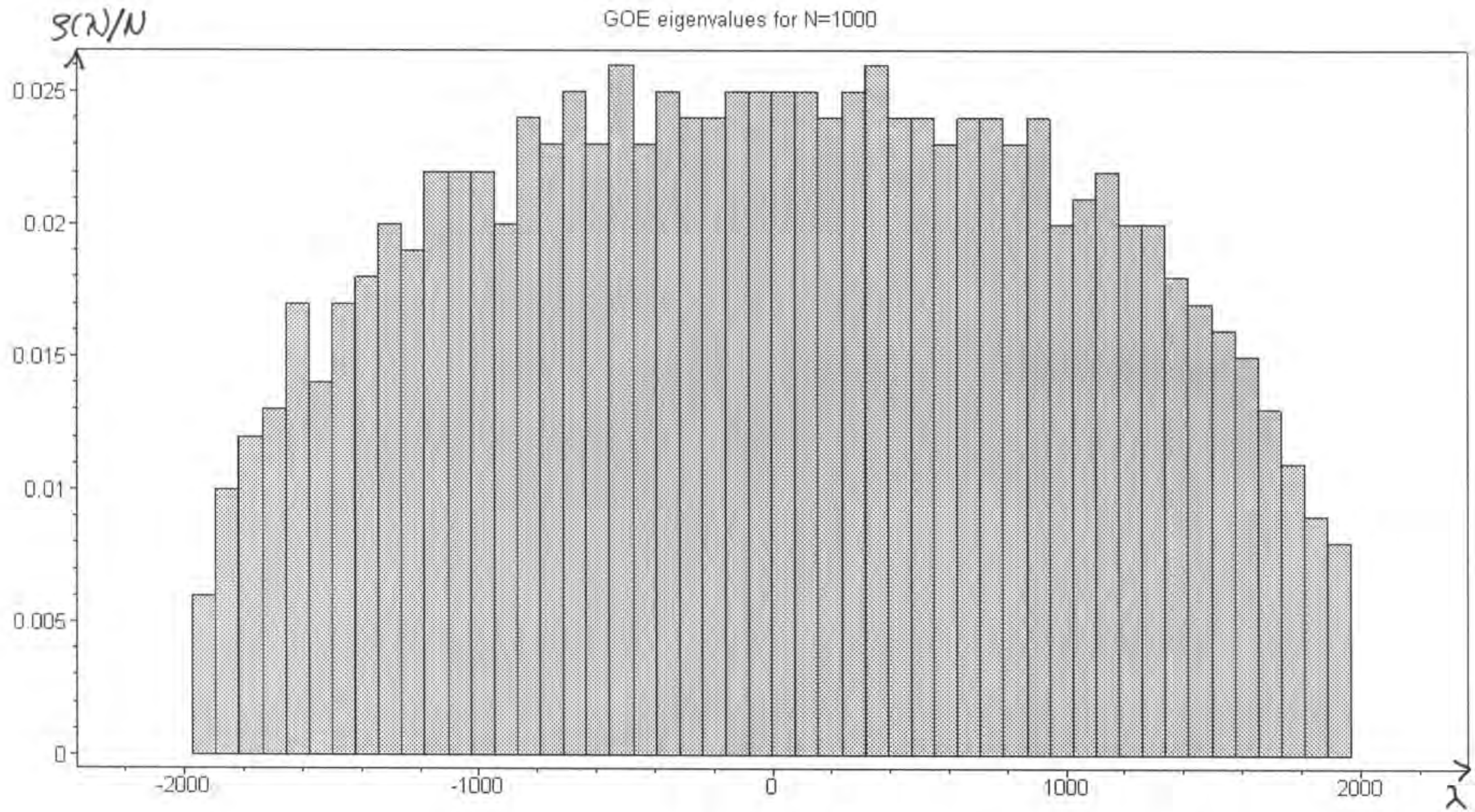
. After Xmas : apply the learned techniques to a few models, which are non random matrix models. Between them one can choose according to the interest of the audience. A few examples are:

- . Longest increasing subsequence problem,
- . Directed percolation on $\mathbb{Z}^2$,
- . Totally asymmetric simple exclusion process,
- . Random tiling of the Aztec diamond,
- . 3D Ising corner model at low temperature,
- . 6-vertex model with domain wall boundary conditions,
- . a stochastic growth model in 1+1 dimension.

. In the next lecture and until Xmas, we start by setting the GUE ensemble, then look at correlation functions of their eigenvalues, and focus on the bulk and edge of the spectrum. At the edge the Tracy-Widom arises, which is linked with a determinantal point process. We will also see multi-matrix extensions of the model and get a limit process, which occurs also in the models listed above.

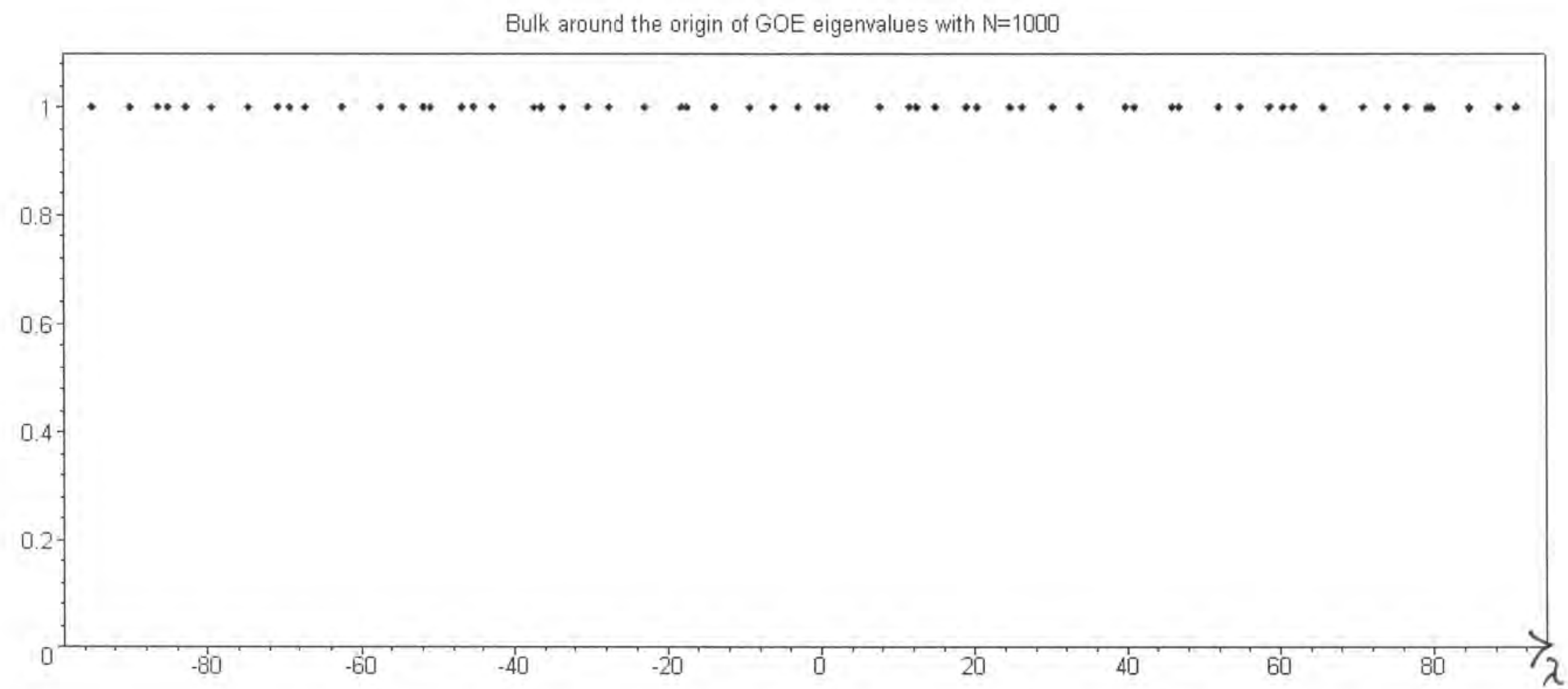
. The exercices are though to complete the lecture by deriving some property which might not be proven in details in the lecture, as well as checking some curiosities (small computations).

Figure 1



Eigenvalue density of a $N \times N$ matrix in the GOE ensemble, $N=1000$.

Figure 2



Eigenvalues in the bulk of the $N \times N$ GOE matrix, $N = 1'000$,

Figure 3

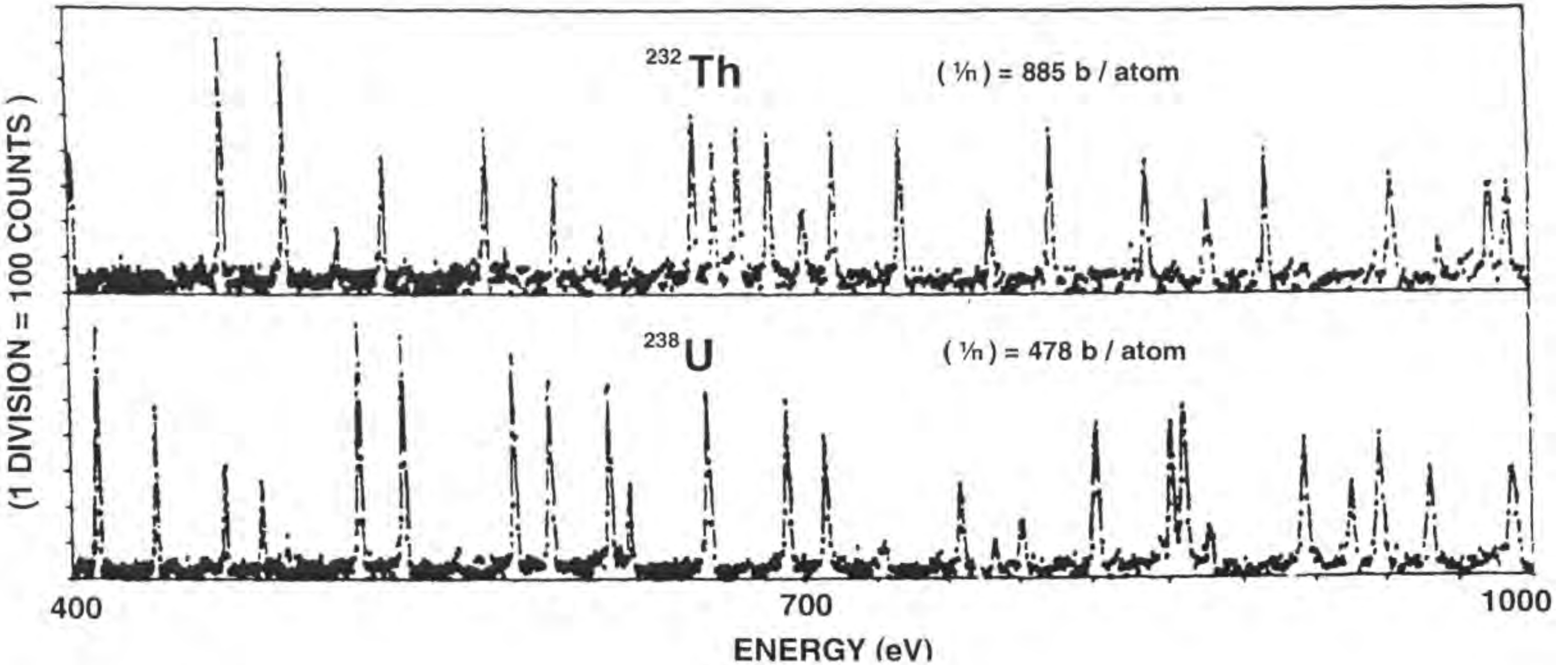


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

. Taken from Mehta book "Random Matrices", page 2.

Figure 4.1

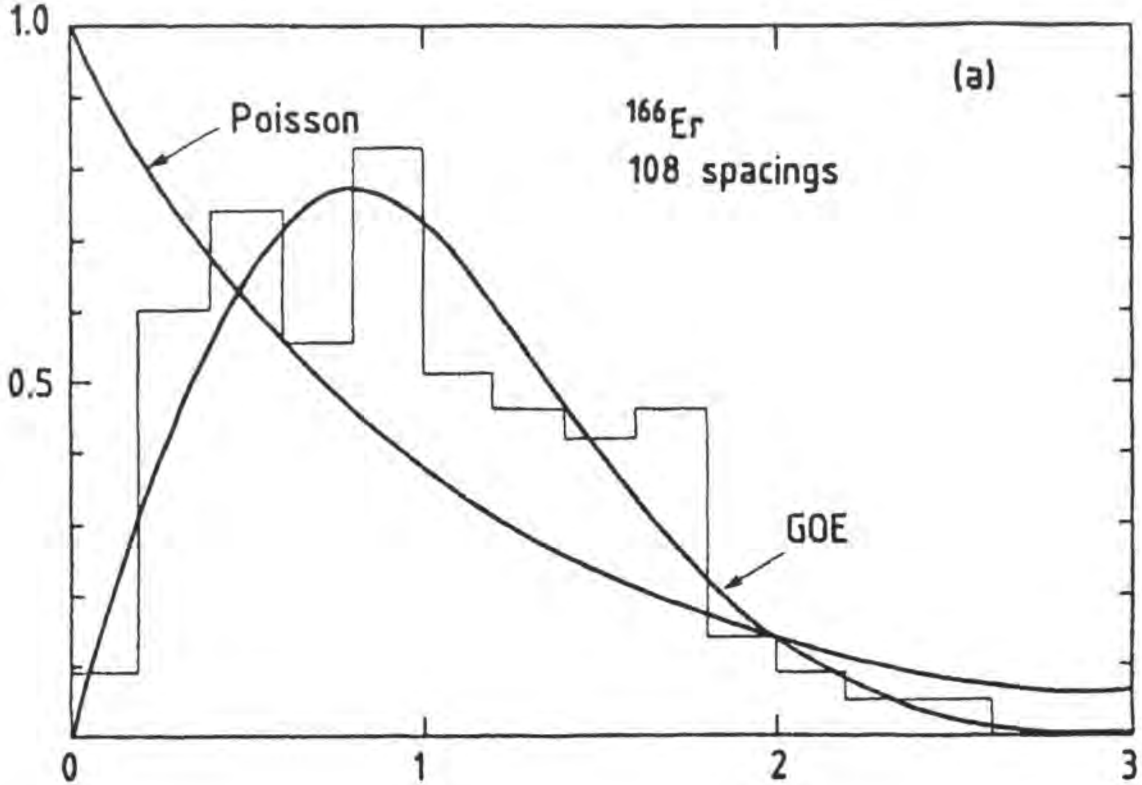


Figure 1.3. The probability density for the nearest neighbor spacings in slow neutron resonance levels of erbium 166 nucleus. The histogram shows the first 108 levels observed. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from The American Physical Society, Liou et al., Neutron resonance spectroscopy data, *Phys. Rev. C* 5 (1972) 974–1001.

. Mehta book, page 12.

Figure 4.2

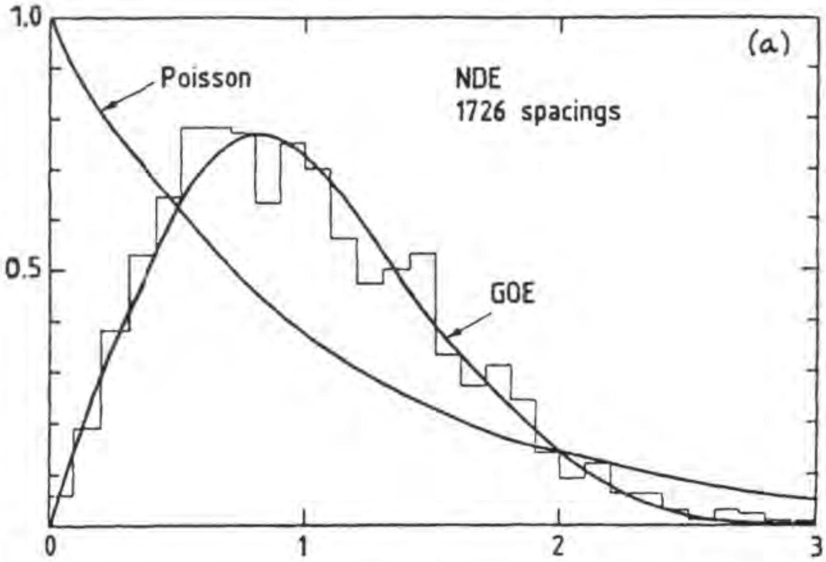


Figure 1.4. Level spacing histogram for a large set of nuclear levels, often referred to as nuclear data ensemble. The data considered consists of 1407 resonance levels belonging to 30 sequences of 27 different nuclei: (i) slow neutron resonances of Cd(110, 112, 114), Sm(152, 154), Gd(154, 156, 158, 160), Dy(160, 162, 164), Er(166, 168, 170), Yb(172, 174, 176), W(182, 184, 186), Th(232) and U(238); (1146 levels); (ii) proton resonances of Ca(44) ($J = 1/2+$), Ca(44) ($J = 1/2-$), and Ti(48) ($J = 1/2+$); (157 levels); and (iii) (n, γ)-reaction data on Hf(177) ($J = 3$), Hf(177) ($J = 4$), Hf(179) ($J = 4$), and Hf(179) ($J = 5$); (104 levels). The data chosen in each sequence is believed to be complete (no missing levels) and pure (the same angular momentum and parity). For each of the 30 sequences the average quantities (e.g. the mean spacing, spacing/mean spacing, number variance μ_2, etc., see Chapter 16) are computed separately and their aggregate is taken weighted according to the size of each sequence. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from Kluwer Academic Publishers, Bohigas O., Haq R.U. and Pandey A., Fluctuation properties of nuclear energy levels and widths, comparison of theory with experiment, in: *Nuclear Data for Science and Technology*, Bökhoff K.H. (Ed.), 809–814 (1983).

. Mehta book, page 13.

Figure 4.3

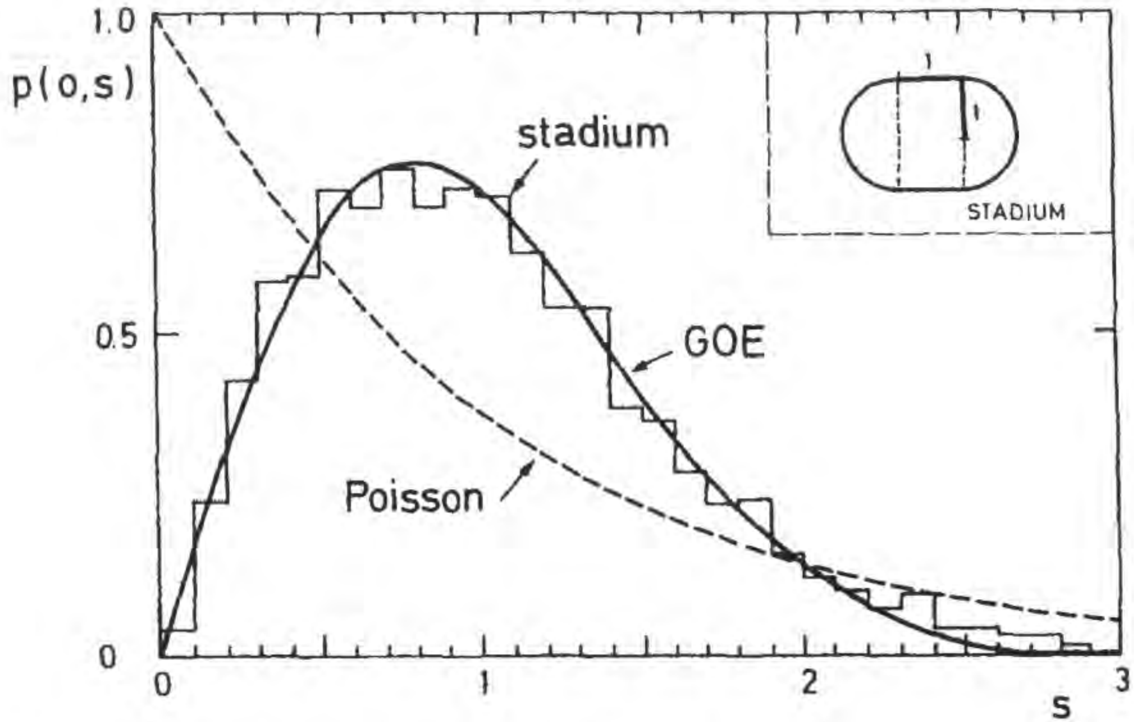


Figure 7.7. Empirical probability density of the nearest neighbor spacings of the possible energies of a particle free to move on the stadium consisting of a rectangle of size 1×2 with semi-circular caps of radius 1, depicted in the right upper corner. The stadium can be defined by the inequalities $|y| \leqslant 1$, and either $|x| \leqslant 1/2$ or $(x \pm 1/2)^2 + y^2 \leqslant 1$. The solid curve represents Eq. (7.3.19) corresponding to the Gaussian orthogonal ensemble (GOE), while the dashed curve is for the Poisson process corresponding to no correlations. Supplied by O. Bohigas, from Bohigas et al. (1984a).

. Mehta book, page 172.

Figure 4.4

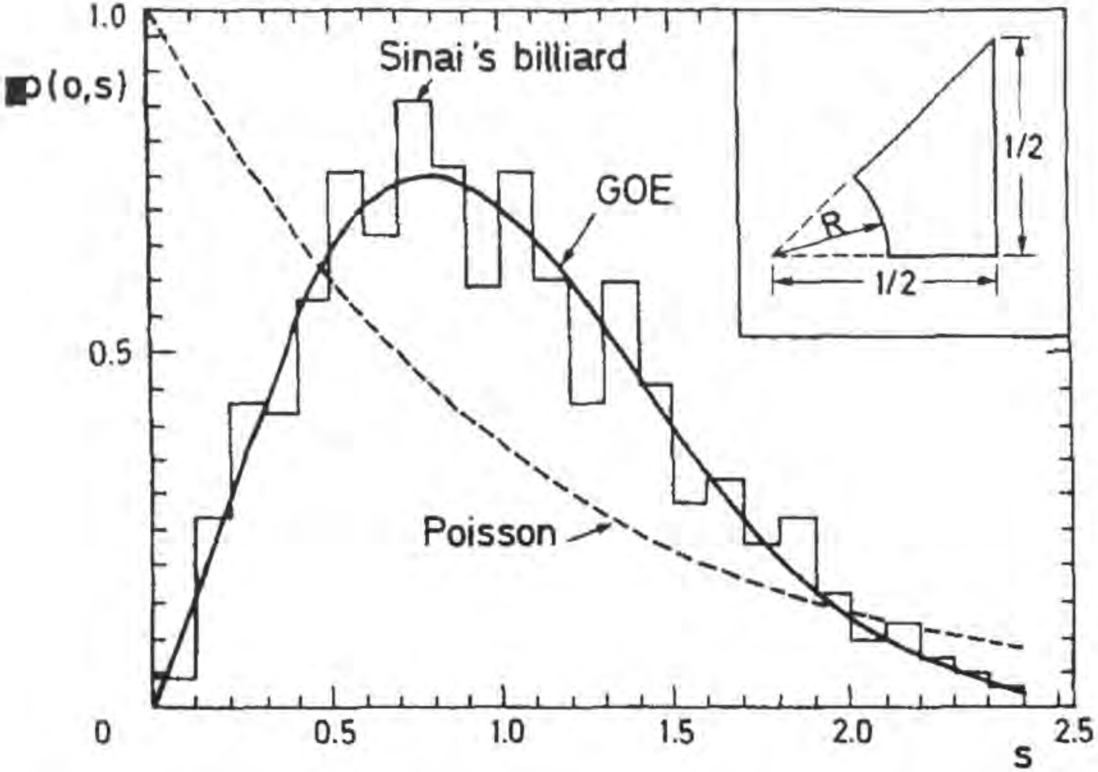


Figure 7.8. Same as Figure 7.7 but when the particle moves on Sinai's billiard table consisting of $1/8$ of a square cut by a circular arc, depicted in the right upper corner. One may define it by the inequalities $y \geqslant 0$, $x \geqslant y$, $x \leqslant 1$ and $x^2 + y^2 \geqslant r$. Only $1/8$th of the square is taken so that all obvious symmetries of the square are disposed of. Supplied by O. Bohigas, from Bohigas et al. (1984a).

. Mehta book, page 173.

Figure 4.5

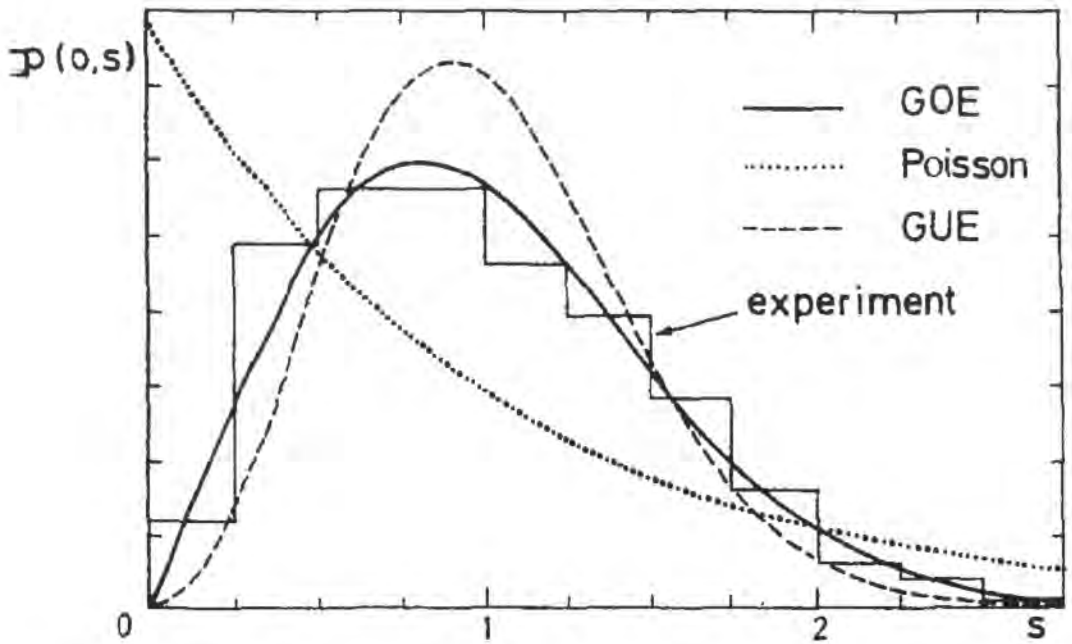


Figure 7.9. Empirical probability density of the nearest neighbor spacings for the ultrasonic frequencies of an aluminium block. The three curves correspond respectively to the Poisson process with no correlations, to the Gaussian orthogonal ensemble (GOE) and to the Gaussian unitary ensemble (GUE). Reprinted with permission from American Institute of Physics, Weaver R.L., Spectral statistics in elastodynamics. *J. Acoust. Soc. Amer.* 85 (1989) 1005–1013.

. Mehta book, page 173.

Figure 5.1

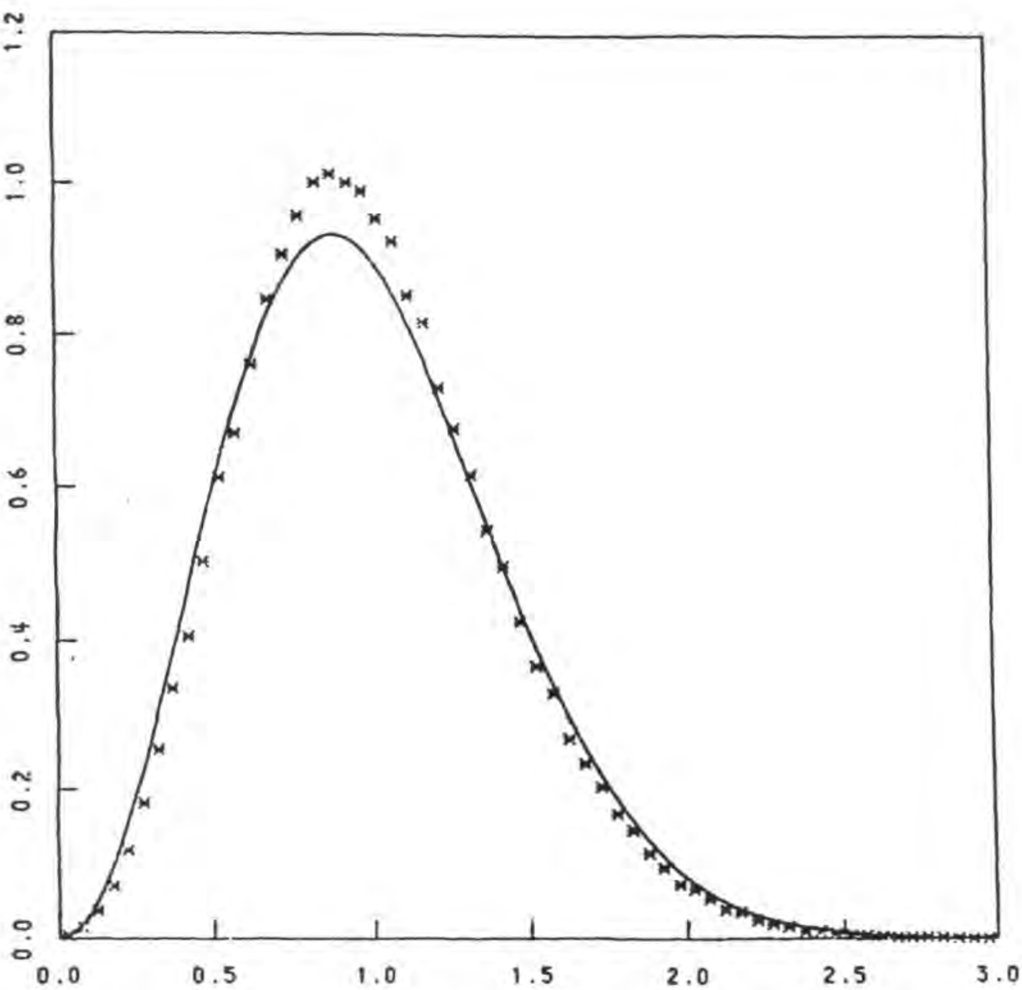


Figure 1.12. Plot of the density of normalized spacings for the zeros $0.5 \pm i\gamma_n$, γ_n real, of the Riemann zeta function on the critical line. $1 < n < 10^5$. The solid curve is the spacing probability density for the Gaussian unitary ensemble, Eq. (6.4.32). From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

. Mehta book, page 24.

Figure 5.2

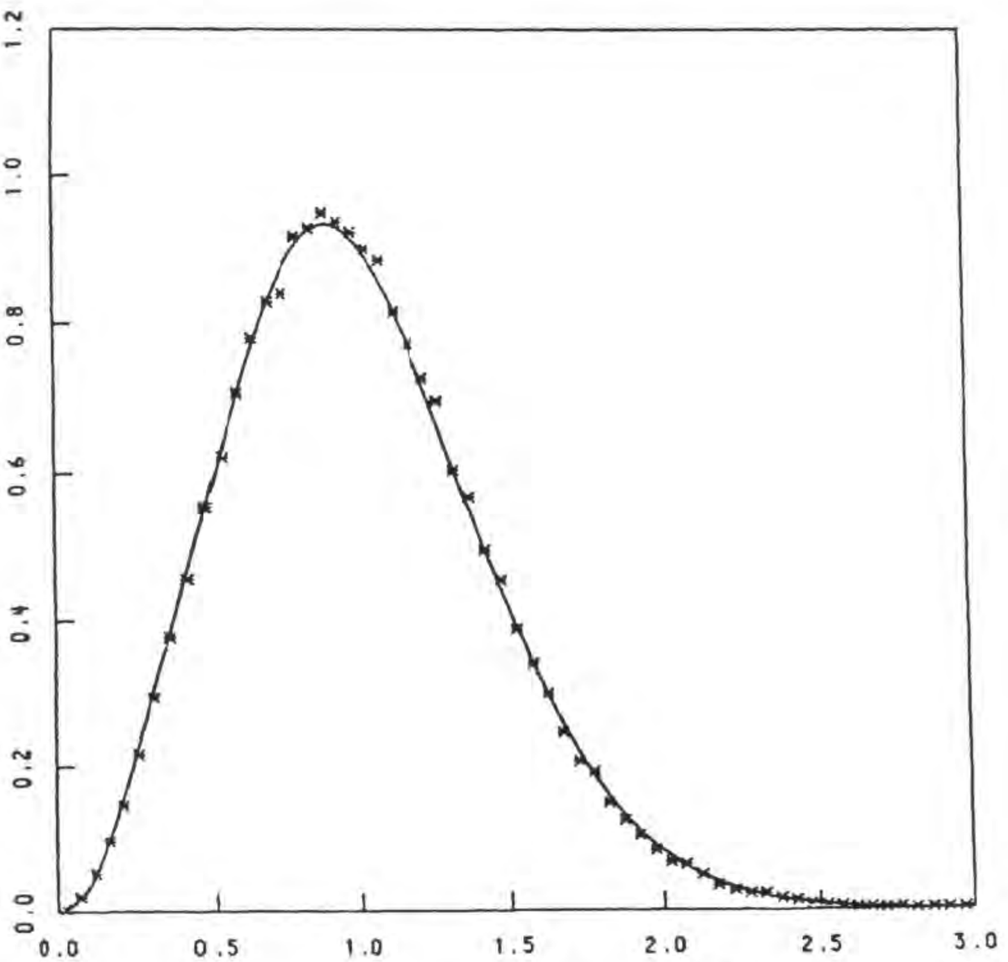


Figure 1.13. The same as Figure 1.12 with $10^{12} < n < 10^{12} + 10^5$. Note the improvement in the fit. From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

. Mehta book, page 25.

Figure 5.3

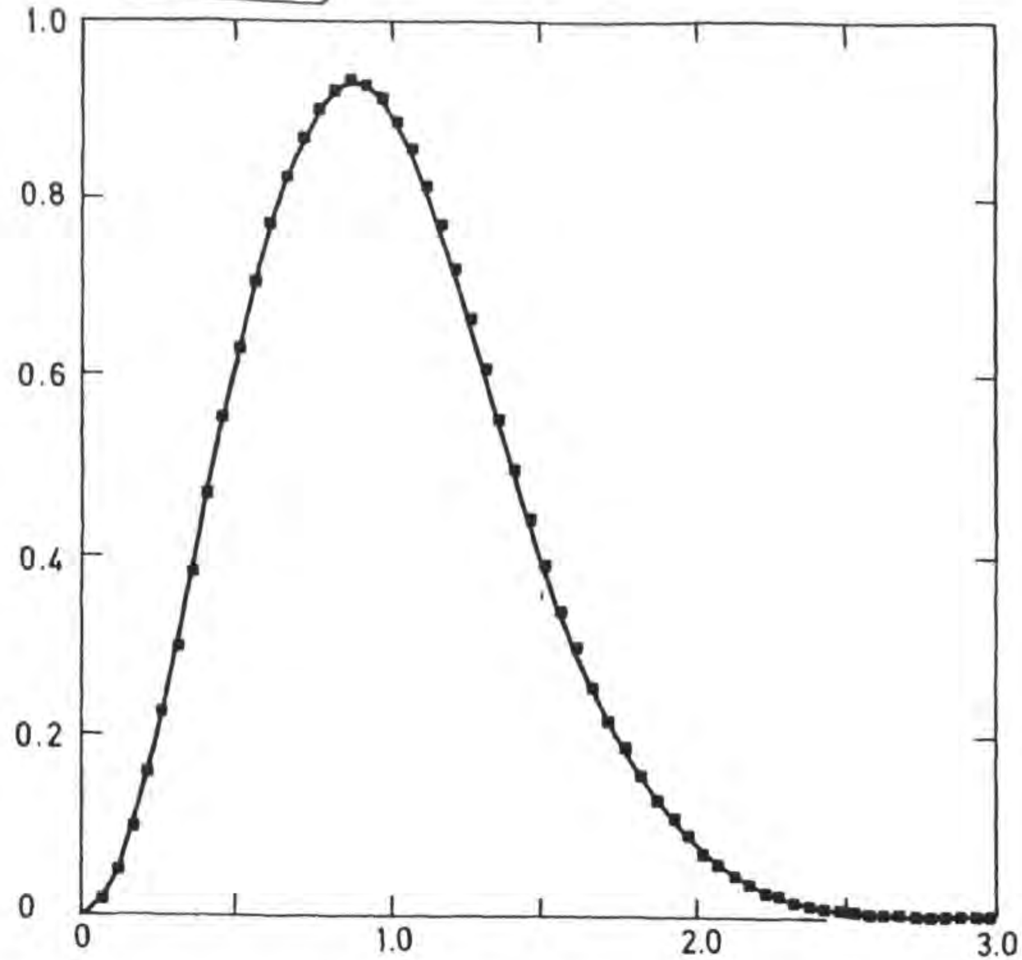


Figure 1.14. The same as Figure 1.12 but for the 79 million zeros around the 10^{20}th zero. From Odlyzko (1989). Copyright © 1989, American Telephone and Telegraph Company, reprinted with permission.

. Mehta book, page 26.

Figure 6

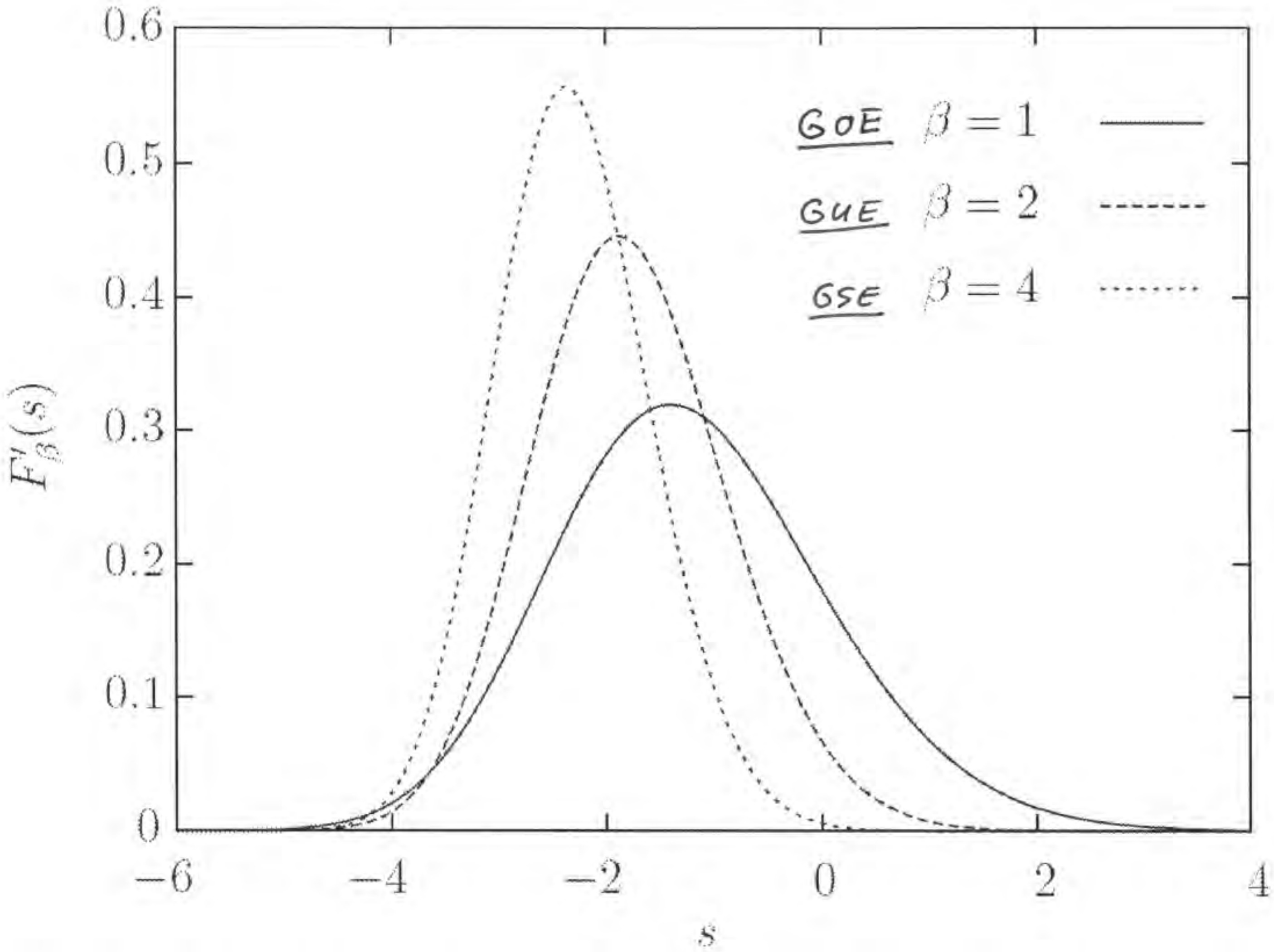


Figure 3.1: Probability densities of the Tracy-Widom distributions generated using [80].

2) The classical ensembles of random matrices.

. In the 60's physicists (Dyson, Wigner, ...) considered random matrix ensembles to model statistics of heavy atom spectral measurements. The result are the classical gaussian ensembles of random matrices. The different ensembles corresponds to different intrinsic symmetries of the system (time reversal, rotation symmetry).

. First we introduce the ensembles of random matrices and derive their eigenvalues' distribution. Then we will come back and discuss more in detail the question of symmetries.

2.1) GOE, GUE, GSE.

. Consider the following classes of matrices:

1 $\beta = 1$: H are real symmetric matrices:

. $H = H^t$ i.e. $H_{ij} = H_{ji} \in \mathbb{R}$.

. There are $\frac{N(N+1)}{2}$ independent entries.

. A real symmetric matric can a priori describe a system which is time reversal and, rotation invariant or with integer magnetic momentum.

. These matrices can be diagonalized by an orthogonal transformation.

2 $\beta = 2$: H are complex hermitian matrices.

. $H = H^*$: $H_{ij} = \overline{H_{ji}} \in \mathbb{C}$, or, equivalently,
$H = H^0 + i H^1$, H^0 real symmetric, H^1 real antisymmetric.

. There are N^2 independent entries.

. A complex hermitian matrix can apriori describe a system which is not time reversal (e.g., in the presence of a magnetic field).

. These matrices can be diagonalized by an unitary transformation.

3 $\beta = 4$: H are real quaternionic matrices:

. What are quaternionic matrices? A simple way to define them is as follows. For $\beta = 1$ each entry was a multiple of "1", for $\beta = 2$ the entries are linear combinations of "1" and "i", and for $\beta = 4$ (the present case), are simply linear combinations of "1", "i", "j", "k", with $i^2 = j^2 = k^2 = -1$, $i \cdot j \cdot k = -1$.

. One way of representing the four basis element of a quaternion number are via 2x2 matrices:

$$"1" \leftrightarrow e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad "i" \leftrightarrow e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$"j" \leftrightarrow e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad "k" \leftrightarrow e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Remark: $e_k = i \cdot \sigma_k$, $k = 1, 2, 3$, σ_k the Pauli matrices.

. Then, ~~the~~ matrices which are called quaternionic real are the ones which writes: $H = H^0 e_0 + H^1 e_1 + H^2 e_2 + H^3 e_3$ with H^0 real symmetric, H^1, H^2, H^3 real antisymmetric ~~NxN~~ matrices.

. There are $N(2N-1)$ independent entries.

. A system which a priori can be described by such matrices have time reversal symmetry but with half-integer (as spins...) magnetic momentum.

. These matrices can be diagonalized by a <u>symplectic</u> unitary transformation.

. <u>Definition:</u> The <u>Gaussian Orthogonal Ensemble (GOE)</u> of random matrices is the set of $N\times N$ real symmetric matrices H with probability measure, p,

(1)
$$p(H)\,dH = \frac{1}{Z_N}\cdot \exp\left(-\frac{\mathrm{Tr}(H^2)}{2N}\right) dH,$$

where dH is the flat reference measure:
$$dH = \prod_{1\leq i\leq j\leq N} dH_{ij},$$
and Z_N a constant.

. <u>Remark:</u> . The name "Orthogonal" comes from the fact that the measure (1) is <u>invariant</u> under <u>orthogonal transformations.</u>
. The factor $\frac{1}{2N}$ is just a normalization, useful at the end of the lectures.

. <u>Definition:</u> The <u>Gaussian Unitary Ensemble (GUE)</u> of random matrices is the set of $N\times N$ complex hermitian matrices H with measure (1), but now
$$dH = \left(\prod_{i=1}^{N} dH_{ii}\right)\cdot \prod_{1\leq i<j\leq N} d\,\mathrm{Re}(H_{ij})\, d\,\mathrm{Im}(H_{ij})$$

. <u>Definition:</u> The <u>Gaussian Symplectic Ensemble (GSE)</u> of random matrices is the set of $N\times N$ quaternionic real matrices N ($\equiv 2N\times 2N$ matrices if one uses the e_i's) with measure (1) and $dH = \left(\prod_{1\leq i\leq j\leq N} dH^0_{ij}\right)\prod_{1\leq i<j\leq N}\left(dH^1_{ij}\, dH^2_{ij}\, dH^3_{ij}\right)$.

Remarks: . "Unitary" is because of invariance under unitary transformations of the measure (1) with hermitian matrices.

. "Symplectic" is because of invariance under symplectic transformations of (1) with real quaternionic matrices.

. The original definition of the gaussian ensembles is different: $p(H)$ is defined to be invariant under the group of symmetry (orthogonal, $O(N)$, for GOE), i.e., no basis is special. This implies that $p(H)$ is a function of $\mathrm{Tr}(H^k)$, $k=1,\dots,N$ only, i.e., of the N eigenvalues $\lambda_1,\dots,\lambda_N$. The second requirement is that the entries of the matrices are independent random variables (up to symmetries). With these two requirement, it can be proven (see, e.g., Chapter 26 of Mehta's book) that

$$p(H) \propto \exp\left[-a\,\mathrm{Tr}(H^2)+b\,\mathrm{Tr}(H)+c\right], \quad a>0,\ b,c\in\mathbb{R}.$$

. c enters in the normalisation, while b can be set to zero by a change in the zero of the energies.

2.2) Eigenvalues' distributions

. Often one is interested in the eigenvalues of the matrix, since are independent of the choice of basis.

Proposition: Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ be the eigenvalues of a Gaussian Ensemble of random matrix (GOE, GUE, or GSE).

Then, the joint distribution on the eigenvalues is given by

$$p(\lambda)\,d\lambda = \frac{1}{Z_N}\left(\prod_{1\leq i<j\leq N}(\lambda_i-\lambda_j)\right)^{\beta}\cdot\prod_{i=1}^{N} d\mu(\lambda_i), \quad d\mu(\lambda_i)=e^{-\frac{\lambda_i^2}{2N}}\,d\lambda_i$$

and Z_N a constant; $\beta=1$ for GOE, $\beta=2$ for GUE, $\beta=4$ for GSE.

Remark: . $\prod_{1\le i<j\le N} (\lambda_i - \lambda_j)$ is called the Vandermonde determinant, and equals, $\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N} =$

$$= \det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{bmatrix}.$$

. Let us prove this Proposition for the orthogonal case, the other two cases can be made as an exercice.

. Given H, $\exists\, g \in O(N)$ s.t. $H = g \cdot \Lambda \cdot g^{-1}$, $\Lambda = \lambda_i \cdot \delta_{ij}$, $i,j = 1,\dots,N$.
Since $M = \frac{N(N+1)}{2}$ are the independent entries of H, we have $M-N$ independent variables $p_1,\dots,p_{M-N}$ in the g's.

. We have to compute the Jacobian of the change of variables $\{H_{ij}, 1\le i\le j\le N\}$ to $\{\lambda_1,\dots,\lambda_N; p_1,\dots,p_{M-N}\}$.

. An infinitesimal transformation of H gives

$$\delta H = \delta g \cdot \Lambda \cdot g^{-1} + g \cdot \delta\Lambda \cdot g^{-1} + g \cdot \Lambda \cdot \delta g^{-1}$$

and sing $\quad g g^{-1} = 1, \ (\delta g)\cdot g^{-1} = -g \cdot \delta g^{-1}$

$$\Rightarrow \delta H = \delta g \cdot \Lambda \cdot g^{-1} + g \cdot \delta\Lambda \cdot g^{-1} - g \cdot \Lambda \cdot g^{-1}(\delta g) g^{-1}$$

$$= g\left[g^{-1}\delta g \cdot \Lambda - \Lambda \cdot g^{-1}\delta g\right] g^{-1} + g\, \delta\Lambda \cdot g^{-1}$$

$$\Rightarrow \delta H = g \cdot \delta\tilde{H} \cdot g^{-1} \quad \text{with} \quad \delta\tilde{H} = \delta\Lambda + [\underbrace{g^{-1}\delta g}_{\doteq\, \delta\Omega}, \Lambda]$$

$\doteq \delta\Omega$, the "angular" variables

. $H \xrightarrow[\text{Jacobian} \,=\, 1]{} \tilde{H} \xrightarrow[\text{Jacobian} \,=\, ?]{} (\Lambda, \Omega)$.

. In components, $\delta \tilde{H}_{i,j} = \delta \Lambda_{i,j} + \sum_{k=1}^{N} \delta \Omega_{i,k} \cdot \delta_{k,j} \lambda_j$

$$- \sum_{k=1}^{N} \lambda_i \delta_{i,k} \cdot \delta \Omega_{k,j}$$

and $\delta \Lambda_{ij} = \delta \lambda_i \cdot \delta_{ij}$

$$\Rightarrow \quad \delta \tilde{H}_{i,j} = \delta \lambda_i \cdot \delta_{i,j} + \delta \Omega_{i,j} \cdot (\lambda_j - \lambda_i)$$

. Thus, the Jacobian between $\tilde{H}$ and (Λ, Ω) is

$$J = \det \left[\frac{\partial(\tilde{H}_{1,1}, \dots, \tilde{H}_{N,N}; \tilde{H}_{1,2}, \dots, \tilde{H}_{1,N}, \dots, \tilde{H}_{N-1,N})}{\partial(\lambda_1, \dots, \lambda_N; \Omega_{1,2}, \dots, \Omega_{1,N}, \dots, \Omega_{N-1,N})} \right]$$

$$= \det \left(\begin{array}{ccc|cccc} 1 & & 0 & & & & \\ & \ddots & & & 0 & & \\ 0 & & 1 & & & & \\ \hline & & & \lambda_1 - \lambda_2 & & & \\ & 0 & & & \lambda_1 - \lambda_3 & & 0 \\ & & & & & \lambda_1 - \lambda_N & \\ & & & & 0 & & \ddots \\ & & & & & & \lambda_{N-1} - \lambda_N \end{array} \right)$$

$$= \prod_{1 \leq i < j \leq N}^{N} (\lambda_i - \lambda_j) \equiv \Delta_N(\lambda).$$

. Therefore, we have $dH = \Delta_N(\lambda) \cdot \underbrace{d\lambda}_{\prod_{i=1}^{N} d\lambda_i} \cdot \underbrace{d\Omega}_{\text{Haar measure on } SO(N)}$.

. On the other hand, from (1), we have

$$p(H)\,dH = \frac{1}{Z_N} e^{-\frac{\mathrm{Tr}(H^2)}{2N}} dH \quad \text{and } \mathrm{Tr}(H^2) = \sum_{k=1}^{N} \lambda_k^2$$

$$= \frac{1}{Z_N} d\Omega \cdot \Delta_N(\lambda) \prod_{k=1}^{N} \left(e^{-\lambda_k^2/2N} d\lambda_k \right)$$

and by integrating out the angular variables, $d\Omega$, we get the result of the Proposition.

. Remark: . The proof for $\beta=2$ and $\beta=4$ are similar. The main difference is that for the non-diagonal terms, instead of 1 we have β variables $\tilde{H}_{ij}$ and $\tilde{\lambda}_{ij}$ too, since for $\beta=2$ we have Real and Imaginary parts of $\tilde{H}_{ij}$, while for $\beta=4$ the $\tilde{H}^1_{ij}, \tilde{H}^2_{ij}, \tilde{H}^3_{ij}$.
. This is the reason why one gets $(\lambda_i-\lambda_j)^\beta$ instead.

2.3) Symmetries in matrices and physical symmetries.

2.3.1) Generalities on symmetries.

. In quantum mechanics, the pure states are given by one-dimensional projections on the Hilbert space $\mathcal{H}$ (describing the space where the system "lives"), called rays: $\hat{\Psi} = \{\varphi \in \mathcal{H},\ \varphi = \alpha\Psi,\ \alpha \in \mathbb{C}\}$.

. Take $\|\Psi\| = 1$, then a one-dimensional projector on the ray containing Ψ is $P_{\hat{\Psi}} = |\hat{\Psi}\rangle\langle\hat{\Psi}|$

. Let A be an observable, a linear operator on $L^2(\mathcal{H})$. Then its expected value on $\hat{\Psi}$ is given by $\langle A\rangle_{\hat{\Psi}} = \mathrm{Tr}(P_{\hat{\Psi}} A)$

$$= \frac{\langle \Psi, A\Psi\rangle}{\langle\Psi|\Psi\rangle},$$

. The transition probability of two pure states $\hat{\Psi}, \hat{\varphi}$ is given by

$$\mathrm{Tr}(P_{\hat{\varphi}} P_{\hat{\Psi}}) = \frac{|\langle\varphi, \Psi\rangle|^2}{\|\varphi\|^2 \cdot \|\Psi\|^2}.$$

. Definition: A symmetry S is an application from $\mathrm{Rays}(\mathcal{H})$ into itself:

$$\begin{cases} - \text{ surjective}: \forall \hat{\varphi}, \exists \hat{\Psi} \text{ st. } \hat{\varphi} = S(\hat{\Psi}). \\ - \text{ Keeping the transition probability invariant}: \\ \quad \mathrm{Tr}(P_{\hat{\varphi}} P_{\hat{\Psi}}) = \mathrm{Tr}(P_{S(\hat{\varphi})} P_{S(\hat{\Psi})}) \end{cases}$$

<u>Theorem (Wigner)</u>: Let S' be a symmetry, then it exists an application $U: \mathcal{H} \to \mathcal{H}$ <u>unitary</u> or <u>antiunitary</u> such that

$S = S_U$ is given by $S_U(\hat{\psi}) = \text{Ray of } (U\psi)$.

. Moreover, U is <u>unique</u> up to a phase factor, i.e., if U' is another application satisfying $S = S_{U'}$, then $U' = \tau U$, with $|\tau| = 1$.

. If $\dim(\mathcal{H}) > 1 \Rightarrow$ the unitary or antiunitary nature of U is determined by S'.

2.3.2) Time reversal symmetry.

. Let T denote the time reversal operator and $|\psi\rangle$ a quantum mechanics state.

. By Wigner's Theorem, $T = K.C$ or $T = K$, K unitary, C is the complex conjugation.

. Let $|\psi(0)\rangle$ be the state at time $t=0$, then at time δt, we have (use $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$) :

$$|\psi(\delta t)\rangle \cong \left(\mathbb{1} - \frac{iH}{\hbar}\delta t\right)|\psi(0)\rangle, \quad H = H^* \text{ the Hamiltonian}$$

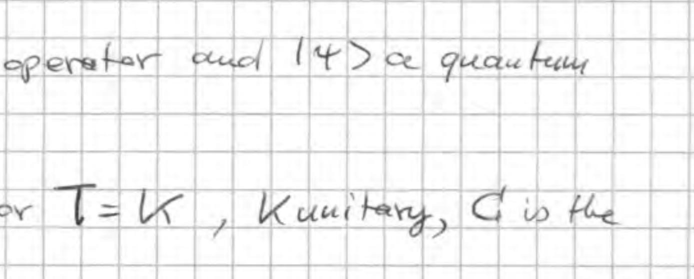
. If the system is time reversal;

$$T\,|\psi(+\delta t)\rangle = e^{-i\frac{H}{\hbar}\delta t}\,T\,|\psi(0)\rangle$$

$$\Rightarrow \left(\mathbb{1} - \frac{iH}{\hbar}\delta t\right)T|\psi(0)\rangle = T\left(\mathbb{1} + \frac{iH}{\hbar}\delta t\right)|\psi(0)\rangle,$$

$\forall\ |\psi(0)\rangle$

$$\Rightarrow -iHT = T(iH)$$



. <u>Q.: Is T unitary or antiunitary?</u>

- Assume T unitary $\Rightarrow T = K$, no conjugation. This means that

$$\{ H, T \} = 0 \qquad (2)$$

- A simple case, free particle, $H = \frac{P^2}{2m}$, P the momentum.

From (2) we get $T^{-1} P^2 T = -P^2$, but time reversal has to satisfy

$T P = -P \quad \Rightarrow T^{-1} P^2 T = P^2$, which is in contradiction with .

Thus T is antiunitary, so $T = K \cdot C \Rightarrow \underline{[T, H] = 0}$.

- Now we have to see what possible choices of K we have.

- A change of representation is a unitary transformation on the states : $|\psi\rangle \longrightarrow U|\psi\rangle$, U unitary.

- A change of representation does not change the scalar product between two states : $\langle \varphi, \psi \rangle = \langle U\varphi, U\psi \rangle = \langle \varphi, U^* U \psi \rangle = \langle \varphi, \psi \rangle$.

Thus also: $\langle \varphi, T\psi \rangle = \langle \varphi, U^* T U \psi \rangle$, $\forall \varphi, \psi \in L^2(\mathcal{H})$, which means that a change of representation transform

T into $U T U^*$.

Now, $T = K \cdot C$ becomes $U K C U^* = U \cdot K \cdot U^t C$

$C(U^* |\psi\rangle) = U^t C |\psi\rangle$ since:

$$C \int U^*(x,y)\, \psi(y)\, dy = C \cdot \int \overline{U(y,x)}\, \psi(y)\, dy = \int U(y,x)\, \overline{\psi(y)}\, dy$$
$$= \int U^t(x,y)\, \overline{\psi(y)}\, dy .$$

which means that the change of representation takes K into $U K U^t$.

- Applying twice T, the physical system has to remain unchanged, therefore $T^2 = \gamma \mathbb{1}$, $|\gamma| = 1$.

$$T^2 = K C K C = K \bar{K} C C = K \bar{K} = \gamma \cdot \mathbb{1} \text{ and } K \text{ is unitary}$$

$$\Rightarrow K K^* = \mathbb{1} \Rightarrow \underbrace{C K C}_{= \bar{K}} \underbrace{C K^*}_{= K^t C} C = \mathbb{1} \Rightarrow \underline{\bar{K} K^t = \mathbb{1}} .$$

$$\Rightarrow K = K(\bar{K}K^t) = \gamma K^t = \gamma(\gamma K^t)^t = \gamma^2 K,$$

thus $\boxed{\gamma = \pm 1.}$

Case (a) $\gamma = +1$: If $K.\bar{K} = \mathbb{1}$, K unitary $\Rightarrow \exists V$ unitary s.t. $K = V.V^t$. (No proof here) check consistency:

$$K\bar{K} = VV^t\,\overline{VV^t} = VV^t\bar{V}V^* = V(\bar{V}^*\bar{V})V^* = VV^* = \mathbb{1}).$$

Then, by an appropriate change of representation,

$|\psi\rangle \longrightarrow V^{-1}|\psi\rangle$, one can take $K = \mathbb{1}$, i.e. $\underline{T = C}$.

· A matrix H is time invariant iff $[H, T] = 0$, i.e.,

$$T^{-1}HT = H$$

With $T = C$, we get $H = CHC = \bar{H}$.

Thus in this case the hamiltonian H can be taken as real symmetric matrix.

Case (b) $\gamma = -1$: If $K.\bar{K} = -\mathbb{1}$, then $\exists V$ s.t. $K \to VKV^t = e_2 \otimes \mathbb{1}_N$.

(No proof; consistency: $(e_2 \otimes \mathbb{1}_N)\overline{(e_2 \otimes \mathbb{1}_N)} = -\mathbb{1}_N$).

· So, in this case we can set $\underline{T = e_2 \otimes \mathbb{1} \cdot C}$

· As exercice, one can check that a matrix of the form

$$H = H^0 e_0 + H^1 e_1 + H^2 e_2 + H^3 e_3 \text{ satisfy}$$

$[H, T] = 0$ iff H^0 is real symmetric, and H^1, H^2, H^3 are real antisymmetric.

· Magnetic momentum and rotations (a remark)

· A system invariant by rotation has an Hamiltonian which commute with the generator of the rotations; the magnetic momentum J (not to be mixed up with spins; J has integer magnetic momentum).

·

Conclusion: . Time reversal symmetry implies that in an appropriate representation we have either a real symmetric matrix or a real quaternionic matrix for the Hamiltonian.

. When time reversal symmetry is not present, one has just the "usual" hermitian matrix.

Exercices: (1) Show the formula for the joint distributions of eigenvalues in the $\beta=2$ and $\beta=4$ cases.

(2) Check the operator identity $CUC=\overline{U}$ where C is the complex conjugation.

(3) Consider the "entropy" functional

$$S(P)=-\int p(H)\ln(p(H))\,dH$$

Show that under the condition $\mathbb{E}\left(\frac{\mathrm{Tr}(H^2)}{2N}\right)=\frac{n}{2}$, where $n=N+\frac{\beta N(N-1)}{2}$ (degrees of freedom), the measure with

$$p(H)=\frac{1}{Z_N}\exp\left(\frac{-\mathrm{Tr}(H^2)}{2N}\right)$$

maximize $S(P)$.

Solutions of the excercices:

① For $\beta = 2$, the strategy is the same as for $\beta = 1$.
We have also in this case $\delta H = g \cdot \delta\tilde{H} \cdot g^{-1}$ with

$$\delta\tilde{H} = \delta\Lambda + [g^{-1}\delta g, \Lambda].$$

In components: $\delta\tilde{H}_{i,i} = \delta\lambda_i$ and

for $j > i$ $\quad \delta(\mathrm{Re}\,\tilde{H}_{ij}) = (\lambda_j - \lambda_i) \cdot \delta(\mathrm{Re}\,\Omega_{ij})$

and $\delta(\mathrm{Im}\,\tilde{H}_{ij}) = (\lambda_j - \lambda_i) \cdot \delta(\mathrm{Im}\,\Omega_{ij})$

Thus, the Jacobian between $\tilde{H}$ and (Λ, Ω) is

$$J = \det \left(\begin{array}{c|ccc|ccc} \begin{matrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{matrix} & & 0 & & & 0 & \\ \hline & \lambda_1 - \lambda_2 & & & & & \\ 0 & & \ddots & & & 0 & \\ & & & \lambda_{N-1} - \lambda_N & & & \\ \hline & & & & \lambda_1 - \lambda_2 & & \\ 0 & & 0 & & & \ddots & \\ & & & & & & \lambda_{N-1} - \lambda_N \end{array} \right)$$

$$= \left[\prod_{1 \le i < j \le N}^{N} (\lambda_i - \lambda_j) \right]^2.$$

. For $\beta = 4$, it is similar. Now, we have instead of only the real and imaginary parts, we have the 4 components of the basis of the quaternionic numbers.

② Let us see how they apply to functions Ψ:

$$(C\,U\,C\,\psi)(x) = C \cdot \int U(x,y)\,\overline{\psi(y)}\,dy = \int \bar{U}(x,y)\,\psi(y)\,dy$$
$$= (\bar{U}\psi)(x).$$

3) Consider the constraint

$$C' = \int p(H)\,\mathrm{Tr}(H^2)\,dH.$$

~~Let~~ $B = -(1+\ln(A))$ and λ be the Lagrange multiplier, i.e.,

$$S(p) = -\int p(H)\ln p(H)\,dH - \lambda\cdot\left(\int p(H)\,\mathrm{Tr}(H^2)\,dH - C_1\right) + (\ln A + 1)\left(\int p(H)\,dH - 1\right)$$

. Let p_0 be the distribution which maximize $S(p)$, then at first order in δp,

$$\delta S(p_0) = S(p_0 + \delta p) - S(p) = 0, \text{ i.e.,}$$

$$-\cancel{1} - \ln(p_0) - \lambda\,\mathrm{Tr}(H^2) + \cancel{1} + \ln(A) = 0,$$

which implies $p_0(H) = A\cdot\exp\left(-\lambda\,\mathrm{Tr}(H^2)\right)$.

. The normalization constant fixes the value of A,

$$A^{-1} = \int e^{-\lambda\mathrm{Tr}(H^2)}\,dH \equiv a(\lambda).$$

. Thus, the second constraint writes

$$C' = \frac{1}{a(\lambda)}\int \mathrm{Tr}(H^2)\cdot e^{-\lambda\cdot\mathrm{Tr}(H^2)}\,dH = -\frac{1}{a(\lambda)}\frac{d\,a(\lambda)}{d\lambda}$$

. Let us get $a(\lambda)$ now. By the change of variable $X = \sqrt{\lambda}\,H$, we obtain

$$a(\lambda) = \lambda^{-n/2}\int e^{-\mathrm{Tr}(X^2)}\,dX,$$ where n is the dimension of the space where the integral is made.

. The integral over dX is independent of λ, thus

$$\frac{da(\lambda)}{d\lambda} = \frac{-n}{2\lambda}\cdot a(\lambda),$$ and finally $\lambda = \frac{n}{2C}$.

Since we want $\lambda = \frac{1}{2N}$, $C = N\cdot n$. This finishes the proof.

3) n-point correlation functions for the GUE eigenvalues

In the previous lecture we determined the joint distribution of the $N\times N$ GUE eigenvalues, $\{\lambda_1,\dots,\lambda_N\}$:

(1) $$P(\lambda_1,\dots,\lambda_N)\,d\lambda_1\dots d\lambda_N = \frac{1}{Z_N}\Big(\prod_{1\le i<j\le N}(\lambda_i-\lambda_j)^2\Big)\prod_{k=1}^{N}\Big(e^{-\frac{\lambda_k^2}{2N}}d\lambda_k\Big)$$

In this lecture we want to obtain the expression of the n-point correlation functions, which we first have to define.

3.1) Correlation functions.

. Consider a measure on $\mathbb{R}^N$ like (1), and let us take any bounded disjoints Borel sets $A_1,\dots,A_n$ of $\mathbb{R}$. Then, let

$$M_n(A_1,\dots,A_n) \doteq \mathbb{E}\Big(\prod_{i=1}^{n}(\#\text{ eigenvalues in } A_i)\Big),$$

where $\mathbb{E}$ is the expectation under the measure on $\mathbb{R}^N$ (e.g., (1) in our case).

Definition: (Correlation functions) If M_n is absolutely continuous with respect to a reference measure on $\mathbb{R}^N$, μ^N, i.e.,

(2) $$M_n(A_1,\dots,A_n) = \int_{A_1\times\dots\times A_n} d\mu(x_1)\dots d\mu(x_n)\,\rho^{(n)}(x_1,\dots,x_n)$$

for all Borel sets A_i in $\mathbb{R}$,

then we call $\rho^{(n)}(x_1,\dots,x_n)$ the n-point correlation function.

. Remarks: (1) From (2) one can think of $\rho^{(n)}$ as Radon–Nikodym derivative on $\mathbb{R}^n$ of the measure M_n.

(2) $\rho^{(n)}(x_1,\dots,x_n) = \rho^{(n)}(x_{\sigma_1},\dots,x_{\sigma_n})$, $\forall\sigma\in S_N$ (permutations of $\{1,\dots,n\}$). This symmetry is obvious since l.h.s. of (2) is independent of the order of the A_i's.

③ We will probably come back later, when we will discuss point processes in general, to the question whether the set of all correlation functions, $\{\rho^{(n)}, n \geq 1\}$ defines uniquely a measure. One not too strong but sufficient condition will be : $\rho^{(n)}(x_1, \dots, x_n) \leq n^{2n} \cdot c^n$ a.s. for some $c > 0$.

④ To speak about n-point correlation functions without specifying the reference measure makes no sense, although when the reference measure is the Lebesgue measure one tends not to specify it.
In our GUE example, we might choose $d\mu_1(x) = e^{-\frac{x^2}{2N}} dx$ and get the n-point correlation functions $\rho_1^{(n)}(x_1, \dots, x_n)$, but also $d\mu_2(x) = dx$ and get $\rho_2^{(n)}(x_1, \dots, x_n)$. Then,

$$\rho_1^{(n)}(x_1, \dots, x_n) = \rho_2^{(n)}(x_1, \dots, x_n) \cdot \prod_{i=1}^{N} e^{-\frac{x_i^2}{2N}}.$$

⑤ Probabilistic interpretation: In the case where a.s. one does not have double points (like in our GUE case, or more generally for simple point processes, see further lectures), then we have the following probabilistic interpretation.
Let $[x_i, x_i + \Delta x_i]$, $i = 1, \dots, n$ be disjoints infinitesimally small sets, then we will have at most one point in each $[x_i, x_i + \Delta x_i]$, and so

$$\rho^{(n)}(x_1, \dots, x_n) = \lim_{\substack{\Delta x_i \to 0 \\ i = 1, \dots, n}} \frac{\mathbb{P}(\text{one particle in each } [x_i, x_i + \Delta x_i], 1 \leq i \leq n)}{\Delta x_1 \cdots \Delta x_n}$$

In particular: $\rho^{(1)}(x)$ is the density of particles (points, eigenvalues) at position x.

⑥ If instead of $\mathbb{R}$ we have $\mathbb{Z}$ or any other discrete sets, then $\rho^{(n)}(x_1,\dots,x_n)$ is the probability of finding particles at $x_1,\dots,x_n$.

<u>Lemma:</u> • A particular situation is when, like in our GUE case, $P_N(x_1,\dots,x_N)$ is a <u>symmetric</u> probability measure on $\mathbb{R}^N$.

Then

$$\rho^{(n)}(x_1,\dots,x_n) = \frac{N!}{(N-n)!} \cdot \int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N \, P_N(x_1,\dots,x_N) \tag{3}$$

<u>Proof:</u> $\rho^{(n)}(x_1,\dots,x_n)$ is the probability density of finding <u>a</u> particle at x_1, <u>a</u> particle at x_2, …, <u>a</u> particle at x_n, but it does not keep the information of which of the N particle is at which of the x_i's.

Using the symmetry of $P_N(x_1,\dots,x_N)$, each possible choice gives a contribution equal to

$$\int_{\mathbb{R}^{N-n}} dx_{n+1}\cdots dx_N \, P_N(x_1,\dots,x_N),$$

and there are $n! \cdot \binom{N}{n} = \frac{N!}{(N-n)!}$ possible choices. #

<u>3.2) Application to GUE</u>

• We will now apply (3) to our measure (1), but before we rewrite (1) in another form.

<u>3.2.1) Orthogonal polynomials.</u>

• Let $w(x) = e^{-\frac{x^2}{2N}}$ and define the orthogonal polynomials $\{q_k(x),\, k=0,\dots,N-1\}$ by the following conditions:

① $q_k(x)$ is a polynomial of degree k, with

(4) $$q_k(x) = u_k \cdot x^k + \dots, \quad u_k > 0,$$

② and they satisfy the orthogonality condition:

(5) $$\int_{\mathbb{R}} dx\, w(x) \cdot q_k(x)\, q_\ell(x) = \delta_{k,\ell}.$$

3.2.2) Kernel K_N.

. Then, the measure (1) can be rewritten using the polynomials $q_k(x)$ (which will be computed at the end of this lecture), namely,

since $\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N} = \text{const} \times \det\left(q_{j-1}(\lambda_i)\right)_{1\le i,j\le N}$,

$$P(\lambda_1, \dots, \lambda_N) = \frac{1}{\tilde{Z}_N} \cdot \left(\prod_{k=1}^{N} w(\lambda_k)\right) \cdot \left(\det\left(q_{i-1}(\lambda_j)\right)_{1\le i,j\le N}\right)^2.$$

(6)
$$= \frac{1}{\tilde{Z}_N} \left(\prod_{k=1}^{N} w(\lambda_k)\right) \cdot \det\left(\sum_{k=1}^{N} q_{k-1}(\lambda_i)\, q_{k-1}(\lambda_j)\right)_{1\le i,j\le N}.$$

matrix multiplication rules

The w's can also be taken into the determinant by using its multilinearity property. We then get:

(7) $$P(\lambda_1, \dots, \lambda_N) = \frac{1}{\tilde{Z}_N} \cdot \det\left[K_N(\lambda_i, \lambda_j)\right]_{1\le i,j\le N},$$

with $$K_N(x,y) = \sqrt{w(x)}\sqrt{w(y)} \cdot \sum_{k=0}^{N-1} q_k(x)\, q_k(y).$$

. Two properties of K_N: (obtained applying (5)).

(8) $$\int_{\mathbb{R}} dx\, K_N(x,x) = N$$

(9) $$\int_{\mathbb{R}} dz\, K_N(x,z)\, K_N(z,y) = K_N(x,y).$$

3.2.3) n-point correlation functions

Lemma: $\rho^{(n)}(\lambda_1,\dots,\lambda_N) = \det\left[K_N(\lambda_i,\lambda_j)\right]_{1\le i,j\le n}$ (10)

Proof: First we determine $\tilde{Z}_N$ in (7).

For $n=N$, $\rho^{(n)}(\lambda_1,\dots,\lambda_N) \overset{(3)}{=} P(\lambda_1,\dots,\lambda_N)\cdot N!$

and: $\int_{\mathbb{R}^N} d\lambda_1\dots d\lambda_N\, \rho^{(n)}(\lambda_1,\dots,\lambda_N) = N!$

$\| (6)$

$$\frac{N!}{\tilde{Z}_N}\cdot\int_{\mathbb{R}^N} d\lambda_1\dots d\lambda_N\; w(\lambda_1)\dots w(\lambda_N)\cdot\det\left(q_{i-1}(\lambda_j)\right)_{1\le i,j\le N}\cdot\det\left(q_{i-1}(\lambda_j)\right)_{1\le i,j\le N}$$

$$\overset{\circledast}{=} \frac{N!}{\tilde{Z}_N}\left[N!\;\det\Big(\underbrace{\int_{\mathbb{R}^N} d\lambda\, w(\lambda)\, q_{i-1}(\lambda)\, q_{j-1}(\lambda)}_{\overset{(5)}{=}\,\delta_{ij}}\Big)_{1\le i,j\le N}\right] = \frac{(N!)^2}{\tilde{Z}_N}.$$

Thus $\tilde{Z}_N = N!$

. In $\circledast$ we used the Cauchy-Binet (or Heine) identity:

$$\boxed{\det\left[\int_\Lambda d\lambda(x)\,\phi_i(x)\,\psi_j(x)\right]_{1\le i,j\le N} = \frac{1}{N!}\cdot\int_{\Lambda^N}\det(\phi_i(x_j))_{1\le i,j\le N}\cdot\det(\psi_i(x_j))_{1\le i,j\le N}\; d\lambda(x_1)\dots d\lambda(x_N)} \quad (11)$$

. Now, by (3),(7) we have

$$(12)\quad \rho^{(n)}(\lambda_1,\dots,\lambda_n) = \frac{1}{(N-n)!}\cdot\int_{\mathbb{R}^{N-n}} d\lambda_{n+1}\dots d\lambda_N\;\det\left[K_N(\lambda_i,\lambda_j)\right]_{1\le i,j\le N}$$

. We need to integrate $N-n$ times, so consider one of the integral to be done, say the one with the determinant of size $m \times m$.

Then,
$$\int_{\mathbb{R}} dx_m \det\left[K_N(x_i, x_j)\right]_{1 \le i,j \le m} = \, ?$$

$$\det\left(K_N(x_i,x_j)\right)_{1\le i,j\le m} = \det \left[\begin{array}{ccc|c} K_N(x_1,x_1) & \cdots & K_N(x_1,x_{m-1}) & K_N(x_1,x_m) \\ \vdots & & \vdots & \vdots \\ K_N(x_{m-1},x_1) & \cdots & K_N(x_{m-1},x_{m-1}) & K_N(x_{m-1},x_m) \\ \hline K_N(x_m,x_1) & \cdots & K_N(x_m,x_{m-1}) & K_N(x_m,x_m) \end{array}\right]$$

$$= K_N(x_m,x_m) \cdot \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} K_N(x_k,x_m) \cdot \det \left[\begin{array}{c} K_N(x_i,x_j) \\ \hline K_N(x_m,x_j) \end{array}\right]_{\substack{1\le i,j\le m-1 \\ \text{and } i\neq k}}$$

$$\underset{\substack{\uparrow \\ \text{by linearity}}}{=} K_N(x_m,x_m) \cdot \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} \cdot \det \left[\begin{array}{c} K_N(x_i,x_j) \\ \hline K_N(x_k,x_m)\cdot K_N(x_m,x_j) \end{array}\right]_{\substack{1\le i,j\le m-1 \\ \text{and } i\neq k}}$$

. We use this decomposition and compute, using (8) and (9),

$$(13) \quad \int_{\mathbb{R}} dx_m \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m} = N \cdot \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m-1}$$

$$+ \sum_{k=1}^{m-1} (-1)^{m-k} \cdot \det \left[\begin{array}{c} K_N(x_i,x_j) \\ \hline K_N(x_k,x_j) \end{array}\right]_{\substack{1\le i,j\le m-1 \\ \text{and } i\neq k}}$$

$$\overset{\substack{\text{reordering the} \\ K_N(x_n,x_j)}}{=} [N-(m-1)] \cdot \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le m-1}.$$

. We plug this result into (12) for $m=N, N-1, \dots, n+1$, and get

$$\rho^{(n)}(\lambda_1,\dots,\lambda_n) = \frac{1.2 \dots \cancel{(N-n)}}{\cancel{(N-n)!}} \cdot \det\left[K_N(\lambda_i,\lambda_j)\right]_{1\le i,j\le n}. \quad \#$$

3.2.4) Another representation of the kernel

. Christoffel-Darboux formula: For orthogonal polynomials one has the three term relationship

$$q_n(x) = (A_n x + B_n)\, q_{n-1}(x) - C_n \cdot q_{n-2}(x), \quad n=2,3,\dots,$$

with $A_n>0$, B_n, $C_n>0$ some constants, which are given in term of the highest coefficient of $q_n(x)$, u_n, by

$$A_n = \frac{u_n}{u_{n-1}}, \quad C_n = \frac{A_n}{A_{n-1}} = \frac{u_n u_{n-2}}{u_{n-1}^2}.$$

Then,

$$(14) \quad \boxed{\sum_{k=0}^{n-1} q_k(x)\, q_k(y) = \begin{cases} \dfrac{u_{n-1}}{u_n} \cdot \dfrac{q_n(x)\cdot q_{n-1}(y) - q_{n-1}(x)\, q_n(y)}{x-y}, & \text{for } x\neq y, \\[2ex] \dfrac{u_{n-1}}{u_n} \cdot \left[q_n'(x)\, q_{n-1}(x) - q_{n-1}'(x)\, q_n(x)\right], & \text{for } x=y. \end{cases}}$$

. Using the Christoffel-Darboux formula (14) we get for our GUE case, that the kernel writes

$$(15) \quad K_N(x,y) = \sqrt{w(x)\cdot w(y)} \cdot \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}$$

3.2.5) Explicit kernel for GUE

. In the GUE case, $w(x) = \exp\left(-\frac{x^2}{2N}\right)$, the orthogonal polynomials $q_k(x)$ are given in terms of Hermite polynomials:

$$(16) \begin{cases} P_k^H(x) \doteq e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, \text{ they satisfy} \\ \int_{\mathbb{R}} P_k^H(x) P_\ell^H(x) e^{-x^2} dx = \sqrt{\pi} \cdot 2^k \cdot k! \cdot \delta_{k,\ell} \\ \text{with } P_k^H(x) = 2^k \cdot x^k + \dots \end{cases}$$

. From (16) one gets, up to rescaling $x \to \frac{x}{\sqrt{2N}}$, that (exercice!)

$$(17) \begin{cases} q_k(x) = \frac{1}{\sqrt[4]{2\pi N}} \cdot \frac{1}{\sqrt{2^k \cdot k!}} \cdot P_k^H(x/\sqrt{2N}), \quad \frac{u_{N+1}}{u_N} = N, \text{ thus} \\ K_N(x,y) = N \cdot e^{-\frac{x^2+y^2}{4}} \cdot \frac{q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y)}{x-y} \end{cases}$$

. Since in (17) the Hermite polynomials enters, this kernel is also called the Hermite Kernel.

Exercices: 1 Prove properties (8) and (9) on the kernel K_N.

2 Prove the Cauchy-Binet formula (11).

3 Prove the Christoffel-Darboux formula (14).

Solutions to the exercices:

1. $$K_N(x,y) = \sqrt{w(x)}\sqrt{w(y)} \sum_{k=0}^{N-1} q_k(x) q_k(y)$$

$$\Rightarrow \int_{\mathbb{R}} dx\, K_N(x,y) = \sum_{k=0}^{N-1} \int_{\mathbb{R}} dx\, w(x)\, q_k(x)^2 \underset{(5)}{=} \sum_{k=0}^{N-1} 1 = N$$

Thus (8) is proven.

$$\int_{\mathbb{R}} dz\, K_N(x,z) K_N(z,y) = \sum_{k,l=0}^{N-1} \sqrt{w(x)}\sqrt{w(y)}\, q_k(x) q_l(y) \cdot \underbrace{\int_{\mathbb{R}} dz\, w(z)\, q_k(z) q_l(z)}_{=\delta_{k,l} \text{ by } (5)}$$

$$= \sum_{k=0}^{N-1} \sqrt{w(x)}\sqrt{w(y)}\, q_k(x) q_k(y) = K_N(x,y),$$

which is property (9).

2. We start with l.h.s. of (11):

$$\det\left[\int_{\Lambda} d\lambda(x)\, \Phi_i(x) \Psi_j(x)\right]_{1\le i,j\le N} \overset{\text{linearity}}{=\!=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N) \det\left[\Phi_i(x_i)\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\text{linearity}}{=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N) \left[\prod_{i=1}^N \Phi_i(x_i)\right] \cdot \det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\substack{\forall \text{ permutation} \\ \sigma\in S_N}}{=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N) \left[\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\right] \cdot \det\left[\Psi_j(x_{\sigma(i)})\right]_{1\le i,j\le N}$$

$$\overset{\substack{\text{antisymmetry} \\ \text{of determinant}}}{=} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N)\, \mathrm{sgn}(\sigma) \cdot \left[\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\right] \cdot \det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

$$\overset{\substack{\text{integral indep.} \\ \text{of } \sigma}}{=} \frac{1}{N!} \int_{\Lambda^N} d\lambda(x_1)\dots d\lambda(x_N) \underbrace{\sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \cdot \left[\prod_{i=1}^N \Phi_i(x_{\sigma(i)})\right]}_{\overset{\text{def of the determinant}}{=\!=} \det\left[\Phi_i(x_j)\right]_{1\le i,j\le N}} \cdot \det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}$$

#

3 . First we prove the three term relation.

- $\frac{q_N(x)}{u_N} - \frac{x \cdot q_{N-1}(x)}{u_{N-1}}$ is a polynomial of degree $N-1$,

thus $\frac{q_N(x)}{u_N} = \frac{x \cdot q_{N-1}(x)}{u_{N-1}} + \sum_{k=0}^{N-1} \alpha_k \cdot q_k(x)$,

with $\alpha_k = \langle \frac{q_N}{u_N} - \frac{x \cdot q_{N-1}}{u_{N-1}}, q_k \rangle$, where the scalar product is defined as: $\langle a, b \rangle = \int_{\mathbb{R}} dx\, w(x)\, a(x)\, b(x)$.

- For $k = 0, \ldots, N-3$, $\alpha_k = 0$. In fact, we use

$\langle x \cdot a, b \rangle = \langle a, x b \rangle$, to see that

$$\alpha_k = \frac{1}{u_N} \underbrace{\langle q_N, q_k \rangle}_{=0} - \frac{1}{u_{N-1}} \cdot \langle q_{N-1}, \underbrace{x \cdot q_k}_{\substack{\text{polynomial,} \\ \text{degree } k+1}} \rangle \quad = 0 \text{ for } k+1 < N-1$$

- For $k = N-2$:

$$\alpha_{N-2} = -\frac{1}{u_{N-1}} \langle q_{N-1}, x \cdot q_{N-2} \rangle$$

and we can write $x\, q_{N-2} = u_{N-2}\, x^{N-1} + \mathrm{poly}_{N-2}(x)$

$$= \frac{u_{N-2}}{u_{N-1}} \cdot q_{N-1}(x) + \mathrm{poly}'_{N-2}(x)$$

$$= -\frac{1}{u_{N-1}} \cdot \frac{u_{N-2}}{u_{N-1}}.$$

- From this, by setting $B_N = \alpha_{N-1} \cdot u_N$, $A_N = \frac{u_N}{u_{N-1}}$, $C_N = \frac{u_N u_{N-2}}{(u_{N-1})^2}$,

we get: $\underline{q_N(x) = (A_N x + B_N) \cdot q_{N-1}(x) - C_N \cdot q_{N-2}(x).}$ (*)

- The second step is to use (*) to show

$$q_{N+1}(x)\, q_N(y) - q_N(x)\, q_{N+1}(y) = (x-y)\, q_N(x)\, q_N(y) \cdot A_{N+1} + C_{N+1} \cdot \left[q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y) \right]$$

- $q_{N+1}(x)\, q_N(y) - q_N(x)\, q_{N+1}(y) =$

$$= \Big[\big(A_{N+1}x + \cancel{B_{N+1}}\big) q_N(x) - C_{N+1}\, q_{N-1}(x)\Big] \cdot q_N(y)$$

$$- q_N(x) \cdot \Big[\big(A_{N+1}y + \cancel{B_{N+1}}\big) q_N(y) - C_{N+1} \cdot q_{N-1}(y)\Big]$$

$$= A_{N+1} \cdot (x-y) \cdot q_N(x)\, q_N(y) + C_{N+1} \cdot \big[q_N(x) q_{N-1}(y) - q_{N-1}(x)\, q_N(y)\big]$$

- Divide this identity by $\underline{(x-y)A_{N+1}}$, and get:

$$\frac{u_N}{u_{N+1}} \cdot \frac{q_{N+1}(x) q_N(y) - q_N(x) q_{N+1}(y)}{x-y} = q_N(x)\, q_N(y) + \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y)}{x-y}$$

$\Rightarrow$ <u>For $k \geq 1$:</u> $$q_k(x)\, q_k(y) = \frac{u_k}{u_{k+1}} \cdot \frac{q_{k+1}(x)\, q_k(y) - q_k(x) q_{k+1}(y)}{x-y} - \frac{u_{k-1}}{u_k} \cdot \frac{q_k(x) q_{k-1}(y) - q_{k-1}(x)\, q_k(y)}{x-y}$$

$$\Rightarrow \sum_{k=0}^{N-1} q_k(x)\, q_k(y) = \frac{u_{N-1}}{u_N} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}.$$

<u>We use:</u> $k=0$: $q_0(x) q_0(y) = u_0^2 = \dfrac{u_0}{u_1} \cdot \dfrac{q_1(x)\, q_0(y) - q_0(x)\, q_1(y)}{x-y}$ since $q_0(x) = u_0$, $q_1(x) = u_1 x + \alpha$.

- For $x=y$, one does just take the limit of the previous formula as $x \to y$.

#

3.3) Universal limits: Sine Kernel (bulk) and Airy Kernel (edge).

. From (17) of the last lecture, we have

(18) $$K_N(x,y) = N \cdot e^{-\frac{x^2}{4N}} \cdot e^{-\frac{y^2}{4N}} \cdot \frac{q_N(x)\, q_{N-1}(y) - q_{N-1}(x)\, q_N(y)}{x-y}$$

with $q_k(x) = \frac{1}{(2\pi N)^{1/4}} \cdot \frac{1}{\sqrt{2^k k!}} \cdot P_k^H(x/\sqrt{2N})$, P_k^H the Hermite polynomials.

3.3.1) Wigner semicircle law.

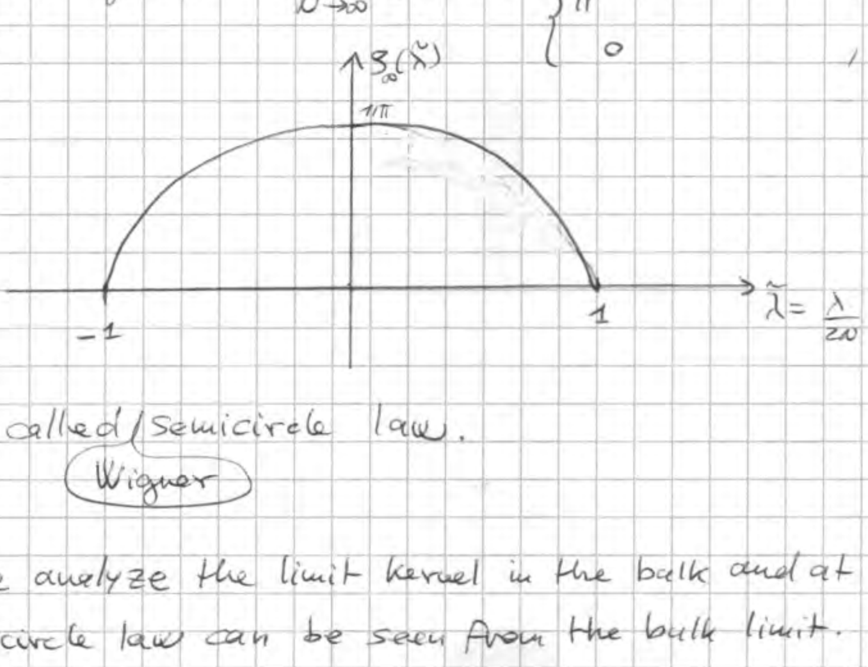
. The largest eigenvalue is close to $2N$ and the smallest eigenvalue close to $-2N$.

. Since there are exactly N eigenvalues, the density of eigenvalues between $-2N$ and $2N$ is of order one also as $N \to \infty$.

. The eigenvalue density can be written in term of the kernel:

(19) $$\underline{\rho_N(\tilde{\lambda}) = K_N(2N\tilde{\lambda}, 2N\tilde{\lambda})}$$

. Computations gives: $\lim_{N\to\infty} \rho_N(\tilde{\lambda}) = \begin{cases} \frac{1}{\pi} \cdot \sqrt{1-\tilde{\lambda}^2}, & \tilde{\lambda} \in [-1,1], \\ 0, & \tilde{\lambda} \notin (-1,1). \end{cases}$



. This is called (Wigner) semicircle law.

. Next we analyze the limit kernel in the bulk and at the edge.
The semicircle law can be seen from the bulk limit.

3.3.2) Limit Kernels (universal)

- There are two limit kernels which appear in a lot of models in an appropriate asymptotic limit (large time or thermodynamic limit or large N matrices). The Sine kernel arises in the bulk of the system, while the Airy kernel at the edge.

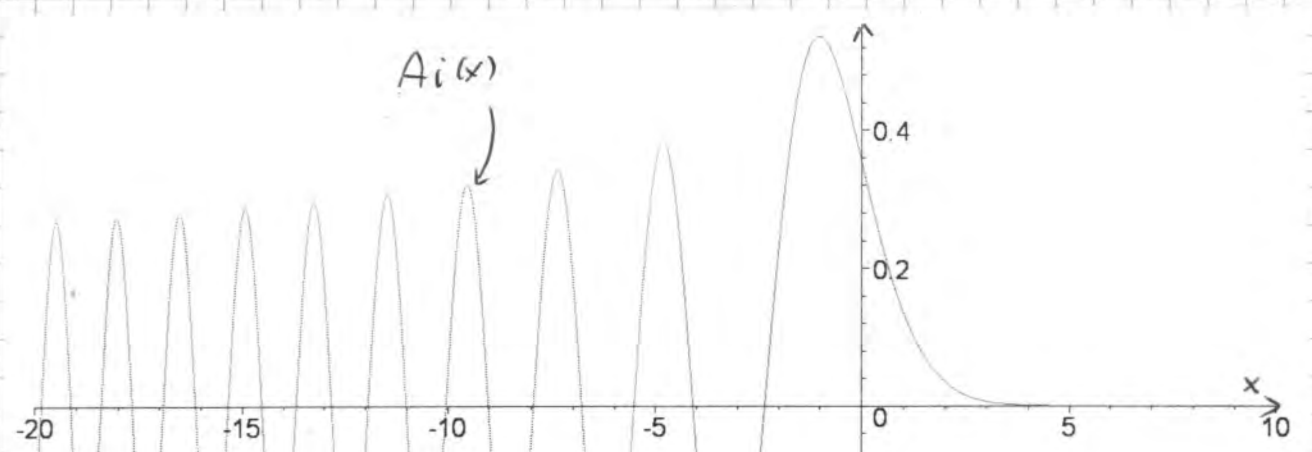
<u>Definition</u>: The <u>sine kernel</u> is defined as $S(x,y) = \dfrac{\sin(\pi(x-y))}{\pi(x-y)}$.

This describes the bulk with density of eigenvalues normalized to one.

<u>Definition</u>: The <u>Airy kernel</u> is defined as $\mathcal{A}(x,y) = \dfrac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$,

where Ai is the Airy function, i.e. solution of

$$y''(x) = x \cdot y(x) \quad , \quad y(x) \sim \frac{\exp(-\frac{2}{3}x^{3/2})}{2\sqrt{\pi}\, x^{1/4}} \text{ as } x \to +\infty$$



3.3.3) Bulk scaling limit.

. Let us consider a $\tilde{\lambda} \in (-1,1)$ and focus around $2N \cdot \tilde{\lambda}$ as follows:

$$(20) \qquad \begin{cases} x = 2N \cdot \tilde{\lambda} + \xi_1 \\ y = 2N \cdot \tilde{\lambda} + \xi_2 \end{cases}$$

<u>Proposition:</u>
$$\lim_{N \to \infty} K_N(2N\tilde{\lambda} + \xi_1, 2N\tilde{\lambda} + \xi_2) = \frac{\sin(\pi \cdot \rho(\tilde{\lambda})(\xi_1 - \xi_2))}{\pi(\xi_1 - \xi_2)} \equiv \rho(\tilde{\lambda}) \cdot S'(\rho(\tilde{\lambda})\xi_1, \rho(\tilde{\lambda})\xi_2)$$
with $\rho(\tilde{\lambda}) = \frac{1}{\pi}\sqrt{1 - \tilde{\lambda}^2}$

. <u>Sketch of the proof:</u> There are known asymptotic expansions of the Hermite polynomials [see e.g. Szegö; Abramowitz-Stegun].

These, translated into our q_n's becomes:

$$(21) \qquad \sqrt{N} \cdot q_{N-h}(x) \cdot e^{-\frac{x^2}{4N}} \underset{x = 2N\tilde{\lambda} + \xi}{\cong} \frac{1}{\pi \cdot \sqrt{\rho(\tilde{\lambda})}} \cdot \sin(\alpha_0 N + \pi \rho(\tilde{\lambda}) \xi + \tilde{\alpha} h),$$
with $\tilde{\alpha} = \arccos(\tilde{\lambda})$.

One uses (21) in (18) and gets

$$(22) \qquad K_N(2N\tilde{\lambda} + \xi_1, 2N\tilde{\lambda} + \xi_2) \cong \frac{1}{\pi^2 \rho(\tilde{\lambda})} \frac{1}{\xi_1 - \xi_2} \cdot \Big[\sin(\alpha_0 N + \pi\rho \cdot \xi_1)\sin(\alpha_0 N + \pi\rho \cdot \xi_2 + \tilde{\alpha}) - \sin(\alpha_0 N + \pi\rho\xi_1 + \tilde{\alpha})\sin(\alpha_0 N + \pi\rho\xi_2)\Big]$$

Then apply the identity: $\sin(a)\sin(b+\alpha) - \sin(a+\alpha)\sin(b) = \sin(\alpha)\sin(a-b)$ and $\sin(\arccos(x)) = \sqrt{1-x^2}$, to get the result. #

3.3.4) Edge scaling limit.

- The largest eigenvalue is around $2N$ and it fluctuates on a $N^{1/3}$ scale. The $1/3$ exponent is a consequence of the square root behavior of the density at the edge of the spectrum.

- Heuristics:

$$\#\text{ eigenvalues} \geq 2N-x \;\approx\; N\cdot\left(\frac{x}{N}\right)^{3/2} = \frac{x^{3/2}}{\sqrt{N}} \longleftrightarrow \text{over a distance } x.$$

$\Rightarrow$ # eigenvalues $\geq 2N-x$ is $\mathcal{O}(1)$, for $x \sim N^{1/3}$, i.e., the top eigenvalues fluctuates over distances $\mathcal{O}(N^{1/3})$.

- Therefore, the scaling limit is as follows:

$$(23)\qquad \begin{cases} x = 2N + \xi_1 N^{1/3} \\ y = 2N + \xi_2 N^{1/3} \end{cases}$$

- Proposition: $\displaystyle\lim_{N\to\infty} N^{1/3} K_N(2N+\xi_1 N^{1/3}, 2N+\xi_2 N^{1/3}) = \mathcal{A}(\xi_1,\xi_2).$

- Remark: The $N^{1/3}$ factor is because of the spacial rescaling (remember that the n-pt. correlation functions, which are densities, are given in terms of K_N).

- Sketch of the proof: One uses asymptotics of Hermite polynomials, which rewrites:

$$N^{1/3} q_{N-n}(x)\cdot e^{-\frac{x^2}{4N}} \underset{x=2N+\xi N^{1/3}}{\cong} Ai\left(\xi + N^{-1/3}\cdot\left(n-\tfrac{1}{2}\right)\right).$$

By replacing it into (18) and taking the $N\to\infty$ limit we get the result. #

Remark: As stated the convergence is pointwise, but by using, for example, double integral representations for the Hermite kernel, one can get some uniform convergence on bounded sets for the bulk and for sets bounded from below for the edge.

Exercices: 1 Consider the GUE ensemble and let

$$E_N(L) = \#\text{ eigenvalues in } [-L, L].$$

. Take the $N \to \infty$ limit first, $E(L) \doteq \lim_{N \to \infty} E_N(L)$.

Prove that: $$\lim_{L \to \infty} \frac{\mathrm{Var}(E(L))}{\ln(L)} = \frac{1}{\pi^2}$$

2 Do the same but for a poisson point process on $\mathbb{R}^d$ with density ρ (box $[0,L]^d$).

3 Compute the probability of not having particles in a box $[0,L]^d$ for a point process on $\mathbb{R}^d$, intensity ρ.

Solutions of exercices:

1. $$\mathrm{Var}\big(E_N(L)\big) = \int_{-L}^{L} dx\, \rho_N^{(1)}(x) + \int_{-L}^{L} dx \int_{-L}^{L} dy\, \rho_N^{(2)}(x,y) - \left(\int_{-L}^{L} dx\, \rho_N^{(1)}(x)\right)^2$$

where $\rho_N^{(1)}(x) = K_N(x,x)$, $\rho_N^{(2)}(x,y) = K_N(x,x)K_N(y,y) - K_N(x,y)K_N(y,x)$.

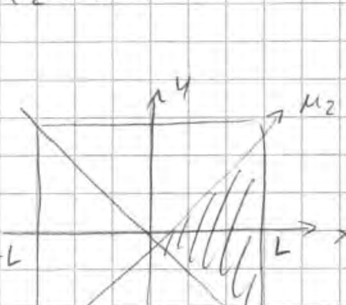
• When $N \to \infty$, our Kernel K_N converges to the sine Kernel with density $\rho = \pi^{-1}$.

Thus, $$\mathrm{Var}(E(L)) = \int_{-L}^{L} dx\, \rho + \int_{-L}^{L} dx \int_{-L}^{L} dy \left[\rho^2 - \frac{\sin^2(\pi\rho(x-y))}{\pi^2 (x-y)^2}\right] - \left(\int_{-L}^{L} dx\, \rho\right)^2$$

$$= 2L\rho - \cancel{(2L\rho)^2} + \cancel{(2L)^2\rho^2} - \frac{1}{\pi^2} \int_{-L}^{L} dx \int_{-L}^{L} dy\, \frac{\sin^2(\pi(x-y)\rho)}{(x-y)^2}$$

Change of variable:
$$\begin{cases} \mu_1 = x - y \\ \mu_2 = x + y \end{cases}$$

$$= 2\rho L - \frac{4}{\pi^2} \int_0^{2L} \frac{d\mu_1}{2} \cdot \frac{\sin^2(\pi\rho\mu_1)}{\mu_1^2} \cdot (2L - \mu_1)$$



$$= 2\rho L - \frac{4}{\pi^2} \cdot L \underbrace{\int_0^{2L} d\mu_1 \frac{\sin^2(\pi\rho\mu_1)}{\mu_1^2}}_{\overset{\text{Maple}}{=} \frac{\pi^2\rho}{2} + O\left(\frac{1}{L}\right)} + \frac{2}{\pi^2} \underbrace{\int_0^{2L} d\mu_1 \frac{\sin^2(\pi\rho\mu_1)}{\mu_1}}_{\overset{\text{Maple}}{=} \frac{1}{2}\ln(2L\pi\rho) + O(1)}$$

$$= \frac{1}{\pi^2} \ln(2L\rho\pi) + O(1)$$

Thus, $$\lim_{L\to\infty} \frac{\mathrm{Var}(E(L))}{\ln L} = \frac{1}{\pi^2}. \quad \#$$

(2) For a point process with density ρ on $\mathbb{R}^d$,

$$\rho^{(n)}(x_1, \dots, x_n) = \prod_{k=1}^{n} \rho(x_k) = \rho^n \quad (\text{if } \rho \text{ is constant}).$$

Thus, $\underline{\mathrm{Var}(E(L))} = \int_0^L dx_1 \dots \int_0^L dx_d\, \rho + \int_{[0,L]^d} d\vec{x} \int_{[0,L]^d} d\vec{y}\, \rho^2 - \left(\int_{[0,L]^d} d\vec{x}\, \rho\right)^2$

$$= \rho \cdot L^d + \rho^2 \cdot L^{2d} - (\rho L^d)^2$$

$$\underline{= \rho \cdot L^d.}$$

$\Rightarrow$ Variance of the number of point in a Poisson point process grows like the Volume of the considered region.

(3) We have to compute a hole probability. Let $\Lambda = [0,L]^d$, then

$$\mathbb{P}(\Lambda \text{ is empty}) = \mathbb{E}\left(\prod_i (1 - \mathbb{1}_\Lambda(x_i))\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} dx_1 \dots dx_n\, \rho^{(n)}(x_1, \dots, x_n)$$

$$\overset{\text{Poisson point process}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \rho^n \cdot |\Lambda|^n = \exp(-\rho|\Lambda|)$$

$$= \exp(-\rho \cdot L^d). \quad \#$$

4) Point processes

. One way of thinking at point processes is as measurable mapping from a probability space into a space of point measures. Otherwise stated, a point process is a random point measure.

. Let us first define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

. Let Λ be the one-particle space, typically will be $\mathbb{R}^d, \mathbb{Z}^d$, or $\mathbb{R} \times \{1, \ldots, n\}$ (in general: a complete separable metric space).

. Let Ω be the space of locally finite particle configurations, i.e., for each configuration $\xi = (x_i)$, $x_i \in \Lambda$, $i \in \mathbb{N}$, and for all bounded set $B \subset \Lambda$, $\xi(B) \doteq (\# x_i \in B) < \infty$.

. σ-algebra, $\mathcal{F}$: $\forall$ bounded set $B \subset \Lambda$, and for any $n \geq 0$, $C_n^B \doteq \{\xi \in \Omega, \xi(B) = n\}$ is a cylinder set. Then, $\mathcal{F}$ is the σ-algebra generated by all cylinder sets and we denote by $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$.

. Secondly we define the space of point measures.

. Let $\mathcal{B}(\Lambda)$ be the Borel σ-algebra of Λ. A point measure on Λ is a positive measure ν on $(\Lambda, \mathcal{B}(\Lambda))$ which is a locally finite sum of Dirac measure, i.e.,

$\nu = \sum_{i \in I} \delta_{x_i}$ with $x_i \in \Lambda$, $I \subset \mathbb{N}$, and for any bounded $B \subset \Lambda$, $x_i \in B$ only for a finite number of $i \in I$.

. Denote by $M_p(\Lambda)$ the space of point measures on Λ and $\mathcal{M}_p(\Lambda)$ the σ-algebra generated by the applications $\nu \to \nu(f)$ of $M_p(\Lambda)$ to $\mathbb{N} \cup \{\infty\}$ obtained when f span $\mathcal{B}(\Lambda)$.

<u>Definition (Point process)</u>: A point process η on Λ is a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(M_p(\Lambda), \mathcal{M}_p(\Lambda))$.
The probability law of this point process is the image of $\mathbb{P}$ by η.

<u>Remark:</u> For the moment we can have $x_i = x_j$ for $i \neq j$ (multiple point).

<u>Definition:</u> A <u>simple point process</u> is a point process s.t.
$\mathbb{P}(\eta(\{x\}) \leq 1) = 1$.

<u>Remark:</u> A simple point process can be identified with the support of the random point measure.

<u>Examples: ① GUE eigenvalues</u>

- For GUE eigenvalues, $\Lambda = \mathbb{R}$ and $\mathbb{P}$ is the probability coming from the eigenvalues density $\frac{1}{Z_N} \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \cdot \prod_{i=1}^{N} \exp\left(-\frac{\lambda_i^2}{2N}\right)$
- Then, point process η then is given:

(1) $$\eta = \sum_{i=1}^{N} \delta_{\lambda_i}$$

<u>② Poisson point process on $\mathbb{R}^d$ with intensity ρ.</u>

- Take $\Lambda = \mathbb{R}^d$ and $\mathbb{P}$ the probability measure s.t. $\forall B, B' \subset \Lambda$, bounded and $n, n' \geq 0$, $\mathbb{P}(C_n^B) = \frac{(\rho|B|)^n}{n!} \cdot e^{-\rho|B|}$ and if $B \cap B' = \emptyset$, $\mathbb{P}(C_n^B \cap C_{n'}^{B'}) = \mathbb{P}(C_n^B)\mathbb{P}(C_{n'}^{B'})$.
- η is a random point measure on $\mathbb{R}^d$ with intensity ρ completely decorrelated, called Poisson point process.

4.1) Correlation functions and moments

. For a point process η on Λ, the total points (counted with multiplicity) in the support of η in a set $A \subset \Lambda$ is given by

(2) $$\eta(\mathbb{1}_A), \text{ with } \mathbb{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

. For a general function f, we write $\eta(f) = \int_\Lambda d\mu(x) f(x) \eta(x)$.

. In 3.1 we already defined the correlation functions. Now we state the explicit relation to factorial moments.

Lemma: Let $A \subset \Lambda$ a subset, then

(3) $$\int_{A^n} d\mu(x_1) \dots d\mu(x_n)\, \rho^{(n)}(x_1, \dots, x_n) = \mathbb{E}\left(\frac{\eta(\mathbb{1}_A)!}{(\eta(\mathbb{1}_A) - n)!} \right)$$

Proof: . For $n=1$, $\eta(\mathbb{1}_A) = \sum_i \mathbb{1}_{[x_i \in A]} \Rightarrow \mathbb{E}(\eta(\mathbb{1}_A)) = \mathbb{E}(\# x_i \in A) = \int_A \rho^{(1)}(x) d\mu(x)$

. For $n=2$, notice that $\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A) - 1) = \sum_i \mathbb{1}_{[x_i \in A]} \sum_{j \neq i} \mathbb{1}_{[x_j \in A]}$.

In fact, the second sum is irrelevant if $i \notin A$, and when $i \in A$, the second sum is $\eta(\mathbb{1}_A) - 1$.

. For general n, we have: $\eta(\mathbb{1}_A)(\eta(\mathbb{1}_A) - 1) \cdots (\eta(\mathbb{1}_A) - n + 1) =$

$$= \sum_{i_1} \sum_{i_2 \neq i_1} \dots \sum_{i_n \neq \{i_1, \dots, i_{n-1}\}} \left(\prod_{k=1}^{n} \mathbb{1}_{[x_k \in A]} \right)$$

. Thus, $$\mathbb{E}\left(\eta(\mathbb{1}_A) \cdots (\eta(\mathbb{1}_A) - n + 1)\right) = \int_{A^n} d\mu(x_1) \dots d\mu(x_n) \cdot \rho^{(n)}(x_1, \dots, x_n).$$ #

• In particular: . $\mathbb{E}(\eta(\mathbb{1}_A)) = \int_A \rho^{(1)}(x) d\mu(x)$

(4)

$$. \operatorname{Var}(\eta(\mathbb{1}_A)) = \int_{A^2} \rho^{(2)}(x,y) d\mu(x) d\mu(y) + \int_A \rho^{(1)}(x) d\mu(x) - \left(\int_A \rho^{(1)}(x) d\mu(x) \right)^2$$

. By the above Lemma, we can compute factorial moments.

If we are interested in the moments, we can use the relation:

(5) $$\mathbb{E}(X^n) = \sum_{k=1}^{n} S(n,k) \cdot \mathbb{E}[X(X-1)\cdots(X-k+1)]$$

where $S(n,k)$ are the Stirling number of the second kind.

$S(n,k) = \#$ of ways to partition a set of n objects into k groups.

. They satisfy the recursion relation: $S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$ for $1 \leq k < n$, and with $S(n,n) = S(n,1) = 1$.

4.2) Linear statistics

. Correlation functions are important in computing expected values of observables. In particular one can consider linear statistics, i.e., consider random variables of the form

(6) $$\sum_i f(x_i) ,$$

for some real function f.

. Define $u(x) = 1 - \exp(f)$. Then,

(7) $$\underline{\mathbb{E}\left[\exp\left(\sum_i f(x_i)\right)\right]} = \mathbb{E}\left(\prod_i (1-u(x_i))\right) = \sum_{n=0}^{\infty} (-1)^n \mathbb{E}\left(\sum_{i_1<\cdots<i_n} \prod_{k=1}^{n} u(x_{i_k})\right)$$

$$\underset{\text{symmetry}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathbb{E}\left(\sum_{\substack{i_1,\ldots,i_n \\ \text{all different}}} \prod_{k=1}^{n} u(x_{i_k})\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \int_{\Lambda^n} d\mu(x_1)\cdots d\mu(x_n)\, \rho^{(n)}(x_1,\ldots,x_n) \cdot \prod_{k=1}^{n} u(x_k).$$

4.3) Determinantal point processes.

. The correlation functions for the GUE eigenvalues had the form $\rho^{(n)}(\lambda_1,\dots,\lambda_n) = \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le n}$. This is an example of what it is called a determinantal point process.

Definition: A point process is called determinantal if the n-point correlation functions are given by

$$\rho^{(n)}(x_1,\dots,x_n) = \det\left(K(x_i,x_j)\right)_{1\le i,j\le n},$$

where $K(x,y)$ is a kernel of an integral operator $K: L^2(\Lambda, d\mu) \to L^2(\Lambda, d\mu)$, non-negative and locally trace-class.

. Remarks: ① Positivity is required because the n-pt. correlation functions are positive.

② Locally trace-class is related to the locally finite number of points ($\leftrightarrow$ point measures).

. Theorem (Soshnikov; Macchi). In the case of Hermitian K: K defines a determinantal point process iff $0 \le K \le 1$.

If the corresponding point process exists, then it is unique.

. For non-hermitian kernels, such a classification is not yet known.

Remarks: ① The probability that the number of particle is finite or infinite is either 0 or 1, depending on whether $\mathrm{Tr}(K)$ is finite or infinite.

② A determinantal point process is simple.

③ The number of particles is n with probability 1 iff K is an orthogonal projector with $\mathrm{rank}(K) = n$. (like in our GUE example).

. For a determinantal point process, (7) becomes

(8) $$\mathbb{E}\left(\prod_i \left(1-u(x_i)\right)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det\left(K(x_i,x_j)\right)_{1\le i,j\le n} \left(\prod_{i=1}^{n} u(x_i)\, d\mu(x_i)\right)$$

$$\equiv \det(\mathbb{1} - uK)_{L^2(\Lambda, d\mu)},$$

where for each $\varphi \in L^2(\Lambda, d\mu)$,

(9) $$[(uK)\varphi](x) \equiv \int_{\Lambda} u(x) K(x,y) \varphi(y)\, d\mu(y).$$

. The determinant in r.h.s. of (8), is called the Fredholm determinant of the operator uK on the space $L^2(\Lambda, d\mu)$. The series is the Fredholm series. We will discuss later Fredholm determinants.

4.3.1) An application: Hole probability

. Compute the hole probability, i.e., the probability of not having particles in a subset B of Λ:

(10) $$\mathbb{P}\left(\eta(\mathbb{1}_B)=0\right) = \mathbb{E}\left(\prod_i \left(1-\mathbb{1}_B(x_i)\right)\right) = \det(\mathbb{1}-K)_{L^2(B,d\mu)}$$

. In particular, for a determinantal point process on $\mathbb{R}$ or $\mathbb{Z}$, which has a last particle, at position x_{max}, its distribution is given by

(11) $$\mathbb{P}(x_{max} \le s) = \mathbb{P}\left(\eta(\mathbb{1}_{(s,\infty)})=0\right) = \det(\mathbb{1}-K)_{L^2((s,\infty),d\mu)}$$

4.3.2) When a measure defines a determinantal point process?

. We have seen that for GUE eigenvalues, the measure

$$\frac{1}{Z_N}\left(\det\left(\lambda_i^{j-1}\right)_{1\le i,j\le N}\right)^2 \cdot \prod_{i=1}^{N} d\mu(\lambda_i), \quad d\mu(\lambda_i) = e^{-\lambda_i^2/2N}\, d\lambda_i$$

induces a determinantal point process. This is a particular case of the following theorem.

Theorem (Borodin; Tracy–Widom for GUE): A measure of the form

$$(12)\qquad \frac{1}{Z_N}\det\left(\varphi_i(x_k)\right)_{1\le i,k\le N}\cdot\det\left(\psi_i(x_k)\right)_{1\le i,k\le N}\cdot d\mu(x_1)\cdots d\mu(x_N)\,,\quad Z_N\neq 0,$$

defines a determinantal point process with kernel

$$(13)\qquad K_N(x,y)=\sum_{i=1}^{N}\psi_i(x)\,[A^{-1}]_{i,j}\,\varphi_j(y)\,,$$

where $A_{i,j}=\int_\Lambda \psi_j(s)\,\varphi_i(s)\,d\mu(s)$.

Remark: One has an explicit formula, but in general to obtain the inverse of A explicitly for large N is not an easy task. Typically one will try first to do a change of basis such that A^{-1} is easy, e.g., if $A=\mathbb{1}$. This is what we made implicitly in the case of GUE matrices, when we introduced the orthogonal polynomials.

• Proof of the theorem: The basic strategy is identical to the GUE case. The difference is that now we have two different set of functions $\{\varphi_i\}$ and $\{\psi_i\}$.

• Notations: $\langle a|b\rangle\equiv\int_\Lambda d\mu(x)\,a(x)\,b(x)$

$|b\rangle$ is a vector with components $\langle x|b\rangle=b(x)$

$\langle a|$ is a covector " " $\langle a|y\rangle=a(y)$.

. Suppose that we can find functions $\xi_k(x),\eta_k(x)$, $k=1,\dots,N$ such that

$$\operatorname{span}(\{\xi_k\})=\operatorname{span}(\{\varphi_k\})\,;\quad \operatorname{span}(\{\eta_k\})=\operatorname{span}(\{\psi_k\})$$

and such that $\langle\xi_k|\eta_\ell\rangle=\delta_{k,\ell}$.

Then, (12) $= \text{const} \times \det(\xi_i(x_j)) \cdot \det(\eta_i(x_j))\, d^N\mu(x)$

$$\overset{\text{as for GUE}}{=} \text{const.} \det\Big(\underbrace{\sum_{k=1}^{N} \eta_k(x_i)\,\xi_k(x_j)}_{=K_N(x_i,x_j)}\Big)\, d^N\mu(x)$$

. The orthogonal relation $\langle \xi_k, \eta_\ell \rangle = \delta_{k,\ell}$ implies

$$\int_\Lambda d\mu(x)\, K_N(x,x) = N \quad ; \quad \int_\Lambda d\mu(z)\, K_N(x,z) K_N(z,y) = K_N(x,y)$$

as (8) and (9) for GUE. By the same argument as GUE we then get

$$\rho^{(n)}(x_1, \dots, x_n) = \det\left(K_N(x_i, x_j)\right)_{1 \le i,j \le N}.$$

. With the bra and ket notations,

$$K_N = \sum_{k=1}^{N} |\eta_k\rangle\langle \xi_k| .$$

. Let S and T be the matrices of change of basis:

$$\phi_i = \sum_j S_{ij}\, \xi_j \quad \text{and} \quad \psi_i = \sum_j T_{ij}\, \eta_j .$$

. Thus, $$K_N(x,y) = \sum_{k=1}^{N} \eta_k(x)\,\xi_k(y) = \sum_{k=1}^{N} \sum_{i,j=1}^{N} (T^{-1})_{ki}\,\psi_i(x) \cdot (S^{-1})_{kj}\,\phi_j(y)$$

$$= \sum_{i,j=1}^{N} \psi_i(x)\,\phi_j(y) \cdot \underbrace{\sum_{k=1}^{N} (T^{-1})_{ki}\,(S^{-1})_{kj}}_{=\sum_k (T^t)^{-1}_{ik} (S^{-1})_{kj} = (S\cdot T^t)^{-1}_{ij}}$$

Let $S \cdot T^t = A$ and compute

. $$\langle \psi_j | \phi_i \rangle = \sum_{k,\ell} T_{jk} \cdot S_{i\ell} \underbrace{\langle \eta_k | \xi_\ell \rangle}_{=\delta_{k,\ell}} = \sum_k T_{jk} \cdot S_{ik} = (S \cdot T^t)_{ij} .$$

. Thus, $A_{ij} = \langle \psi_j | \phi_i \rangle$. #

<u>Remark:</u> A determinantal point process with kernel $K(x,y)$ is the same as the one with kernel $\tilde{K}(x,y) = \frac{f(x)}{f(y)} K(x,y)$ for some f with $f(x) \neq 0,\ \forall x \in \Lambda$.

. We say that K and $\tilde{K}$ are <u>conjugate kernels</u>.

5) Fredholm determinant

. In the last lecture we saw that the "hole probability" is given by a series. We called it "Fredholm expansion" of an object called "Fredholm determinant".

(1) We had: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} dx_1 \dots dx_n \det(K(x_i,x_j))_{1\le i,j\le n} =: \det(\mathbb{1}-K)_{L^2(\Lambda,dx)}$

. There are essentially two point of views:

ⓐ Take the series in (1) as definition of r.h.s. As soon as the series / integrals are absolutely summable / integrable, everything is well defined.

ⓑ One can think at (1) as the Fredholm determinant of an operator, with eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq 0$. Then, the Fredholm determinant will be equal to $\prod_{n\geq 0}(1-\lambda_n)$.

Now we describe the two approaches:

5.1) Kernel approach.

. The idea is the following. Assume that we can bound

$|K(x,y)| \leq f(x)\cdot f(y)\cdot C$, with C a finite constant and

that $\int_\Lambda f(x)^2 dx < \infty$.

. Then $\left|\det(\mathbb{1}-K)_{L^2(\Lambda,dx)}\right| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} dx_1 \dots dx_n\, C^n \prod_{i=1}^n f(x_i)^2 \cdot \det\left[\frac{K(x_i,x_j)}{C\cdot f(x_i) f(x_j)}\right]_{1\le i,j\le n}$

. The entries of the $n\times n$ determinant have absolute value less or equal to 1, and by Hadamard's bound, it can be then bounded by $n^{n/2}$.

. From this will follows that $|\det(\mathbb{1}-K)| < \infty$.

. More generally, let us do the above statement precise.

. Consider (M,μ) a measure space and $A(x)$ a positive, continuous function on M satisfying $\frac{1}{A(x)} \in L^2(M,\mu)$. [$\frac{1}{A(x)}$ plays the role of $f(x)$].

. Then one defines thin and thick sets:

→ A measurable set $S \subset M\times M$ is thin if $\forall x_0, y_0 \in M$,

$$\mu(\{x\in M \mid (x,y_0)\in S\})=0,\ \mu(\{y\in M \mid (x_0,y)\in S\})=0,\ \mu(\{x\in M \mid (x,x)\in S\})=0.$$

→ A thick subset of $M\times M$ is a subset of $M\times M$ which is not thin.

Definition: A function $K(x,y)$ on $M\times M$ is a kernel if:

(a) $K(x,y)$ is measurable,

(b) for some thick open subset $U\subset M\times M$, $K(x,y)$ is continuous on U,

(c) $\|K\|_A \doteq \sup_{(x,y)\in M\times M} A(x)A(y)|K(x,y)| < \infty$.

. The class of kernels forms a vector space with the norm $\|\cdot\|_A$.

Define, for any kernel K and $n>0$:

$$\Delta_n(K) \doteq \int_{M^n} d\mu(x_1)\cdots d\mu(x_n)\, \det\left[K(x_i,x_j)\right]_{1\le i,j\le n} \quad , n\geq 1 \tag{2}$$

and $\Delta_0(K) \doteq 1$.

. Hadamard bound: Let T be a $n\times n$ matrix with entries satisfying $|T_{ij}|\le 1$. Then, $|\det(T)| \le n^{n/2}$.

: Lemma: $|\Delta_n(K)| \le C^n \cdot (\|K\|_A)^n \cdot n^{n/2}$ for $C = \|A^{-1}\|_2$.

Proof: $|\Delta_n(K)| \le \int_{M^n} d^n\mu(x)\, \|K\|_A^n \left(\prod_{k=1}^n \frac{1}{A(x_k)^2}\right) \left|\det(\tilde{K}(x_i,x_j))\right|$, $\tilde{K}(x,y) = \frac{K(x,y)A(x)A(y)}{\|K\|_A}$

$$\overset{\text{Hadamard}}{\le} \|K\|_A^n \cdot n^{n/2} \left(\int_M d\mu(x) A(x)^{-2}\right)^n . \#$$

· Using this Lemma, we then define the Fredholm determinant attached to the kernel K by

$$(3) \qquad \Delta(K) \doteq \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \Delta_n(K).$$

· Remark: $|\Delta(K)| \leq \sum_{n \geq 0} \|K\|_A^n \cdot \|\bar{A}^{-1}\|_2^n \cdot \frac{n^{n/2}}{n!} < \infty$, because

$n! \cong n^n \cdot e^{-n}$ for large n, thus,

$|\Delta(K)| \leq \text{const.} \sum_{n \geq 1} \frac{\tilde{C}^n}{\sqrt{n!}} < \infty.$

5.2) Operator approach.

- The operator approach is more focused around the spectral properties of the operators. Here we assume some knowledge on functional analysis, see an appendix to this lecture.

- The idea can be understood starting with finite-rank operators, whose closure gives the compact operators. Consider a finite-rank operator K with rank $(K) = n < \infty$ ($\simeq$ a $n \times n$ matrix).
- Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of K.

Then, $\det(\mathbb{1} + K) = \prod_{k=1}^{n} (1 + \lambda_k)$.

$$(4) \qquad = 1 + \sum_i \lambda_i + \sum_{i<j} \lambda_i \lambda_j + \sum_{i_1<i_2<i_3} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} + \dots + \lambda_1 \cdots \lambda_n.$$

· "Obviously", $|\det(\mathbb{1} + K)| \leq \sum_{k=0}^{n} \left(\sum_{i=1}^{n} |\lambda_i| \right)^k \frac{1}{k!} \leq \exp\left(\mathrm{Tr}(|K|)\right)$

use: $\sum_i |\lambda_i| \leq \sum_i \mu_i$, where μ_i are the e.v. of $|K|$.

. The question is to give a sense of

$$\det(\mathbb{1}+K) = \prod_{k\geq 1}(1+\lambda_k) \tag{5}$$

also for more general compact operators.

. The previous bound indicates that for trace-class operators it should not be a problem. Indeed, the Fredholm determinant will be well defined for trace-class operators.

. Remark: It is possible to define (5) for operators which are only Hilbert-Schmidt in some cases, but it will not be made here [see, e.g., Simon's book "Trace Ideals and Their Applications].

. Before going into the construction, let us give an example where the operator approach can be useful.

. Consider our example of GUE eigenvalues. Let $\lambda_{max}^{(N)}$ be the largest eigenvalue for $N\times N$ matrices. Then,

$$\mathbb{P}_N\left(\lambda_{max}^{(N)} \leq 2N + s\cdot N^{1/3}\right) = \det(\mathbb{1}-K_N)_{L^2((s,\infty),dx)}$$

for a kernel K_N.

Question: Does $\lim_{N\to\infty} \det(\mathbb{1}-K_N) = \det(\mathbb{1}-K_\infty)$, where

$$K_\infty = \lim_{N\to\infty} K_N \,?$$

. Answer: To answer to this question we can apply the bound:

$$\left|\det(\mathbb{1}-K_N) - \det(\mathbb{1}-K_\infty)\right| \leq \|K_N - K_\infty\|_1 \cdot e^{1+\|K_N\|_1+\|K_\infty\|_1} \tag{6}$$

. Thus we need to prove that K_N, K_∞ are trace-class and that $K_N \to K_\infty$ in trace-class norm.

5.2.1) (Antisymmetric) tensor product.

. First we construct the tensor product.

. Let $\mathcal{H}$ be an Hilbert space, then the <u>tensor product</u> of $\mathcal{H}$, n times, denoted by $\otimes^n \mathcal{H}$ is the vector space of multilinear functionals on $\mathcal{H}$:

(a) for given $\varphi_1, \dots, \varphi_n \in \mathcal{H}$, $\varphi_1 \otimes \dots \otimes \varphi_n \in \bigotimes^n \mathcal{H}$ satisfies

$$(\varphi_1 \otimes \dots \otimes \varphi_n)(\eta_1, \dots, \eta_n) = \prod_{k=1}^{n} (\varphi_k, \eta_k), \quad \forall (\eta_1, \dots, \eta_n) \in \mathcal{H} \times \dots \times \mathcal{H}$$

(b) inner product:

$$(\varphi_1 \otimes \dots \otimes \varphi_n, \psi_1 \otimes \dots \otimes \psi_n) = \prod_{k=1}^{n} (\varphi_k, \psi_k)$$

(c) for any operator $A \in \mathcal{L}(\mathcal{H})$, $\exists$ an operator $\Gamma_n(A) \in \mathcal{L}(\otimes^n \mathcal{H})$ with $\Gamma_n(A)(\varphi_1 \otimes \dots \otimes \varphi_n) = A\varphi_1 \otimes \dots \otimes A\varphi_n$.

. Γ_n satisfies $\Gamma_n(A \cdot B) = \Gamma_n(A) \cdot \Gamma_n(B)$.

. <u>Basis</u>: If $\{\phi_k\}_k$ is an orthonormal basis of $\mathcal{H}$, then $\{\phi_{k_1} \otimes \dots \otimes \phi_{k_n}\}_{k_1, \dots, k_n}$ is an orthonormal basis of $\otimes^n \mathcal{H}$.

. The space we are actually looking for is a subspace of $\otimes^n \mathcal{H}$, namely its <u>antisymmetric subspace, $\Lambda^n \mathcal{H}$</u>.

. Let S_n denote the group of permutation of $\{1, \dots, n\}$. Then, for given $\psi_1, \dots, \psi_n \in \mathcal{H}$,

$$\underline{\psi_1 \wedge \dots \wedge \psi_n \doteq \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^{|\sigma|} \cdot \psi_{\sigma(1)} \otimes \dots \otimes \psi_{\sigma(n)}} \tag{7}$$

belongs to $\Lambda^n \mathcal{H}$.

. Indeed, <u>$\Lambda^n \mathcal{H}$ is spanned by the vectors $\psi_1 \wedge \dots \wedge \psi_n$</u>, when $\psi_1, \dots, \psi_n$ span over $\mathcal{H}$.

. For $n=0$, we define <u>$\Lambda^0 \mathcal{H} = \mathbb{C}$</u>.

Lemma 1: $(\phi_1 \wedge \dots \wedge \phi_n, \psi_1 \wedge \dots \wedge \psi_n) = \det\left((\phi_i, \psi_j)\right)_{1\le i,j\le n}$

Proof: $(\phi_1 \wedge \dots \wedge \phi_n, \psi_1 \wedge \dots \wedge \psi_n) = \frac{1}{n!} \sum_{\sigma,\sigma' \in S_n} (-1)^{|\sigma|} (-1)^{|\sigma'|} \cdot$

$$\cdot (\phi_{\sigma(1)} \otimes \dots \otimes \phi_{\sigma(n)}, \psi_{\sigma'(1)} \otimes \dots \otimes \psi_{\sigma'(n)})$$

$$= \frac{1}{n!} \sum_{\sigma,\sigma' \in S_n} (-1)^{|\sigma|} (-1)^{|\sigma'|} \prod_{k=1}^{n} (\underbrace{\phi_{\sigma(k)}}_{\doteq j}, \psi_{\sigma'(k)})$$

let $\pi \equiv \sigma' \circ \sigma^{-1}$

$$= \frac{1}{n!} \sum_{\sigma,\pi \in S_n} (-1)^{|\pi|} \prod_{j=1}^{n} (\phi_j, \psi_{\pi(j)}) = \det\left((\phi_i, \psi_j)\right)_{1\le i,j\le n} \quad \#$$

Lemma 2: Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the eigenvalues of A on $\mathcal{H}$, $\dim \mathcal{H} = N < \infty$. Then, $\Lambda^n(A) \equiv \underbrace{A \otimes \dots \otimes A}_{n \text{ times}}$ applied on $\Lambda^n \mathcal{H}$, satisfy

(8) $$\underline{\mathrm{Tr}\left[\Lambda^n(A)\right]} = \sum_{i_1 < \dots < i_n} \underline{\lambda_{i_1} \cdot \dots \cdot \lambda_{i_n}}.$$

In particular, $\mathrm{Tr}(\Lambda^N(A)) = \Lambda^N(A) = \det(A)$; $\Lambda^N \mathcal{H} = \mathbb{C}$.

Proof: Let $\{e_1, \dots, e_N\}$ be a basis of $\mathcal{H}$, e_i eigenvector of A with eigenvalue λ_i. Then, $\{e_{i_1} \wedge \dots \wedge e_{i_n}\}_{i_1 < \dots < i_n}$ is a basis of $\Lambda^n \mathcal{H}$ and $\mathrm{Tr}(\Lambda^n(A)) = \sum_{i_1 < \dots < i_n} (e_{i_1} \wedge \dots \wedge e_{i_n}, Ae_{i_1} \wedge \dots \wedge Ae_{i_n}) = $ r.h.s. (8) #

By Lemma 2, comparing with (4), we see the identity

(9) $$\sum_{n=0}^{N} \mathrm{Tr}(\Lambda^n(A)) = \det(\mathbb{1} + A) = \prod_{k=1}^{N} (1 + \lambda_k(A)).$$

The next step is to see that (9) holds also when $N = \infty$, for trace-class operators.

5.2.2) Traces and determinants

. Looking at (9), the first actually non-trivial point is to prove that, if $A \in \mathcal{J}_1$ and $\{\lambda_n(A)\}$ are its eigenvalues, then

(10) $$\sum_n \lambda_n(A) = \mathrm{Tr}(A).$$ [Lidskii's equality]

· The proof of (10) gives at the same time the justification for (9).

. The final theorem is namely:

Theorem 1: For any $A \in \mathcal{J}_1$ (trace-class),

(11) $$\det(\mathbb{1} + z \cdot A) = \prod_n \left(1 + z \cdot \lambda_n(A)\right).$$

. In particular (10) holds.

· The proof of Theorem 1 goes along the following steps.

Theorem 2: If A is trace class on an Hilbert space $\mathcal{H}$, then $\Lambda^k(A)$ is a trace class operator on $\Lambda^k \mathcal{H}$ with

(12) $$\|\Lambda^k(A)\|_1 \leq \frac{\|A\|_1^k}{k!}.$$

· In particular, the series

(13) $$\det(\mathbb{1} + zA) \doteq \sum_{k=0}^{\infty} z^k \,\mathrm{Tr}(\Lambda^k(A)) \qquad (z \in \mathbb{C})$$

defines an entire function satisfying

(14) $$|\det(\mathbb{1} + zA)| \leq \exp(|z| \cdot \|A\|_1).$$

. Moreover, $\forall \varepsilon > 0$, $\exists\, C_\varepsilon < \infty$ s.t. $|\det(\mathbb{1} + zA)| \leq C_\varepsilon \cdot \exp(\varepsilon \cdot |z|)$.

. This theorem is important because tell us that $\mathrm{Tr}(\Lambda^k(A))$ in (9) are well defined. The second step is to prove that (13) is a continuous function, as stated in the following thm. It provides at the same time the criteria cited at page ④.

Theorem 3: $A \longmapsto \det(\mathbb{1} + A)$ is a continuous function on $\mathcal{J}_1$. In particular,

(15) $$|\det(\mathbb{1} + A) - \det(\mathbb{1} + B)| \leq \|A - B\|_1 \cdot \exp\left[1 + \|A\|_1 + \|B\|_1\right].$$

. The next theorem is important since it tell us that the series (13) has the right zeros with the right multiplicity.

Theorem 4:

ⓐ For any $A, B \in \mathfrak{I}_1$,

(16) $$\det(\mathbb{1}+A+B+AB)=\det(\mathbb{1}+A)\cdot\det(\mathbb{1}+B)$$

ⓑ For $A\in\mathfrak{I}_1$, $\det(\mathbb{1}+A)\neq 0$ iff $\mathbb{1}+A$ is invertible.

ⓒ $\forall A\in\mathfrak{I}_1$ and $z_0=-\lambda^{-1}$ with λ an eigenvalue of algebraic multiplicity n, $\det(\mathbb{1}+zA)$ has a zero of order n at $z=z_0$.

. From this theorem, we know that $\det(\mathbb{1}+zA)$ has to be proportional to $\prod_n (1+z\lambda_n(A))$. One has still to see that it is this expression.

Theorem 5: Let $f(z)$ be an entire function with zeros at $z_1, z_2, \ldots$ (repeated by their multiplicity). Suppose $f(0)=1$ and $\sum_{n\geq 1}\frac{1}{|z_n|}<\infty$, and that $\forall\varepsilon>0$, $|f(z)|\leq C_\varepsilon\cdot\exp(\varepsilon\cdot|z|)$.

Then

(17) $$f(z)=\prod_{n\geq 1}\left(1-\frac{z}{z_n}\right).$$

. The proofs of Theorems 1–4 can be found in [Simon: "Traces Ideals and their Applications"] Chapter 3.

. Maybe they will be added for next lecture.

5.2.3) Explicit formulas.

- Suppose that

$$(18) \qquad (Af)(x) = \int_a^b K(x,y) f(y)\,dy$$

on $L^2(a,b)$, with $-\infty < a < b < \infty$ and with K continuous on $[a,b]\times[a,b]$.

- Theorem: Let $A \in \mathcal{J}_1$ as (18). Then

$$(19) \qquad \mathrm{Tr}(A) = \int_a^b K(x,x)\,dx.$$

- Theorem: Let $A \in \mathcal{J}_1$ as (18). Then

$$(20) \qquad \det(\mathbb{1}+A) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} \quad \text{with } \alpha_n = \int_{[a,b]^n} dx_1 \dots dx_n \det\left[K(x_i,x_j)\right]_{1\leq i,j\leq n}.$$

- This last expression tell us that the Fredholm determinant defined by (13) can be written in series, which corresponds to the one given in (1).

- Final remark: Sometimes it is either difficult to check or not true that the operator considered is trace-class. Alternative definition for operators which are only Hilbert-Schmidt exist and have the series expansion (20). We do not enter in details here, since we do not plan to use it in this Lecture.

Functional analysis appendix.

- Here we collect some analysis background. The Hilbert spaces are complex (and separable) with $\langle \psi, a\varphi_1 + b\varphi_2 \rangle = a\langle \psi, \varphi_1\rangle + b\langle \psi, \varphi_2\rangle$

$$\langle a\psi_1 + b\psi_2, \varphi\rangle = \bar{a}\langle \psi_1, \varphi\rangle + \bar{b}\langle \psi_2, \varphi\rangle$$

- An operator A is positive iff $\langle \phi, A\phi\rangle \geq 0$, $\forall \phi$ (Notation: $A \geq 0$).
- If $A \geq 0$, then $A = A^*$.
- If $A \geq 0$, then $\exists!\ B \geq 0$ s.t. $B^2 = A$. Then we write $B = \sqrt{A}$.
- For any bounded operator A, $A^*A \geq 0$ and one define $|A| \doteq \sqrt{A^*A}$.
- Polar decomposition: Given an operator A, there exists an unique U s.t.

  ⓐ $A = U \cdot |A|$

  ⓑ $\|U\psi\| = 0$ for $\psi \in \operatorname{Ker} A$

  ⓒ $\|U\psi\| = \|\psi\|$ for $\psi \in (\operatorname{Ker} A)^\perp$
- Norm: $\|A\| = \sup_{\|\varphi\|_2 = 1} \|A\varphi\|_2 \approx$ the largest e.v..

- A bounded operator A is called finite rank if $\dim(\operatorname{Im}(A)) < \infty$, and $\dim(\operatorname{Im}(A))$ is the rank of A.
- A bounded operator is compact iff it is a limit of a finite rank operator (limit ≡ norm limit).

- Spectral theorem: Let A be a compact operator on a Hilbert space $\mathcal{H}$. Then:

  ⓐ $\sigma(A)$, the spectrum of A, is a discrete set with at most zero as accumulation point,

  ⓑ all $\lambda \in \sigma(A) \setminus \{0\}$ is an eigenvalue with finite (algebraic ~~geometric~~) multipicity

  ⓒ $\forall \lambda \in \sigma(A)\setminus\{0\}$, $\exists$ finite rank projection P_λ s.t.

  $$AP_\lambda = P_\lambda A,\ \sigma(A|_{P_\lambda \mathcal{H}}) = \{\lambda\},\ \sigma(A|_{(\mathbb{1}-P_\lambda)\mathcal{H}}) = \sigma(A)\setminus\{\lambda\}$$

  . $\operatorname{Dim}(P_\lambda)$ is the (geometric ~~algebraic~~) multiplicity of λ.

- Theorem (o.n. basis): Let A be a self-adjoint compact operator on $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis of eigenvectors for A.

(Hilbert space)

Remark: Just to remind the difference between algebraic and geometric multiplicity, consider a finite rank operator A.

. Take its Jordan bloc decomposition J_A, then the algebraic multiplicity is the order of the zero of the characteristic polynomial $\det(\lambda \mathbb{1} - A) = 0$, and is equal to the number of times that "λ" appears on the diagonal of J_A.

. The geometric multiplicity of "λ" is the number of blocs with eigenvalue "λ" $\equiv \operatorname{Ker}(A - \lambda \mathbb{1})$.

- A is diagonalizable iff $\text{geom}(\lambda) = \text{algebr}(\lambda)$, $\forall \lambda$.

. Theorem (canonical expansions):

Let A be a compact operator. Then A has a norm-convergent expansion

$$A = \sum_{n \geq 1} \mu_n(A) |\psi_n\rangle\langle\phi_n| ,$$

each $\mu_n(A) > 0$, $\mu_1(A) \geq \mu_2(A) \geq \ldots$, and the $\{\phi_n\}_n$, $\{\psi_n\}_n$ are orthonormal sets (not necessarily complete, i.e., not necessarily bases).

. Moreover, $\mu_n(A)$ are uniquely determined and ϕ_n, ψ_n are essentially determined [i.e., up to a change of basis in the subspaces of given μ_n's].

. Notation: $\mu_n(A)$ are called singular values of A.

- The $\mu_n(A)$ are the non-zero eigenvalues of $|A|$.

. Weyl's inequality: For any compact operator A and for all n,

$$\sum_{k=1}^{n} |\lambda_k(A)| \leq \sum_{k=1}^{n} \mu_n(A) ,$$

where $\lambda_n(A)$ are ordered satisfying $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots$ (counted up to their algebraic multiplicity).

Proof of the canonical expansion.

Using the polar decomposition, we have $|A| = U^* A$. Since $|A|$ is compact and self-adjoint, by the o.n.basis theorem, we have

$$|A| = \sum_{n \geq 1} \mu_n(A) |\phi_n\rangle\langle\phi_n|, \qquad (\mu_n(A) > 0)$$

where $\mu_n(A)$ are the nonzero e.v. of $|A|$ and ϕ_n the associated $\vec{e.v.}$.
Since U is an isometry on $\mathrm{Im}(|A|)$, $\psi_n \doteq U\phi_n$ are orthonormal.
The uniqueness of $\mu_n(A)$ is due to the fact that $\mu_n(A)^2$ are the non-zero e.v. of A^*A and A^*A is positive. $\{\phi_n\}$ are the $\vec{e.v.}$ of A^*A and $\{\psi_n\}$ are the $\vec{e.v.}$ of AA^*. Those vectors are determined uniquely, up to basis transformation in the subspaces of given μ_n's. #

Trace-class operators.

Theorem A: Let $\mathcal{H}$ be a separable Hilbert space and $\{\varphi_n\}_{n \geq 1}$ an orthonormal basis of $\mathcal{H}$. Then, for any positive operator $A \in \mathcal{L}(\mathcal{H})$,

(1) $$\mathrm{Tr}(A) \doteq \sum_{n \geq 1} \langle \varphi_n, A\varphi_n \rangle,$$

is independent of basis. $\mathrm{Tr}(A)$ is called the trace of A.

Definition: An -operator A is called trace-class iff $\mathrm{Tr}(|A|) < \infty$. The family of all trace-class operators is denoted by $\mathcal{J}_1$.

Theorem B: $\mathcal{J}_1$ is a $*$-ideal in $\mathcal{L}(\mathcal{H})$, i.e.,

(a) $\mathcal{J}_1$ is a vector space,

(b) If $A \in \mathcal{J}_1$ and $B \in \mathcal{L}(\mathcal{H})$, then $AB \in \mathcal{J}_1$ and $B A \in \mathcal{J}_1$.

(c) If $A \in \mathcal{J}_1$, then $A^* \in \mathcal{J}_1$

. Remarks: . $\|\cdot\|_1$ norm: let $\|\cdot\|_1$ be defined on $\mathcal{J}_1$ by $\|A\|_1 \doteq \mathrm{Tr}(|A|)$. Then $\mathcal{J}_1$ is a Banach space with norm $\|\cdot\|_1$ and $\|A\| \leq \|A\|_1$.

. $A \mapsto \mathrm{Tr}(A)$ is a linear bounded functional on $\mathcal{J}_1$ and for $A \in \mathcal{J}_1$, $B \in \mathcal{L}(\mathcal{H})$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

. Every $A \in \mathcal{J}_1$ is compact.

. Theorem C: let K be a continuous, positive function on $M \times M$. If $\int_M K(x,x)\,d\mu(x) < \infty$, then $\exists!$ operator $A \in \mathcal{J}_1$ s.t.
$(A\phi)(x) = \int K(x,y)\,\phi(y)\,d\mu(y)$ and $\|A\|_1 = \int K(x,x)\,d\mu(x)$.

. A simple case when an operator A is trace-class, is when it is a product of two Hilbert-Schmidt operators.

. Hilbert-Schmidt operators.

. Definition: An operator $A \in \mathcal{L}(\mathcal{H})$ is called Hilbert-Schmidt iff $\mathrm{Tr}(A^*A) < \infty$. The family of Hilbert-Schmidt operators is denoted by $\mathcal{J}_2$.

. Theorem D: $\mathcal{J}_2$ is a $*$-ideal.

. Theorem E: let $\mathcal{H} = L^2(M, d\mu)$. If $A \in \mathcal{J}_2$, then there exists a unique function $K \in L^2(M \times M, d\mu \otimes d\mu)$ with

(2) $$(A\phi)(x) = \int K(x,y)\,\phi(y)\,d\mu(y).$$

. Conversely, any $K \in L^2(M\times M, d\mu \otimes d\mu)$ defines an operator A by (2) which is Hilbert-Schmidt and

(3) $$\|A\|_2 = \|K\|_2 \doteq \left(\int |K(x,y)|^2\,d\mu(x)\,d\mu(y) \right)^{1/2}.$$

<u>Remarks</u>: . J_2 with inner product $\langle \cdot, \cdot \rangle_2$ is an Hilbert Space.

. <u>Every $A \in J_2$ is compact.</u>

. $A \in J_1$ iff $A = B \cdot C$ for some $B, C \in J_2$.

<u>Theorem F</u>: Let B a bounded operator on $\mathcal{H}$ and $\{\phi_n\}$, $\{\psi_n\}$ be orthonormal basis. Then,

$$\sum_{n=1}^{\infty} \|B\phi_n\|^2 \quad \text{and} \quad \sum_{m,n=1}^{\infty} |\langle \psi_m, A\phi_n \rangle|^2 \tag{4}$$

are independent of the basis and are equal. (4) are finite iff $B \in J_2$ and in that case, $= (4) = \|B\|_2^2$

5.2.4) Proof of the theorems 1 – 4.

① **Proof of theorem 2:** . One starts by noticing that $|\Lambda^k(A)| = \Lambda^k(|A|)$,

since $A = U\,|A|$ (polar decomposition).

. Thus, $\Lambda^k(A)$ has singular values $\mu_{i_1}(A)\cdots\mu_{i_k}(A)$ with $i_1 < \cdots < i_k$. Therefore, $\|\Lambda^k(A)\|_1 \left(= \mathrm{Tr}(|A|)\right) = \sum_{i_1<\cdots<i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$. is finite and bounded by $\sum_{i_1,\dots,i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A) = \dfrac{\|A\|_1^k}{k!}$.

. This bound implies that the series $\sum_{k\geq 0}^{\infty} z^k\,\mathrm{Tr}(\Lambda^k(A))$ is entire and the bound $\exp(|z|\cdot\|A\|_1)$ too.

. Also, $\|\Lambda^k(A)\|_1 = \sum_{i_1<\cdots<i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$ implies that

$$|\det(\mathbb{1}+zA)| \leq \sum_{k=0}^{\infty} |z|^k\, \underbrace{\mathrm{Tr}(|\Lambda^k(A)|)}_{=\|\Lambda^k(A)\|_1} \underset{\boxed{\text{since all terms are positive}}}{=} \prod_{k\geq 1}\left(1+|z|\cdot\mu_k(A)\right).$$

. Let us choose N s.t. $\sum_{k\geq N+1} \mu_k(A) \leq \frac{\varepsilon}{2}$, then

since $(1+x)\leq e^x$, $$|\det(\mathbb{1}+zA)| \leq \underbrace{\left(\prod_{k=1}^{N}(1+|z|\mu_k(A))\right)}_{\boxed{\leq C_\varepsilon\, e^{|z|\frac{\varepsilon}{2}}}}\cdot e^{|z|\frac{\varepsilon}{2}} \leq C_\varepsilon\, e^{\varepsilon\cdot|z|}.$$ #

② **Proof of theorem 3:** Denote $F(A) = \det(\mathbb{1}+A)$. The function $g(z) = F(C+zD)$ is analytic in z since $\mathrm{Tr}\left[\Lambda^k(C+zD)\right]$ is a polynomial of degree k in z. Define $g(z) = F\left[\frac{A+B}{2} + z\cdot(A-B)\right]$. Then,

$$\begin{aligned}
|F(A)-F(B)| &= |g(\tfrac12) - g(-\tfrac12)| \leq \sup_{-\frac12\leq t\leq\frac12} |g'(t)| \\
&\leq R^{-1}\cdot \sup_{|z|\leq R+\frac12} |g(z)| \quad \text{, by using a Cauchy estimate.} \\
&\overset{R^{-1}=\|A-B\|_1}{=} \|A-B\|_1 \cdot \sup_{|z|\leq\frac12+\|A-B\|_1^{-1}} \left|\det\left(\mathbb{1}+\frac{A+B}{2}+z\cdot(A-B)\right)\right| \\
&\leq \|A-B\|_1 \sup_{|z|\leq\cdots} \exp\left(\left\|\frac{A+B}{2}+z(A-B)\right\|_1\right) \\
&\leq \|A-B\|_1\cdot\exp\left(\frac12\|A\|_1+\frac{\|B\|_1}{2}+\frac12\|A-B\|_1+\cancel{\|A-B\|_1^{-1}}\cdot\cancel{\|A-B\|_1}^{\,1}\right).
\end{aligned}$$ #

. The Cauchy estimate used here is the following.

$f'(a) = \frac{1}{2\pi i}\oint dz \frac{f(z)}{z-a}$; $f(z)$ analytic $\Rightarrow$ no pole of $\frac{f(z)}{z-a}$ in any finite region except $z=a$.

$\Rightarrow |f'(a)| \le \max_{|z|=R+a} \frac{|f(z)|}{R}$. $\boxed{\text{For } a\in[-\frac{1}{2},\frac{1}{2}],\ |f'(a)| \le \max_{|z|\le R+\frac{1}{2}} \frac{|f(z)|}{R}.}$

③ Proof of Theorem 4: Part ⓐ is proven by first showing the identity for finite rank operators (it is just the finite dimensional determinant relation $\det(AB)=\det(A)\det(B)$) and then, by continuity (Thm. 3) it follows that the same relation holds for trace-class operators.

ⓑ If $\mathbb{1}+A$ is not invertible, then A has an e.v. at -1 and by ⓒ $\det(\mathbb{1}+A)=0$. If $\mathbb{1}+A$ is invertible, let $B \doteq -A(\mathbb{1}+A)^{-1}$. Thus, $\mathbb{1}+A+B+AB=\mathbb{1}$, so that by (16), $\det(\mathbb{1}+A)\neq 0$.

ⓒ Let P_λ be the spectral projection on λ, eigenvalue of A. Then,

$(P_\lambda A)\cdot((\mathbb{1}-P_\lambda)A)=0$, thus

$$\det(\mathbb{1}+z\cdot A) = \det(\mathbb{1}+z(P_\lambda A)+z\cdot(\mathbb{1}-P_\lambda)A) \overset{ⓐ}{=}$$
$$= \det(\mathbb{1}+z\cdot P_\lambda A)\cdot\det(\mathbb{1}+z(\mathbb{1}-P_\lambda)A)$$

. let $z_0=-\lambda^{-1} \Rightarrow \mathbb{1}+z_0\cdot(\mathbb{1}-P_\lambda)A$ is invertible since $\lambda\notin\sigma((\mathbb{1}-P_\lambda)A)$.

. By the spectral theorem for compact operators, $P_\lambda A$ is a finite rank operator with eigenvalue λ of multiplicity, say n, finite. Thus,

$$\det(\mathbb{1}+z\cdot P_\lambda A) \underset{\text{nice basis}}{=} \det\left(\mathbb{1}+z\cdot\begin{pmatrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{pmatrix}\right) = (1+\lambda\cdot z)^n = \left(1-\frac{z}{z_0}\right)^n.$$

#

④ Proof of Theorem 1: let $f(z)=\det(\mathbb{1}+z\cdot A)$. By Thm. 4, it has zeros at points $z_n=-\lambda_n^{-1}$. By $\sum_{n=1}^{M}|\lambda_n| \le \sum_{n=1}^{M}\mu_n$, for $M\to\infty$, we get $\sum_{n\geq 1}|\lambda_n| = \sum_{n\geq 1}\frac{1}{|z_n|}<\infty$, since A is trace-class.

Also, $f(0)=1$ and by Thm. 2, $|f(z)| \le C_\varepsilon\cdot\exp(\varepsilon\cdot|z|)$. Thus we can apply Thm. 5 which gives $\det(\mathbb{1}+zA)=\prod_n(1+z\lambda_n(A))$. If one checks the first term of expansion in z, one gets: $\mathrm{Tr}(A)=\sum_n\lambda_n(A)$.

#

6) The Tracy-Widom distribution for $\beta=2$.

. Let us remind a few results we obtained for GUE eigenvalues.

. Let H be $N\times N$ Hermitian matrices distributed according to $\exp\left(-\frac{\mathrm{Tr}(H^2)}{2N}\right)$. Then the eigenvalues of H, $\lambda_1,\dots,\lambda_N$, have joint distribution given by $\frac{1}{Z_N}\Delta_N(\lambda)^2\prod_{k=1}^{N}(e^{-\lambda_k^2/2N}\,d\lambda_k)$, with $\Delta_N(\lambda)=\det(\lambda_i^{j-1})_{1\le i,j\le N}$, the Vandermonde determinant.

. We then computed the n-point correlation functions:

$$\rho^{(n)}(\lambda_1,\dots,\lambda_n)=\det\left[K_N(\lambda_i,\lambda_j)\right]_{1\le i,j\le n},$$

with K_N the Hermite kernel:

$$K_N(x,y)=e^{-\frac{x^2}{4N}}\cdot e^{-\frac{y^2}{4N}}\sum_{k=0}^{N-1}q_k(x)q_k(y)$$

$$=N\cdot e^{-\frac{x^2}{4N}}e^{-\frac{y^2}{4N}}\cdot\frac{q_N(x)q_{N-1}(y)-q_{N-1}(x)q_N(y)}{x-y},$$

with $q_k(x)=\frac{1}{\sqrt[4]{2\pi N}}\cdot\frac{1}{\sqrt{2^k k!}}\cdot P_k^H(x/\sqrt{2N})$, $P_k^H(x)$ the Hermite polynomials of degree k.

. The particular structure of the correlation functions allows us to write the distribution of the largest eigenvalue as a Fredholm determinant:

$$\mathbb{P}(\lambda_{max}^{(N)}\le u)=\det(\mathbb{1}-P_u\cdot K_N\cdot P_u)_{L^2(\mathbb{R},dx)},$$

$$\text{with } P_u(x)=\begin{cases}1, & x>u,\\ 0, & x\le u.\end{cases}$$

. We also saw that as $N\to\infty$, the scaling limit relevant for the largest eigenvalue is $\frac{\lambda_{max}^{(N)}-2N}{N^{1/3}}$, and the rescaled kernel converges to the Airy kernel:

$$K_N^{resc}(x,y)\doteq N^{1/3}\cdot K_N(2N+x\cdot N^{1/3},2N+y\cdot N^{1/3})\underset{N\to\infty}{\longrightarrow}\mathcal{A}(x,y)\equiv\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$$

. Therefore,

$$\mathbb{P}\left(\lambda^{(N)}_{max} \leq 2N + s\cdot N^{1/3}\right) = \det\left(\mathbb{1} - P_{2N+sN^{1/3}} \cdot K_N \cdot P_{2N+sN^{1/3}}\right)$$

$$\underset{\substack{\text{change of} \\ \text{variables:} \\ x \to \frac{x-2N}{N^{1/3}}}}{=} \det\left(\mathbb{1} - P_s \cdot K_N^{resc} \cdot P_s\right)_{L^2(\mathbb{R}, dx)}.$$

. It is possible to prove (but we do not do here the computations), that $K_N^{resc} \to \mathcal{A}$ in trace-norm as $N \to \infty$. Thus, we have

$$F_2(s) \doteq \lim_{N\to\infty} \mathbb{P}\left(\lambda^{(N)}_{max} \leq 2N + s\cdot N^{1/3}\right) = \det\left(\mathbb{1} - P_s \cdot \mathcal{A} \cdot P_s\right)_{L^2(\mathbb{R}, dx)}$$

. F_2 is called the Tracy-Widom distribution (for GUE eigenvalues), and it is one of the universal laws arising in a lot of different models (in the 1+1 KPZ class).

. In the following we want to discuss F_2 and relate it with solutions of Painlevé-II equation, for which high precision numerical solutions are available [Michael Prähofer's homepage].

6.1). Properties of the Airy kernel.

① $\mathcal{A}(u,v) = \int_0^\infty d\lambda\, Ai(u+\lambda) Ai(v+\lambda)$

Proof: We need to show the identity:

$$Ai(u)Ai'(v) - Ai'(u)Ai(v) = (u-v)\cdot \int_0^\infty d\lambda\, Ai(u+\lambda)Ai(v+\lambda).$$

$$= \int_0^\infty d\lambda\, (u+\lambda) Ai(u+\lambda) Ai(v+\lambda) - \int_0^\infty d\lambda\, (v+\lambda) Ai(v+\lambda) Ai(u+\lambda)$$

$$\overset{Ai''(x) = x\cdot Ai(x)}{=} \int_0^\infty d\lambda\, Ai''(u+\lambda) Ai(v+\lambda) - \int_0^\infty d\lambda\, Ai(u+\lambda) Ai''(v+\lambda)$$

$$\overset{\int \text{by parts}}{=} Ai(v+\lambda)Ai'(u+\lambda)\Big|_0^\infty - Ai(u+\lambda)Ai'(v+\lambda)\Big|_0^\infty$$

$$- \cancel{\int_0^\infty d\lambda\, Ai'(u+\lambda) Ai'(v+\lambda)} + \cancel{\int_0^\infty d\lambda\, Ai'(u+\lambda) Ai'(v+\lambda)}$$

$$= -Ai'(u)Ai(v) + Ai(u)Ai'(v) \qquad \#$$

② $\mathbb{A}^2 = \mathbb{A}$: $\int_0^\infty d\mu \int_0^\infty d\lambda\, Ai(u+\lambda) \underbrace{\int_{\mathbb{R}} dz\, Ai(\lambda+z) Ai(\mu+z)}_{= \delta(\mu-\lambda) \text{ (completeness relation)}} \cdot Ai(v+\mu) =$

$$= \int_0^\infty d\lambda\, Ai(u+\lambda) Ai(v+\lambda).$$

③ $H = -\frac{d^2}{dx^2} + x$ is the Airy operator (regarded as self-adjoint on $L^2(\mathbb{R})$).

. Generalized eigenfunctions: $\psi_\lambda(x) \doteq Ai(x-\lambda) \Rightarrow H\psi_\lambda = \lambda \cdot \psi_\lambda$. Then, the Airy kernel is the spectral projection onto $\{H \leq 0\}$:

$$\mathbb{A}(u,v) = \int_{\mathbb{R}_-} d\mu\, Ai(u-\mu) Ai(v-\mu).$$

④ $\mathbb{A}$ is locally trace-class and $\| P_s \mathbb{A} P_s \| < 1$, $\forall s > -\infty$ (defines a det pp.).

6.2). Tracy-Widom distribution on a fixed Hilbert space.

. A convenient way for the next part, is to consider $F_2(s)$ as a Fredholm determinant on a fixed Hilbert space, $L^2(\mathbb{R}_+, dx)$, but with kernel depending on "s".

• Define the kernel $B_s(u,v) \doteq Ai(u+v+s)$, then

$$F_2(s) = \det(\mathbb{1} - B_s^2)_{L^2(\mathbb{R}_+, dx)}$$

. In fact, $B_s^2(u,v) = \int_0^\infty d\lambda\, B_s(u,\lambda) B_s(\lambda,v) = \int_0^\infty d\lambda\, Ai(u+\lambda+s) Ai(v+\lambda+s)$.

$$\Rightarrow \det(\mathbb{1} - B_s^2)_{L^2(\mathbb{R}_+, dx)} \underset{\boxed{\substack{u+s \to u \\ v+s \to v}}}{=} \det(\mathbb{1} - \mathbb{A})_{L^2((s,\infty), dx)} = F_2(s).$$

. Remark: ① B_s is not a positive operator.

② B_s is Hilbert-Schmidt on $L^2(\mathbb{R}_+, dx)$, $\forall s > -\infty$ (easy computation)

$\Rightarrow$ ③ B_s^2 is Trace-Class on $L^2(\mathbb{R}_+, dx)$, $\forall s > -\infty$.

. To check ② it is enough to consider the simple bound:

$$Ai(x) \le e^{-x},\ x \in \mathbb{R}.$$

. A few properties of the (GUE)-Tracy-Widom distribution:

mean	variance	Skewness	Kurtosis
-1,77109	0,8132	0,224	0,094

,

where: $\text{mean} = \mathbb{E}(X)$

$\text{variance} = \mathbb{E}(X-\mathbb{E}(X))^2$: spread of distribution

$\text{skewness} = \dfrac{\mathbb{E}(X-\mathbb{E}(X))^3}{\left[\mathbb{E}((X-\mathbb{E}(X))^2)\right]^{3/2}}$; degree of asymmetry (0 for Gaussian)

$\text{Kurtosis} = \dfrac{\mathbb{E}(X-\mathbb{E}(X))^4}{\left[\mathbb{E}((X-\mathbb{E}(X))^2)\right]^2} - 3$: degree to which a distribution is peaked/flat (0 for Gaussian).

. $\ln(F_2'(s)) \sim -\frac{4}{3}\cdot s^{3/2}$, $s \gg 1$ and $\ln(F_2'(s)) \sim -\frac{1}{12}\cdot|s|^3$, $s \ll -1$.

6.3) Tracy-Widom distribution and Painlevé-II equation.

. Theorem [Tracy-Widom]: $F_2(s) = \exp\left(-\int_s^{\infty}(x-s)\,q^2(x)\,dx\right)$,

where $q(x)$ is the unique solution of the Painlevé-II equation $q''(x) = s\cdot q(x) + 2\cdot q^3(x)$ satisfying the asymptotic condition $q(s) \simeq Ai(s)$ as $s \to +\infty$.

. Proof: . Define K_s on $L^2(\mathbb{R}_+, dx)$ by $K_s = B_s^2$. We also have $\|K_s\| < 1$.

. Write $A_s(x) \doteq Ai(x+s)$, then,

$$\frac{\partial}{\partial s}K_s(x,y) = \frac{\partial}{\partial s}\left(\int_0^{\infty} d\lambda\, Ai(x+\lambda+s)\,Ai(y+\lambda+s)\right)$$

$$= \int_0^{\infty} d\lambda\, Ai'(x+\lambda+s)\,Ai(y+\lambda+s) + \int_0^{\infty} d\lambda\, Ai(x+\lambda+s)\,Ai'(y+\lambda+s)$$

$$= Ai(x+\lambda+s)\,Ai(y+\lambda+s)\Big|_0^{\infty} - \cancel{\int_0^{\infty} d\lambda\, Ai(x+\lambda+s)\,Ai'(y+\lambda+s)} + \cancel{\text{II}}$$

$$= -A_s(x)\,A_s(y).$$

In bra-ket notations: $\frac{\partial}{\partial s} K_s = -|A_s\rangle\langle A_s|$. (eq. 1)

. Let $u(s) = \frac{\partial}{\partial s} \ln(\det(\mathbb{1} - K_s))$. (eq. 2)

By the identity : $\underline{\det(\mathbb{1}+A) = \exp(\mathrm{Tr}(\ln(\mathbb{1}+A)))}$ $(\|A\|<1)$

we get $\underline{u(s)} = \frac{\partial}{\partial s} \mathrm{Tr}(\ln(\mathbb{1} - K_s))$

$= -\mathrm{Tr}((\mathbb{1}-K_s)^{-1} \cdot \frac{\partial}{\partial s} K_s)$: used cyclicity of the trace

$= \mathrm{Tr}((\mathbb{1}-K_s)^{-1} |A_s\rangle\langle A_s|)$: rank-one operator

$= \underline{\langle A_s | (\mathbb{1}-K_s)^{-1} A_s \rangle}$ (eq. 3)

. Another expression for $u(s)$ is

$$\underline{u(s)} = \frac{\partial}{\partial s} \sum_{n \geq 1} \frac{-1}{n} \cdot \mathrm{Tr}(K_s^n)$$

$$= -\sum_{n\geq 1} \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \mathrm{Tr}(K_s^{n-1} \cdot \frac{\partial}{\partial s} K_s) = \sum_{n \geq 1} \mathrm{Tr}(K_s^{n-1} |A_s\rangle\langle A_s|)$$

$$= \sum_{n\geq 1} \langle A_s | K_s^{n-1} | A_s \rangle$$

$|A_s\rangle = B_s|\delta_0\rangle$, $K_s = B_s^2$ $\Rightarrow$ $= \sum_{n \geq 1} \langle \delta_0 | K_s^n \delta_0 \rangle = \underline{\langle \delta_0 | K_s (\mathbb{1}-K_s)^{-1} \delta_0 \rangle}$ (eq. 4)

$\underline{\text{. Define}}$
$$\begin{cases} \cdot\ q(s) = \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle \\ \cdot\ p(s) = \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A'_s \rangle \\ \cdot\ v(s) = \langle A_s | (\mathbb{1}-K_s)^{-1} A'_s \rangle \end{cases}$$

<u>Lemma</u>:

ⓐ $\frac{\partial u(s)}{\partial s} = -q^2(s)$

ⓑ $q^2(s) = u^2(s) - 2 \cdot v(s)$

ⓒ $\frac{\partial q(s)}{\partial s} = p(s) - q(s) \cdot u(s)$

ⓓ $\frac{\partial p(s)}{\partial s} = s \cdot q(s) - 2q(s)v(s) + p(s)u(s)$

Proof of the lemma: (a) $\frac{\partial u(s)}{\partial s} = \frac{\partial}{\partial s} \langle \delta_0 | (\mathbb{1}-K_s)^{-1} \delta_0 \rangle$

$$\overset{(*)}{=} \langle \delta_0 | (\mathbb{1}-K_s)^{-1} \frac{\partial K_s}{\partial s} (\mathbb{1}-K_s)^{-1} \delta_0 \rangle$$

$$\overset{(eq.1)}{=} -(\langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle)^2 = -q^2(s) .$$

(*) : use the identity : $\frac{d}{d\alpha} (\mathbb{1}-K)^{-1} = (\mathbb{1}-K)^{-1} \frac{dK}{d\alpha} (\mathbb{1}-K)^{-1}$ (eq.5)
for any operator with $\|K\| < 1$ depending on a parameter α.
[can be easily be proven by writing the Neumann series].

(b) We compute $\frac{\partial u(s)}{\partial s}$ using the representation (eq. 3).

$$\frac{\partial u(s)}{\partial s} \underset{\substack{\uparrow \\ (eq.5) \\ (eq.3)}}{=} 2 \cdot \langle A_s | (\mathbb{1}-K_s)^{-1} A_s' \rangle - \langle A_s | (\mathbb{1}-K_s)^{-1} A_s \rangle \langle A_s | (\mathbb{1}-K_s)^{-1} A_s \rangle$$

$$= \underline{2 \cdot v(s) - u(s)^2 \overset{(a)}{=} -q^2(s)}$$

(c) $\frac{\partial q(s)}{\partial s} \underset{\substack{\uparrow \\ (eq.5) \\ (eq.3)}}{=} \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s' \rangle - \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle \langle A_s | (\mathbb{1}-K_s)^{-1} A_s \rangle$

$$= p(s) - q(s) \cdot u(s).$$

(d) For this identity we will also need:

(eq. 6) : $[L, (\mathbb{1}-K)^{-1}] = (\mathbb{1}-K)^{-1} [L, K] (\mathbb{1}-K)^{-1}$

(eq. 7) : $[Q, K_s] = |A_s\rangle\langle A_s'| - |A_s'\rangle\langle A_s|$,
where Q is the multiplication operator of the position.

- $\frac{\partial p(s)}{\partial s} = - \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s \rangle \langle A_s | (\mathbb{1}-K_s)^{-1} A_s' \rangle + \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s'' \rangle .$

- Using $Ai''(x+s) = (x+s) Ai(x+s)$, we obtain

$$A_s'' = (Q+s) A_s$$

$$\Rightarrow \langle \delta_0 | (\mathbb{1}-K_s)^{-1} A_s'' \rangle = s\cdot\langle \delta_0|(\mathbb{1}-K_s)^{-1}A_s\rangle + \langle\delta_0|(\mathbb{1}-K_s)^{-1} Q A_s\rangle$$

$$= s\cdot q(s) + \underbrace{\langle\delta_0|Q(\mathbb{1}-K_s)^{-1}A_s\rangle}_{\equiv 0} - \langle\delta_0|[Q,(\mathbb{1}-K_s)^{-1}]A_s\rangle$$

(eq.6) with $L=Q$ and (eq.7)

$$\Downarrow\ \ s\cdot q(s) - \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s\rangle\langle A_s'|(\mathbb{1}-K_s)^{-1}A_s\rangle + \langle\delta_0|(\mathbb{1}-K_s)^{-1}A_s'\rangle\langle A_s|(\mathbb{1}-K_s)^{-1}A_s\rangle$$

$$= s\cdot q(s) - q(s)\cdot v(s) + p(s)\cdot u(s)$$

$$\Rightarrow \frac{\partial p(s)}{\partial s} = s\cdot q(s) - 2\cdot q(s)v(s) + p(s)u(s).$$ # of Lemma.

. By Lemma, we have:

$$\frac{d^2 q(s)}{ds^2} = p'(s) - q'(s)u(s) - q(s)u'(s)$$
$$= s\cdot q(s) - 2q(s)v(s) + \underline{p(s)u(s)} - \underline{p(s)u(s)} + q(s)u^2(s) + q^3(s)$$
$$= s\cdot q(s) + q(s)\cdot\underbrace{[q^2(s) + u^2(s) - 2v(s)]}_{=2\cdot q^2(s)}$$

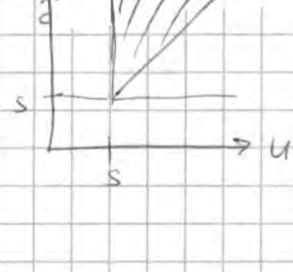
. Moreover, as $s\to\infty$, $(\mathbb{1}-K_s)^{-1}\to\mathbb{1}$, thus $q(s)\to Ai(s)$.

. To get the final formula, we need to integrate twice:

$$\cdot\ \frac{\partial u(s)}{\partial s} = \frac{\partial^2}{\partial s^2}\ln\left(F_2(s)\right) = -q^2(s).$$

$$\Rightarrow -\int_s^\infty dt\, q^2(t) = \int_s^\infty dt\,\frac{d^2}{dt^2}\left(\ln F_2(t)\right) = \frac{d}{dt}\ln F_2(t)\Big|_s^\infty = -\frac{d}{ds}\ln F_2(s)$$

$$\Rightarrow -\int_s^\infty du\int_u^\infty dt\, q^2(t) = -\int_s^\infty du\,\frac{d}{du}\ln F_2(u) = -\ln F_2(u)\Big|_s^\infty = \ln F_2(s)$$



$$\hookrightarrow = -\int_s^\infty dt\, q^2(t)\cdot\int_s^t du = -\int_s^\infty dt\,(t-s)q^2(t)$$

$$\Rightarrow F_2(s) = \exp\left[-\int_s^\infty dt\,(t-s)q^2(t)\right]$$ #

7) Dynamics on point processes: "extended point processes"

. Until now we considered only point processes "at a fixed time", e.g., eigenvalues' point process of a given random matrix ensemble. In the application to physical system that we will consider after Xmas, this corresponds to focus on a one-point random variable, like the height in a stochastic growth model at a fixed position. However, in the growth model it is natural to be interested in the spacial structure of correlations $\Rightarrow$ need to do some extension of the framework.

. In this chapter we will consider non-intersecting Brownian Motions which lead to extended point processes. Then we give a discrete version (on a graph) which is quite useful in applications. We will also discuss Dyson's Brownian Motion on matrices.

7.1) Karlin-McGregor theorem.

. The original work was in continuous time but discrete space (jump processes). Here we present something similar but for Brownian Motions.

. Consider N Brownian paths on $\mathbb{R}$. Let $\{x_i(t)\}_{i=1}^{N}$ be their positions at time t. The question is: "What is the probability that the N Brownian Motions start at $t=0$ for $y_1 > y_2 > \cdots > y_N$ and reach at time t the positions $x_1 > x_2 > \cdots > x_N$ and that they are non-intersecting"?

. The answer is "simple":

Theorem (K-McG): Let $y_1 > y_2 > \cdots > y_N$, $x_1 > x_2 > \cdots > x_N$. Then

$$
(1) \qquad \mathbb{P}_{\text{non-int}}\left(x_1(t)=x_1, \ldots, x_N(t)=x_N \mid x_1(0)=y_1, \ldots, x_1(N)=y_N \right) =
$$
$$
= \det\left[\mathbb{P}\left(x(t)=x_i \mid x(0)=y_j \right) \right]_{1\leq i,j\leq N} ,
$$

where $\mathbb{P}(X(t)=x_i \mid X(0)=y_j)$ is the transition probability of a single Brownian Motion $X(t)$. ②

<u>Remark:</u> This is not the conditional proba. that they do not intersect, but the proba. that they go from y_i's to x_i <u>and</u> do not intersect.

. <u>Proof:</u> To understand the theorem one need to keep in mind the two key properties: ① Brownian Motion is a <u>Markov</u> process $\Rightarrow$ "weight" is the product of "weights" on smaller intervals.

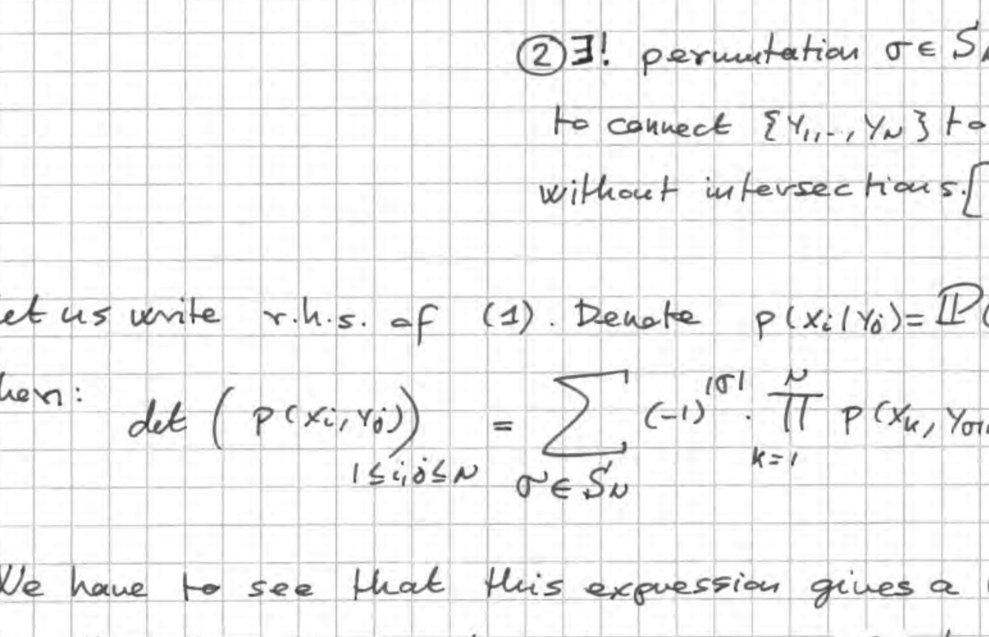
② $\exists!$ permutation $\sigma \in S_N$ s.t. it is possible to connect $\{Y_1, \dots, Y_N\}$ to $\{X_{\sigma(1)}, \dots, X_{\sigma(N)}\}$ without intersections. [$\sigma = id$ in our case].

. let us write r.h.s. of (1). Denote $p(x_i \mid y_j) = \mathbb{P}(X(t)=x_i \mid X(0)=y_j)$.

then:

$$\det\left(p(x_i, y_j)\right)_{1\le i,j\le N} = \sum_{\sigma\in S_N} (-1)^{|\sigma|} \cdot \prod_{k=1}^{N} p(x_k, y_{\sigma(k)}) \qquad (2)$$

. We have to see that this expression gives a measure zero to all configurations with intersections. Graphically:



. <u>Permutations:</u> σ $\qquad$ $\tilde{\sigma} = \sigma$ plus one transposition

. <u>Prefactor in (2):</u> $(-1)^{|\sigma|}$ $\qquad$ $(-1)^{|\tilde{\sigma}|} = (-1)^{|\sigma|+1}$.

. <u>Weights:</u> identical by Markov property.

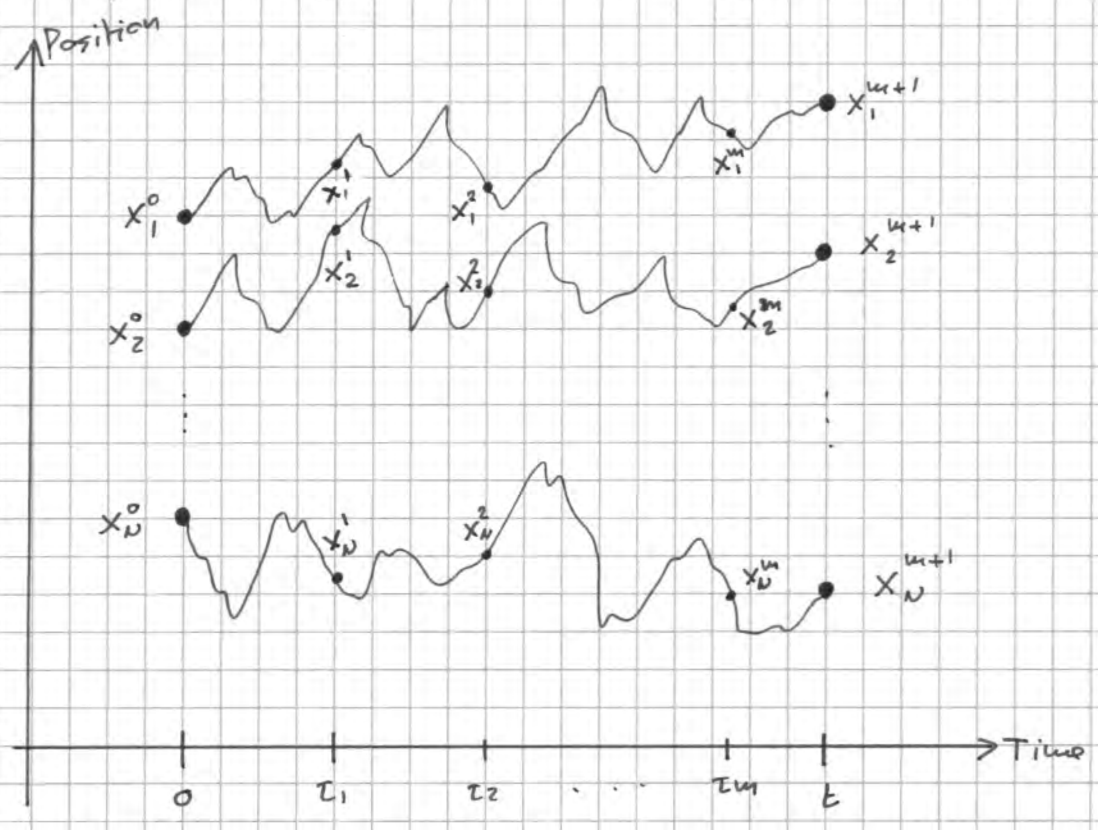
$\Rightarrow$ zero <u>contribution.</u>

. Therefore, all contributions coming from configurations with intersections is identically equal to zero, and

$$(2) = \mathbb{P}_{\text{non-int}}\left(\{X_i(t)=x_i\} \mid \{X_i(0)=y_i\}\right). \qquad \#$$

. Let now consider $0<\tau_1<\tau_2<\dots<\tau_m<t$. Then the measure of non-intersecting Brownian Motions at times $\tau_1, \tau_2, \dots, \tau_m$ is obtained by the previous theorem. Introduce the notations:

$$\begin{cases} \cdot\, x_k^n = \text{position of B.M. number } k \text{ at time } \tau_n \equiv X_k(\tau_n). \\ \cdot\, x_k^0 \text{ and } x_k^{m+1} \text{ be } \underline{\text{fixed}} \text{ initial and final positions.} \end{cases}$$



. Then, the measure on $\{x_k^\ell,\ 1\le k\le N,\ 1\le \ell\le m\}$ is given by:

$$(3)\ \cdot\ \mathbb{P}_{N,m}(\{x_k^\ell\}) = \frac{\det\left(\phi_{0,1}(x_i^0, x_j^1)\right)_{1\le i,j\le N}}{Z_{N,m}\cdot (N!)^m} \cdot \left[\prod_{k=1}^{m-1} \det\left(\phi_{k,k+1}(x_i^k, x_j^{k+1})\right)_{1\le i,j\le N}\right] \cdot \det\left(\phi_{m,m+1}(x_i^m, x_j^{m+1})\right)_{1\le i,j\le N}$$

where $\phi_{k,k+1}(x,y) =$ transition probability of one B.M. from $(x, t=\tau_k)$ to $(y, t=\tau_{k+1})$.

. Assume $Z_{N,m}\neq 0$.

Theorem: The space-time correlation functions for a measure of the form (3) is determinantal with <u>extended kernel</u>:

$$K_{N,m}(r,x;s,y) = -\phi_{r,s}(x,y) + \sum_{i,j=1}^{N} \phi_{r,m+1}(x, x_i^{m+1}) [A^{-1}]_{ij} \phi_{0,s}(x_j^0, y), \tag{4}$$

with $\phi_{r,s}(x,y) = \begin{cases} (\phi_{r,r+1} * \dots * \phi_{s-1,s})(x,y) & , \text{for } r < s, \\ 0 & , \text{for } r \geq s, \end{cases}$

and $A = [a_{ij}]_{1 \le i,j \le N}$, with $a_{ij} = \phi_{0,m+1}(x_i^0, x_j^{m+1})$.

Remark: By applying Cauchy-Binet identity several times, one obtains $Z_{N,m} = \det(A) \neq 0$ by assumption, so A is invertible.

. The above theorem follows from the following one:

Theorem: Let g be a function on $\mathbb{R} \times \{1,\dots,m\}$ such that $K_{N,m} \cdot g$ is trace-class. Then,

$$\int_{\mathbb{R}^{N\times m}} \prod_{r=1}^{m} \prod_{j=1}^{N} (1 + g(r, x_j^r)) \cdot P_{N,m}(X)\, dX^{N\times m} = \tag{5}$$

$$= \det\left(\mathbb{1} + K_{N,m} \cdot g\right)_{L^2(\mathbb{R}\times\{1,\dots,m\}, dx)}$$

where with $X = (x_1^1, \dots, x_1^m; x_2^1, \dots, x_2^m; \dots; x_N^1, \dots, x_N^m)$.

Proof: . L.h.s. of (5) is the expected value of $(1+g)$ with respect to the measure (3). A general expansion in terms of correlation functions leads to a series (see chapter 3) and in the particular case when the correlation functions are determinantal, this series is a Fredholm expansion of a Fredholm determinant. Thus, proving (5) will imply (4).

. Denote $w_{N,m}(x) = Z_{N,m} \cdot P_{N,m}(x)$. Then, set

$$Z_{N,m}(g) \doteq \frac{1}{(N!)^m} \int_{\mathbb{R}^{N\times m}} dx \cdot w_{N,m}(x) \cdot \prod_{r=1}^{m} \prod_{j=1}^{N} (1+g(r,x_j^r)).$$

. With this notation: $Z_{N,m} = \det(A) \equiv Z_{N,m}(0)$.

$$. \; Z_{N,m}(g) = \frac{1}{(N!)^m} \int_{\mathbb{R}^{N\times m}} dx \left(\prod_{r=1}^{m} \prod_{j=1}^{N} (1+g(r,x_j^r)) \right) \cdot \prod_{r=0}^{m} \det\left(\phi_{r,r+1}(x_i^r, x_j^{r+1}) \right)_{1\le i,j\le N}$$

$$\underset{m \text{ times}}{\overset{\text{Cauchy-Binet}}{=}} \det\left[\int_{\mathbb{R}^m} \prod_{r=1}^{m} (1+g(r,z_r)) \cdot \phi_{0,1}(x_i^0, z_1) \cdot \prod_{r=1}^{m-1} \phi_{r,r+1}(z_r, z_{r+1}) \cdot \phi_{m,m+1}(z_m, x_j^{m+1}) \cdot d^m z \right]_{1\le i,j\le N} \qquad (6)$$

. We expand: $\displaystyle\prod_{r=1}^{m} (1+g(r,z_r)) = 1 + \sum_{e=1}^{m} \sum_{1\le r_1<r_2<\dots<r_e\le m} g(r_1,z_{r_1}) \cdots g(r_e,z_{r_e})$

and plug back in (6).

. The restriction of ordering of the r_i in can be dropped since $\phi_{r,s} = 0$ for $r \geq s$.

. Also, we denote $\psi^{(0)}(r,x;s,y) = \delta_{r,s} \cdot \delta(x-y)$

and, $e \geq 1$, $\displaystyle\psi^{(e)}(r,x;s,y) = \sum_{u=1}^{m} \int_{\mathbb{R}} dz \, \psi(r,x;u,z) \cdot \psi^{(e-1)}(u,z;s,y)$, where

$\boxed{\psi(r,x;u,z) \doteq \phi_{r,u}(x,z) \cdot g(u,z).}$ $\boxed{. \, \psi^{(e)} \text{ is the } e\text{-th convolution of } \psi.}$

$$. \text{ We have: } Z_{N,m}(g) = \det\left[a_{ij} + \sum_{e=1}^{m} \sum_{r_1,\dots,r_e=1}^{m} \int_{\mathbb{R}^e} dz_1 \dots dz_e \, \phi_{0,r_1}(x_i^0, z_1) \cdot g(r_1,z_1) \cdot \left(\prod_{s=1}^{e-1} \psi(r_s,z_s;r_{s+1},z_{s+1}) \right) \cdot \phi_{r_e,m+1}(z_e, x_j^{m+1}) \right]_{1\le i,j\le N}.$$

$$\underset{\substack{r_1\equiv r \\ z_1\equiv x \\ r_e\equiv s \\ z_e\equiv y}}{=} \det\left[a_{ij} + \sum_{r,s=1}^{m} \int_{\mathbb{R}^2} dx\,dy \, \phi_{0,r}(x_i^0, x) \cdot g(r,x) \cdot \left(\sum_{e=1}^{m} \psi^{(e-1)}(r,x;s,y) \right) \cdot \phi_{s,m+1}(y, x_j^{m+1}) \right]_{1\le i,j\le N}$$

. What we need to compute is $\dfrac{Z_{N,m}(g)}{\det(A)} = \dfrac{Z_{N,m}(g)}{Z_{N,m}(0)}$.

$$\frac{Z_{N,m}(g)}{Z_{N,m}(0)} = \det\Big[\delta_{ij} + (a\cdot b)_{ij}\Big]_{1\le i,j\le N}$$

with $\displaystyle a(i;r,x) \doteq \sum_{k=1}^{N} [A^{-1}]_{ik}\cdot \phi_{0,r}(x_k^0,x)\cdot g(r,x)$

and $\displaystyle b(r,x;j) \doteq \sum_{s=1}^{N}\int_{\mathbb{R}} dy \left(\sum_{e=1}^{m} \psi^{(e-1)}(r,x;s,y)\right)\cdot \phi_{s,m+1}(y,x_j^{m+1})$.

. At this point we use the identity:

$$(7)\qquad \det(\mathbb{1} + (a\cdot b))_{\ell^2(\{1,\dots,N\})} = \det(\mathbb{1} + (b\cdot a))_{L^2(\mathbb{R}\times\{1,\dots,m\},\, d\mathring{x})}.$$

$$.\ (b\cdot a)(r,x;s,y) = \sum_{j=1}^{N}\sum_{e=1}^{m}\sum_{\tilde{s}=1}^{N}\int_{\mathbb{R}} d\tilde{y}\, \psi^{(e-1)}(r,x;\tilde{s},\tilde{y})\cdot \phi_{\tilde{s},m+1}(\tilde{y},x_j^{m+1}) \cdot \sum_{k=1}^{N}[A^{-1}]_{jk}\cdot \phi_{0,s}(x_k^0,y)\cdot g(s,y).$$

$$\Rightarrow\ b\cdot a = \left(\sum_{e=1}^{m} \psi^{(e-1)}\right)\cdot \tilde{K}_{N,m}\cdot g\,,\ \text{where}$$

$$\tilde{K}_{N,m}(r,x;s,y) = K_{N,m}(r,x;s,y) + \phi_{r,s}(x,y).$$

. Finally, since $\phi_{r,s} = 0$ for $r \geq s$, we have

$$\det(\mathbb{1} - \psi) = 1 \ \text{ and } \ \underbrace{\psi^{(e)} \equiv 0}_{\text{for } e\geq m}\ (\text{Nilpotent}).\ \text{Thus}$$

$$\det(\mathbb{1}-\psi)\det(\mathbb{1}+b\cdot a) = \det\Big(\mathbb{1}-\psi + (\mathbb{1}-\psi)\cdot\sum_{e=1}^{m}\psi^{(e-1)}\tilde{K}_{N,m}\, g\Big)$$

$$= \det\Big(\mathbb{1}-\psi + \underbrace{\Big(\sum_{e=1}^{m}(\mathbb{1}-\psi)\psi^{(e-1)}\Big)}_{\equiv \mathbb{1}}\tilde{K}_{N,m}\cdot g\Big)$$

$\equiv \mathbb{1}$. Indeed: $= (\mathbb{1}-\psi)(\mathbb{1}+\psi+\dots+\psi^{m-1}) = \mathbb{1} - \psi^{(m)} = \mathbb{1}$.

$$= \det(\mathbb{1} + K_{N,m}\cdot g)$$

//

7.2) Application to non-intersecting Brownian Bridges.

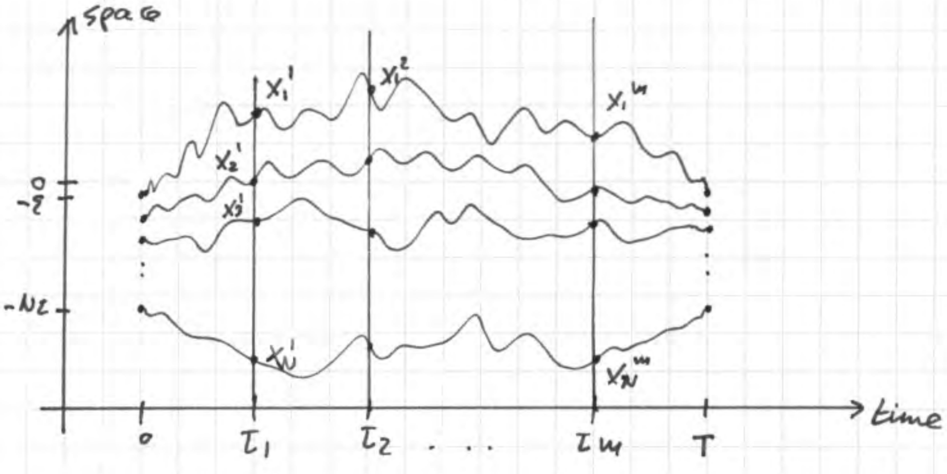
. For Brownian paths, $\phi_{r,s}(x,y) = \dfrac{\exp\left(-\frac{(y-x)^2}{2(\tau_s-\tau_r)}\right)}{\sqrt{2\pi(\tau_s-\tau_r)}} \mathbb{1}_{[\tau_s>\tau_r]}$.

. By the above theorem, the point process with support on $\{x_k^n,\ 1\le k\le N,\ 1\le n\le m\}$ is determinantal in space-time provided the initial and final positions are fixed.

. We apply to N non-intersecting Brownian Bridges from time $t=0$ to time $t=T$. We do the usual construction:

① $x_i^0 = x_i^{m+1} = -\varepsilon \cdot i$

② $\lim_{\varepsilon\to 0}$.



. let us see what happens as $\varepsilon\to 0$ to the first / last term in (3).

First term: $\det(\phi_{0,1}(x_i^0, x_j^1)) = \det\left[\dfrac{1}{\sqrt{2\pi\tau_1}} \cdot e^{-\frac{(x_j^1+\varepsilon i)^2}{2\tau_1}}\right]_{1\le i,j\le N}$

$$= \left(\prod_{k=1}^{N} \frac{e^{-(x_k^1)^2/2\tau_1}}{\sqrt{2\pi\tau_1}}\right) \cdot \underbrace{\det\left(e^{-\frac{i\varepsilon x_j^1}{\tau_1}} \cdot e^{-\frac{i^2\varepsilon^2}{2\tau_1}}\right)_{1\le i,j\le N}}_{\circledast}$$

. Now: $\circledast = \det\left[1 - \dfrac{i\cdot\varepsilon x_j^1}{\tau_1} + \dfrac{1}{2}\dfrac{(i\varepsilon x_j^1)^2}{\tau_1^2} + \dots + \dfrac{(-1)^{N-1}}{(N-1)!}\cdot\dfrac{(i\varepsilon x_j^1)^{N-1}}{\tau_1^{N-1}} + \mathcal{O}(\varepsilon^N)\right]_{1\le i,j\le N} \cdot$

$$\cdot\left(\prod_{i=1}^{N} e^{-\frac{i^2\varepsilon^2}{2\tau_1}}\right)$$

• The product $\prod_{i=1}^{N} e^{-\frac{i^2\varepsilon^2}{2\tau_1}} \longrightarrow 1$ as $\varepsilon \to 0$.

• It remains the determinant:

$$\det\left(1-\left(\frac{i\varepsilon x_j^1}{\tau_1}\right)+\dots+\left(\frac{-i\varepsilon x_j^1}{\tau_1}\right)^{N-1}\frac{1}{(N-1)!}+\mathcal{O}(\varepsilon^N)\right)_{1\le i,j\le N}$$ to evaluate.

• We can apply linear combinations and get [Check for example 4x4...]

$$\text{const}\times\det\left[\begin{pmatrix}1 + \mathcal{O}(\varepsilon^N)\\ \varepsilon x_j^1 + \mathcal{O}(\varepsilon^N)\\ (\varepsilon x_j^1)^2 + \mathcal{O}(\varepsilon^N)\\ \vdots \\ (\varepsilon x_j^1)^{N-1} + \mathcal{O}(\varepsilon^N)\end{pmatrix}_{1\le j\le N}\right] = \text{const}\cdot\varepsilon^{\frac{N(N-1)}{2}}\underbrace{\det\left[\begin{pmatrix}1+\mathcal{O}(\varepsilon^N)\\ x_j^1+\mathcal{O}(\varepsilon^{N-1})\\ \vdots\\ (x_j^1)^{N-1}+\mathcal{O}(\varepsilon)\end{pmatrix}_{1\le j\le N}\right]}_{**}$$

• As $\varepsilon \to 0$, $**$ $\to$ Vandermonde determinant in $(x_1^1,\dots,x_N^1)$.

Therefore, $$\lim_{\varepsilon\to 0}\frac{\det\left(\phi_{0,1}(-i\varepsilon, x_j^1)\right)_{1\le i,j\le N}}{\varepsilon^{N(N-1)/2}} = \text{const}\cdot\Delta_N(x_1^1,\dots,x_N^1).$$

• Last term: Exactly in the same way.

• The factors $\varepsilon^{\frac{N(N-1)}{2}}$ will be compensate by the normalization constant, so, in the $\varepsilon\to 0$ limit, the measure on $\{x_k^n\}$ is given by

$$P(\{x_k^n\}) = \text{const}\times \Delta_N(\{x_k^1\})\cdot\Delta_N(\{x_k^{m-1}\})\cdot\prod_{k=1}^{N}\frac{e^{-(x_k^1)^2/2\tau_1}}{\sqrt{2\pi\tau_1}}\cdot\prod_{\ell=1}^{N}\frac{e^{-(x_k^{m-1})^2/2(T-\tau_{m-1})}}{\sqrt{2\pi(T-\tau_{m-1})}}\cdot$$
$$\cdot\prod_{n=1}^{m-1}\det\left(\frac{e^{-\frac{(x_j^{n+1}-x_i^n)^2}{2(\tau_{n+1}-\tau_n)}}}{\sqrt{2\pi(\tau_{n+1}-\tau_n)}}\right)_{1\le i,j\le N}.$$
$$= \text{const}\times\det\left[\frac{e^{-(x_j^1)^2/2\tau_1}}{\sqrt{2\pi\tau_1}}\cdot P_{i-1}(x_j^1)\right]_{1\le i,j\le N}\cdot\prod_{n=1}^{m-1}\det\left[\frac{e^{-\frac{(x_j^{n+1}-x_i^n)^2}{2(\tau_{n+1}-\tau_n)}}}{\sqrt{2\pi(\tau_{n+1}-\tau_n)}}\right]_{1\le i,j\le N}\cdot$$
$$\times\det\left[\frac{e^{-(x_j^{m-1})^2/2(T-\tau_{m-1})}}{\sqrt{2\pi(T-\tau_{m-1})}}\cdot\tilde{P}_{i-1}(x_j^{m-1})\right]_{1\le i,j\le N}$$

. We can still choose the p_i's and $\tilde{p}_i$'s. No surprise: they are given in terms of Hermite polynomials, defined as:

$$\int_{\mathbb{R}} dx\, e^{-x^2} H_k(x)\, H_l(x) = \sqrt{\pi}\cdot 2^k k!\cdot \delta_{k,l}$$

. Let us define: $$\underline{\Phi_i(r,y) = \frac{\sqrt[4]{2\pi T}}{\sqrt{i!}\, 2^{i/2}} \cdot \left(\frac{T-\tau r}{\tau r}\right)^{i/2} \cdot H_i\left(\frac{y}{\sqrt{2\tau r(T-\tau r)/T}}\right) \frac{e^{-\frac{y^2}{2\tau r}}}{\sqrt{2\pi\tau r}}}$$

and $$\underline{\Psi_j(s,x) = \frac{\sqrt[4]{2\pi T}}{\sqrt{j!}\cdot 2^{j/2}} \cdot \left(\frac{\tau s}{T-\tau s}\right)^{j/2} \cdot H_j\left(\frac{x}{\sqrt{2\tau s(T-\tau s)/T}}\right) \cdot \frac{e^{-\frac{x^2}{2\tau s}}}{\sqrt{2\pi\tau s}}}$$

. Then, these functions satisfy (see page 4):

$$(*)\begin{cases} \displaystyle\int_{\mathbb{R}} dx\, \Phi_i(r,x)\, \phi_{r,s}(x,y) = \Phi_i(s,y) \quad \text{and} \\ \displaystyle\int_{\mathbb{R}} dz\, \phi_{r,s}(x,z)\, \Psi_j(s,z) = \Psi_j(r,x)\,. \end{cases}$$

. Moreover, $\{\Phi_i(1,x)\}_{i=0}^{N-1}$ generates $\{e^{-\frac{x^2}{2\tau 1}} p_{i-1}(x)\}_{i=1}^{N}$

and $\{\Psi_j(m,x)\}_{j=0}^{N-1}$ generates $\{e^{-\frac{x^2}{2(T-\tau m)}}\cdot p_{j-1}(x)\}_{j=1}^{N}$

. Finally, with this choice, the matrix A to be inverted has entries:

$$a_{ij} \overset{\text{use }(*)}{=} \int_{\mathbb{R}} dx\, \Phi_i(1,x)\,\Psi_j(1,x) = \int_{\mathbb{R}} dx\, H_i\left(\frac{x}{\sqrt{2\tau(T-\tau)/T}}\right) H_j\left(\frac{x}{\sqrt{2\tau(T-\tau)/T}}\right) \cdot \frac{e^{-\frac{x^2}{\tau}}}{2\pi\tau} \cdot \frac{\sqrt{2\pi T}}{\sqrt{j!\,i!}\, 2^{\frac{i+j}{2}}} \cdot \left(\frac{T-\tau}{\tau}\right)^{\frac{i-j}{2}}$$

$$\overset{\tau=\frac{T}{2}}{=} \underbrace{\int_{\mathbb{R}} dx\, H_i\left(\frac{x}{\sqrt{T/2}}\right) H_j\left(\frac{x}{\sqrt{T/2}}\right) \cdot e^{-\frac{x^2 2}{T}}}_{=\delta_{ij}\sqrt{\frac{T}{2}}\cdot\sqrt{\pi}\cdot 2^i\, i!} \cdot \frac{1}{\sqrt{j!\,i!}\, 2^{\frac{i+j}{2}}} \cdot \frac{\sqrt{T}\, 2}{\sqrt{2\pi}\cdot T}$$

$$= \delta_{ij}.$$

. The consistency relations $(*)$ are implied by the following computation.

. A computation $\Rightarrow$ consistency of $\Phi_i(r,x)$, $\Psi_0(r,x)$.

. The n-th Hermite polynomials, H_n, is given by

$$H_n(x) = \frac{n!}{2\pi i} \oint_{\Gamma_0} dw \, \frac{e^{2xw - w^2}}{w^{n+1}}.$$

$$\Rightarrow \boxed{\int_{\mathbb{R}} dx \, \frac{e^{-\frac{x^2}{2\tau_1}}}{\sqrt{2\pi\tau_1}} \cdot \frac{e^{-\frac{(y-x)^2}{2(\tau_2-\tau_1)}}}{\sqrt{2\pi(\tau_2-\tau_1)}} \cdot H_n\left(\frac{x}{\sqrt{2\tau_1(T-\tau_1)/T}}\right) =}$$

$$= \frac{n!}{2\pi i} \oint_{\Gamma_0} dw \, \frac{e^{-w^2}}{w^{n+1}} \cdot \underbrace{\int_{\mathbb{R}} dx \, \frac{e^{-\frac{x^2}{2\tau_1}}}{\sqrt{2\pi\tau_1}} \cdot \frac{e^{-\frac{(y-x)^2}{2(\tau_2-\tau_1)}}}{\sqrt{2\pi(\tau_2-\tau_1)}} \cdot e^{2x \cdot w \cdot \alpha}}_{= \frac{e^{-\frac{y^2}{2\tau_2}}}{\sqrt{2\pi\tau_2}} \cdot \exp\left[\frac{2yw\sqrt{\tau_1 T}}{\sqrt{2(T-\tau_1)}\,\tau_2} + \frac{w^2 \cdot T(\tau_2-\tau_1)}{(T-\tau_1)\tau_2}\right]}, \quad \alpha = \frac{1}{\sqrt{\frac{2\tau_1(T-\tau_1)}{T}}}$$

$w \doteq z \sqrt{\frac{(T-\tau_1)\tau_2}{(T-\tau_2)\tau_1}}$

$$= \frac{n!}{2\pi i} \oint_{\Gamma_0} dz \, \frac{e^{-z^2}}{z^{n+1}} \cdot e^{\frac{2yz}{\sqrt{2(T-\tau_2)\tau_2/T}}} \cdot \left(\frac{(T-\tau_2)\tau_1}{(T-\tau_1)\tau_2}\right)^{n/2}$$

$$= \boxed{\left(\frac{(T-\tau_2)\tau_1}{(T-\tau_1)\tau_2}\right)^{n/2} \cdot H_n\left(\frac{y}{\sqrt{2\tau_2(T-\tau_2)/T}}\right)}$$

. Now we have all ingredients to get the kernel for N non-intersecting Brownian Bridges:

$$\boxed{K_N(r,x;s,y) = -\phi_{r,s}(x,y) + \sum_{k=0}^{N-1} \Psi_k(r,x) \cdot \Phi_k(s,y)}$$

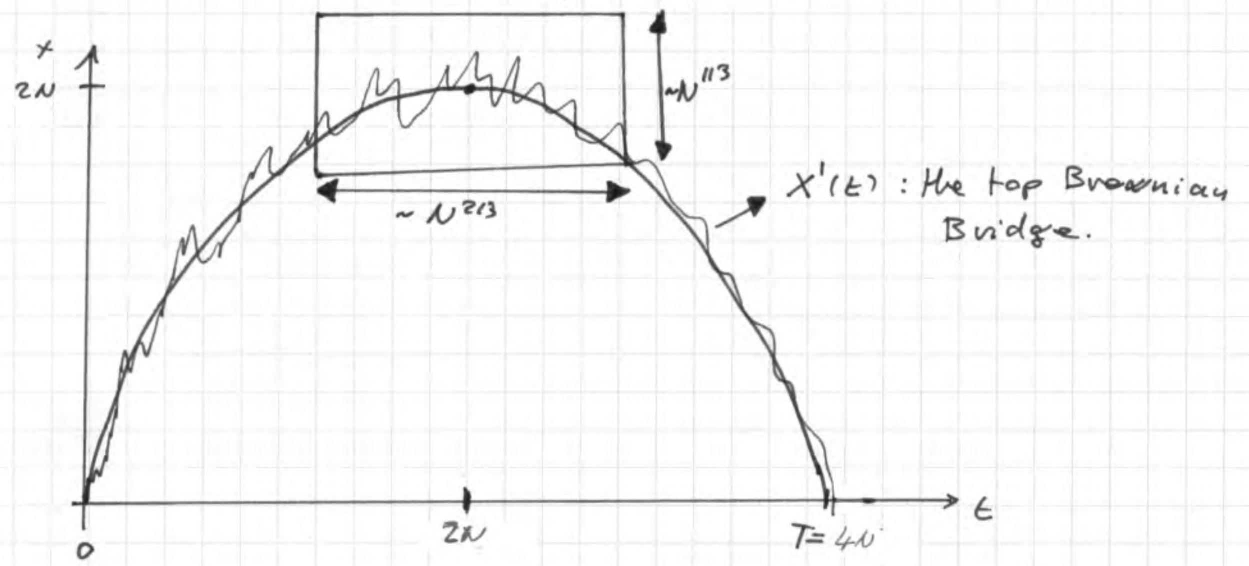
with $\Phi_i(r,x)$, $\Psi_j(s,y)$ defined at page 3 and

$$\phi_{r,s}(x,y) = \frac{\exp\left[-\frac{(y-x)^2}{2(\tau_s-\tau_r)}\right]}{\sqrt{2\pi(\tau_s-\tau_r)}} \mathbb{1}_{[r<s]}$$

7.3) Edge-scaling and Airy process.

- Consider now the following rescaling:

  - $T = 4N$ and focus around
  - $t \cong N$: $\begin{cases} \tau_i = 2N + u_i \cdot N^{2/3} & , u_1 < u_2 < \ldots < u_m \text{ fixed,} \\ x_i = 2N - \frac{u_i^2}{4} \cdot N^{1/3} + s_i \cdot N^{1/3}. \end{cases}$

- 

Remark: In the rescaling of x_i; $2N - \frac{u_i^2}{4} N^{1/3}$, is the position of the limit shape and $s_i N^{1/3}$ is the deviation from it.

- The rescaled Kernel is then:

$$K_N^{resc}(u_1, s_1; u_2, s_2) \doteq N^{1/3} \cdot K_N(\tau_1, x_1; \tau_2, x_2).$$

- Let us compute the <u>asymptotics of $\Psi_{N-k}(r, x)$</u>:

$$\Psi_{N-k}(r,x) = \frac{\sqrt[4]{2\pi \cdot 4N}}{\left((N-k)!\, 2^{N-k}\right)^{1/2}} \cdot \left(\frac{2N + u \cdot N^{2/3}}{2N - u \cdot N^{2/3}}\right)^{\frac{N-k}{2}} \cdot \frac{e^{-\frac{\left[2N + (s - \frac{1}{4}u^2) N^{1/3}\right]^2}{2 \cdot \left[2N + u N^{2/3}\right]}}}{\sqrt{2\pi \cdot (2N + u N^{2/3})}} \cdot$$

$$\cdot H_{N-k}\Bigg[\underbrace{\frac{2N + (s - \frac{u^2}{4}) N^{1/3}}{\left(2(2N + uN^{2/3})(2N - uN^{2/3})/4N\right)^{1/2}}}_{\cong \sqrt{2N} + N^{-1/6} \cdot \frac{s}{\sqrt{2}} = \frac{2N + s N^{1/3}}{\sqrt{2N}}}\Bigg]$$

. The asymptotics of Hermite polynomials are known, compare with 3.3.4 of this lecture series:

$$H_{N-k}\left(\frac{2N+sN^{1/3}}{\sqrt{2N}}\right) \cong (2\pi N)^{1/4} \cdot \sqrt{2^{N-k}\cdot(N-k)!} \cdot N^{-1/3} \cdot$$

$$\underline{\cdot\, e^{+\frac{(2N+sN^{1/3})^2}{4N}} \cdot Ai\left(s+N^{-1/3}\cdot\left(k-\tfrac{1}{2}\right)\right).}$$

Therefore: $\underline{\Psi_{N-\lambda N^{1/3}}(s,x)} \cong N^{-1/3} \cdot Ai(s+\lambda) \cdot \quad \cdot \exp\left[+\frac{(2N+sN^{1/3})^2}{4N}\right]$

$$\cdot \exp\left[-\frac{(2N+sN^{1/3}-\frac{u^2}{4}N^{1/3})^2}{4N+2uN^{2/3}}\right]\cdot$$

$$\cdot \exp\left[-\frac{N-\lambda N^{1/3}}{2}\cdot \ln\left(\frac{2N+uN^{2/3}}{2N-uN^{2/3}}\right)\right]$$

$$\underline{\cong N^{-1/3}\cdot Ai(s+\lambda)\cdot \exp\left[\frac{\lambda u}{2}\right]\cdot f(s,u)}$$

$$\underline{\text{with } f(s,u) = \exp\left[-\frac{u^3}{24}+\frac{su}{2}\right]}$$

. Asymptotics of $\underline{\Phi_{N-k}(s,y)}$: Similar.

$$\cdot\ \Phi_{N-\lambda N^{1/3}}(s,y) \cong N^{-1/3}\cdot Ai(s+\lambda)\cdot \exp\left[\frac{\lambda u}{2}\right]\cdot f(s,u)^{-1}$$

. Therefore, the main part of the kernel, i.e. $N^{1/3}\sum_{k=0}^{N-1} \Psi_k(s,x)\,\Phi_k(r,y) =$

$$\cong \int_0^\infty d\lambda\, Ai(s_1+\lambda)Ai(s_2+\lambda)\cdot e^{-\frac{\lambda}{2}(u_2-u_1)} \cdot \underline{\frac{f(s_1,u_1)}{f(s_2,u_2)}}.$$

. Asymptotics of the transition term:

$$N^{1/3}\cdot \phi_{v,s}(x,y) = \frac{\cancel{N^{1/3}}\cdot \exp\left[-\frac{\left(s_1-\frac{u_1^2}{4}-s_2+\frac{u_2^2}{4}\right)^2 \cancel{N^{2/3}}}{2(u_2-u_1)\cancel{N^{2/3}}}\right]}{\sqrt{2\pi(u_2-u_1)\cancel{N^{2/3}}}}$$

$$= \frac{1}{\sqrt{2\pi(u_2-u_1)}}\cdot \exp\left[-\frac{(u_2-u_1)(u_2+u_1)^2}{32} - \frac{u_1+u_2}{4}\cdot(s_1-s_2) - \frac{(s_2-s_1)^2}{2(u_2-u_1)}\right]$$

$$= \frac{1}{\sqrt{4\pi\left(\frac{u_2-u_1}{2}\right)}} \cdot \exp\left[-\frac{(u_2-u_1)}{2}\cdot\left(\frac{u_2+u_1}{2}\right)^2\cdot\frac{1}{4} - \frac{u_2+u_1}{2}\cdot\frac{s_1-s_2}{2} - \frac{(s_2-s_1)^2}{4\cdot\frac{u_2-u_1}{2}}\right]$$

$$= \frac{f(s_1,u_1)}{f(s_2,u_2)} \cdot \frac{1}{\sqrt{4\pi\frac{u_2-u_1}{2}}}\cdot\exp\left[-\frac{(s_2-s_1)^2}{4\cdot\frac{u_2-u_1}{2}} + \frac{1}{12}\left(\frac{u_2-u_1}{2}\right)^3 - \frac{(u_2-u_1)(s_1+s_2)}{4}\right].$$

. The common factor $\frac{f(s_1,u_1)}{f(s_2,u_2)}$ cancels out in the determinant.

. Moreover, since we have a lot of factors $\frac{1}{2}$ in front of u, we redefine $u \to 2u$. We have derived the following result.

Thm.: . Let N Brownian Bridges from $t=0$ to $t=4N$ be conditioned on non-intersecting (excepts at the origin at $t=0$, $t=4N$).

. Let $\begin{cases} \tau = 2N + 2u\,N^{2/3} \\ x = 2N - u^2 N^{1/3} + s\,N^{1/3} \end{cases}$

. Then, in the $N \to \infty$ limit, we have an extended determinantal point process with kernel:

$$K_2(u,s;u',s') = -\frac{\mathbb{1}(u'>u)}{\sqrt{4\pi(u'-u)}}\cdot\exp\left[-\frac{(s'-s)^2}{4(u'-u)} + \frac{1}{12}(u'-u)^3 - \frac{(u'-u)(s+s')}{2}\right]$$
$$+ \int_0^\infty d\lambda\, Ai(s+\lambda)\,Ai(s'+\lambda)\cdot e^{\lambda(u'-u)}$$

. One can also check that $\int_{\mathbb{R}} d\lambda\, Ai(s+\lambda) Ai(s'+\lambda)\, e^{\lambda(u'-u)} =$

$$\overset{u'>u}{=} \frac{1}{\sqrt{4\pi(u'-u)}}\cdot\exp\left[-\frac{(s'-s)^2}{4(u'-u)} + \frac{1}{12}(u'-u)^3 - \frac{(u'-u)(s+s')}{2}\right].$$

. Under the assumption that not only the Kernel converges but control on the decay is good enough (it can be made) the above result can be stated as follows.

Corollary: let $X'(t)$ be the trajectory of the top Brownian Bridge. Then, define the rescaled process

$$Y_N(u) \doteq \frac{X'(2N+2uN^{2/3})-(2N-u^2N^{1/3})}{N^{1/3}}.$$

Then, $\lim_{N\to\infty} Y_N(u) = \mathcal{A}_2(u)$

in the sense of finite-dimensional distributions, where $\underline{\mathcal{A}_2(u)}$ is called the <u>Airy process.</u>

. <u>Definition: Airy process:</u>

. The Airy process is defined via the finite-dimensional distributions given by (set $u_1 < u_2 < \dots < u_m$)

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \mathcal{A}_2(u_k) \leq s_k\right) = \det\left(\mathbb{1} - \chi_s K_2 \chi_s\right)_{L^2(\mathbb{R}\times\{u_1,\dots,u_m\})}$$

with $\chi_s(x, u_k) = \mathbb{1}_{[x \geq s_k]}$ and K_2 is the extended Airy Kernel:

$$K_2(u,s;u',s') = \begin{cases} \int_0^\infty d\lambda\, Ai(s+\lambda) Ai(s'+\lambda) e^{\lambda(u'-u)}, & \text{if } u' \leq u, \\ -\int_{-\infty}^0 d\lambda\, Ai(s+\lambda) Ai(s'+\lambda) e^{\lambda(u'-u)}, & \text{if } u' > u. \end{cases}$$

. Some properties of the Airy Process, $\mathcal{A}_2$.

. From the formula of the extended Kernel, it is immediate that the Airy Process is stationary.

. The one-point distribution is given by the $\beta=2$ Tracy–Widom distribution:

$$\mathbb{P}(\mathcal{A}_2(0) \leq s) = F_2(s).$$

. Covariance: $$\mathrm{Cov}(\mathcal{A}_2(0), \mathcal{A}_2(u)) = \begin{cases} \mathrm{Var}(\mathcal{A}_2(0)) - u + \mathcal{O}(u^2), & u \ll 1, \\ \frac{1}{u^2} + \mathcal{O}(u^{-4}), & u \gg 1. \end{cases}$$

. The Airy process is not a Markov process.

7.4) Dyson's Brownian Motion.

. Consider matrices H in one of the GOE/GUE/GSE ensembles. Then, the independent parameter are:

$$\begin{cases} H_{ii}, & 1 \leq i \leq N \\ H_{ij}^{(\lambda)}, & 1 \leq i < j \leq N, \ 0 \leq \lambda \leq \beta - 1 \end{cases}$$

$\Rightarrow$ $p = N + \frac{\beta}{2} N(N-1)$ independent real "entries", which we denote by H_μ, $\mu = 1, \dots, p$.

. Dyson's Brownian Motion on matrices consists in independent Ornstein-Uhlenbeck processes on H_μ (i.e., Brownian Motions in a quadratic potential) as follows:

$$(1) \quad \begin{cases} dH_\mu = -\alpha \cdot H_\mu dt + \sigma_\mu dB_\mu, \quad \sigma_\mu = \begin{cases} 1, & \mu = (i,i) \\ \frac{1}{2}, & \text{otherwise} \end{cases} \\ \text{where } dB_\mu \text{ are independent standard B.M.} \end{cases}$$

. Denote by $P(H_1, \ldots, H_p; t)$ the probability density at time t. Then, one can check that P satisfies the Smoluchowski equation:

$$(2) \quad \frac{\partial P}{\partial t} = \sum_{\mu=1}^{P} \left[\frac{1}{2} \sigma_\mu \cdot \frac{\partial^2}{\partial H_\mu^2} P + \alpha \cdot \frac{\partial}{\partial H_\mu} (H_\mu P) \right]$$

. Moreover, the solution of (2) with initial condition $H(0)$ at $t=0$ is given by

$$(3) \begin{cases} P(H, t) = \dfrac{\text{const}}{(1-q_t^2)^{p/2}} \cdot \exp\left[- \dfrac{\alpha \operatorname{Tr}((H - q_t H(0))^2)}{(1-q_t^2)} \right] \\ \text{where} \quad q_t = \exp(-\alpha \cdot t) \end{cases}$$

. In particular, $P^{stat}(H) = \text{const} \cdot \exp[-\alpha \operatorname{Tr}(H^2)]$ is the stationary solution of (3) [take $t \to \infty$ in (3)].

. <u>Question:</u> What happens to the eigenvalues when the matrices evoles according to (1)?

. <u>Answer:</u> The N eigenvalues, $\lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_N(t)$ satisfy the system of SDE:

$$(4) \quad \begin{cases} d\lambda_i = \left[-\alpha \lambda_i + \dfrac{\beta}{2} \displaystyle\sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \right] dt + db_i, \quad i = 1, \ldots, N \\ \text{where } db_i \text{ are independent standard B.M.} \end{cases}$$

. Let us now determine (4). First of all, notice that the evolution (1) does not depends on the choice of basis. In fact, the solution (3) involves only a trace, which is representation-independent.

. Therefore we choose a representation such that at time t, $H(t)$ is diagonal and see what happens at $t+dt$.

We have: $H(t)|\psi_i\rangle = \lambda_i|\psi_i\rangle$ with the eigenvalues a.s. all different. The problem can be formulate as follows:

at time $t+dt$, we have an Hamiltonian $H(t+dt) = H(t) + \delta H$ with $\delta H = O(\sqrt{dt}) + O(dt)$, in any case with δH small.

$\Rightarrow$ Goal : find the perturbation of the eigenvalues.

. Result: (see below)
$$\lambda_i(t+dt) \overset{*}{=} \lambda_i(t) + \delta H_{ii}^{(0)} + \sum_{j\neq i}\sum_{\lambda=0}^{\beta-1} \frac{(\delta H_{ij}^{(\lambda)})^2}{\lambda_i - \lambda_j} + O((\delta H)^3)$$

Since $\delta H_{ii}^{(0)} = -\alpha.\lambda_i(t)dt + dB_{ii}$

and $(\delta H_{ij}^{(\lambda)})^2 = \frac{dt}{2} \quad \left(=\frac{1}{2}(dB_{ij}^{(\lambda)})^2\right)$.

. Therefore, $d\lambda_i = -\alpha\lambda_i dt + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}dt + dB_{ii}$.

. Now we just have to justify $*$:

Lemma: Let H a $N\times N$ matrix, V another $N\times N$ matrix. Suppose $H|\psi_i\rangle = \lambda_i|\psi_i\rangle$ with λ_i all distincts: $\{|\psi_i\rangle\}_{i=1}^N$ form an o.n. basis.
Then $(H+\varepsilon V)|\psi_i'\rangle = \lambda_i'|\psi_i'\rangle$, with
$$\lambda_i' = \lambda_i + \varepsilon\langle\psi_i, V\psi_i\rangle + \varepsilon^2 \sum_{j\neq i}\frac{|\langle\psi_i, V\psi_j\rangle|^2}{\lambda_i - \lambda_j} + O(\varepsilon^3)$$

Proof.: We use $\{|\psi_i\rangle, i=1,\dots,N\}$ as basis and write
$$\begin{cases} |\psi_i'\rangle = |\psi_i\rangle + \varepsilon\sum_j A_i^{\,j}|\psi_j\rangle + \varepsilon^2\sum_j B_i^{\,j}|\psi_j\rangle + O(\varepsilon^3), \text{ and} \\ \lambda_i' = \lambda_i + \varepsilon.a_i + \varepsilon^2.b_i + O(\varepsilon^3) \end{cases}$$

$\Rightarrow (H+\varepsilon V)|\psi_i'\rangle = \lambda_i'|\psi_i'\rangle$ becomes, order by order;

. $O(\varepsilon^0)$: $H|\psi_i\rangle = \lambda_i|\psi_i\rangle$: v.

. $O(\varepsilon^1)$: $V.|\psi_i\rangle = a_i|\psi_i\rangle + (\lambda_i - H)\sum_j A_i^{\,j}|\psi_j\rangle$,

multiplied by $\langle\psi_k| \Rightarrow \langle\psi_k, V\psi_i\rangle = \delta_{k,i}\,a_i + (\lambda_i - \lambda_k)\ A_i^k$

In particular, for $k=i \Rightarrow \underline{\langle \psi_i, V\psi_i\rangle = a_i}$

and for $k \neq i$: $\langle \psi_k, V\psi_i\rangle = (\lambda_i - \lambda_k) A_i^k$

. $\underline{\mathcal{O}(\varepsilon^2)}$: $V\sum_j A_i^j |\psi_j\rangle = b_i|\psi_i\rangle + a_i \sum_j A_i^j |\psi_j\rangle + (\lambda_i - H)\sum_j B_i^j |\psi_j\rangle$

multiplied by $\langle \psi_k|$

$$\Rightarrow \sum_j A_i^j \langle \psi_k, V\psi_j\rangle = b_i \cdot \delta_{k,i} + a_i \cdot A_i^k + (\lambda_i - \lambda_k) B_i^k$$

$$\Rightarrow \text{For } k=i: \quad b_i = \sum_j A_i^j \langle \psi_i, V\psi_j\rangle - a_i \cdot A_i^i$$

$$= \sum_{j\neq i} A_i^j \langle \psi_i, V\psi_j\rangle + \cancel{A_i^i \langle \psi_i, V\psi_i\rangle} - \cancel{\langle \psi_i, V\psi_i\rangle A_i^i}$$

$$= \sum_{j\neq i} \frac{\langle \psi_i, V\psi_j\rangle \cdot \langle \psi_j, V\psi_i\rangle}{\lambda_i - \lambda_j}. \quad \#$$

7.4.1) Multimatrix model and Airy process (for Hermitian case).

. Now we consider Dyson's Brownian Motion for $\beta = 2$ and $\alpha = \frac{\beta}{4N} = \frac{1}{2N}$. Let $H(t=0)$ be distributed according to the stationary measure $e^{-\frac{\mathrm{Tr}(H_0^2)}{2N}} dH_0$. Consider m times $0 < t_1 < t_2 < \dots < t_m$. Then, by (3), the measure on matrices at these times is given by

$$(5) \quad \frac{1}{Z_{N,m}} \cdot e^{-\frac{\mathrm{Tr}(H_0^2)}{2N}} \cdot \prod_{j=0}^{m-1} e^{-\frac{\mathrm{Tr}(H_{j+1} - q_j H_j)^2}{2N(1-q_j^2)}} dH_0 \cdots dH_m,$$

where $q_j \doteq \exp[-(t_{j+1} - t_j)/2N]$ and H_j is the matrix at time t_j.

. (5) can be rewritten as

$$(6) \quad \frac{1}{Z_{N,m}} \cdot e^{-\frac{\mathrm{Tr}(H_0^2)}{2N(1-q_0^2)}} \prod_{j=1}^{m-1} e^{-\frac{\mathrm{Tr}(H_j^2)}{2N}\left[\frac{1}{1-q_{j-1}^2} + \frac{q_j^2}{1-q_j^2}\right]} \cdot e^{-\frac{\mathrm{Tr}(H_m^2)}{2N(1-q_{m-1}^2)}}$$

$$\cdot \prod_{j=0}^{m-1} e^{\frac{q_j}{N(1-q_j^2)} \mathrm{Tr}(H_j \cdot H_{j+1})} dH_0 \dots dH_m.$$

. Denote by $\alpha_k \doteq \dfrac{1}{2N(1-q_k^2)}$, $\gamma_k \doteq \dfrac{1-q_{k-1}^2 q_k^2}{2N(1-q_{k-1}^2)(1-q_k^2)}$, and

$$\beta_k \doteq \frac{q_k}{N(1-q_k^2)}.$$

. Then, (6) writes

$$(7)\quad \frac{1}{Z_{N,m}} \cdot e^{-\alpha_0 \cdot \mathrm{Tr}(H_0^2)} \cdot \left(\prod_{k=1}^{m-1} e^{-\gamma_k \cdot \mathrm{Tr}(H_k^2)}\right) \cdot e^{-\alpha_{m-1} \cdot \mathrm{Tr}(H_m^2)}$$

$$\cdot \prod_{k=0}^{m-1} e^{\beta_k \cdot \mathrm{Tr}(H_k \cdot H_{k+1})} \, dH_0 \dots dH_m.$$

. As in the case of a single GUE matrix, if we are interested only in the eigenvalues, we have to integrate out the angular variables. We have seen that

$$dH_k = \Delta_N^2(\lambda_k) \cdot d\lambda_k \cdot dU_k \quad ,$$

where $\lambda_k = (\lambda_{k,1}, \dots, \lambda_{k,N})$, are the eigenvalues of H_k and $H_k = U_k \Lambda_k U_k^{-1}$ with $U_k \in U(N)$ and $\Lambda_k = \begin{pmatrix} \lambda_{k,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{k,N} \end{pmatrix}$.

. Replacing $H_k = U_k \Lambda_k U_k^{-1}$ in (7), the only terms in which the U_k do not disappear is the last product:

$$(8)\quad \prod_{k=0}^{m-1} e^{\beta_k \cdot \mathrm{Tr}(U_k \Lambda_k U_k^{-1} U_{k+1} \Lambda_{k+1} U_{k+1}^{-1})}.$$

Defining $V_k \doteq U_k^{-1} U_{k+1}$, we get $(8) = \prod_{k=0}^{m-1} e^{\beta_k \mathrm{Tr}(\Lambda_k V_k \Lambda_{k+1} V_k^{-1})}$.

The problem is to integrate over the unitary group $U(N)$ the expressions.

<u>Lemma</u>: [Harish-Chandra / Itzykson-Zuber formula]:

$$(9)\quad \int_{U(N)} dU \exp\left[\beta \cdot \mathrm{Tr}(\Lambda_1 U \Lambda_2 U^{-1})\right] = \frac{1}{\beta^{N(N-1)/2}} \left(\prod_{p=1}^{N-1} p!\right) \cdot \frac{\det\left[e^{\beta \lambda_{1,i} \cdot \lambda_{2,j}}\right]_{1 \le i,j \le N}}{\Delta_N(\lambda_1)\Delta_N(\lambda_2)}$$

if $\Lambda_1 = \begin{pmatrix} \lambda_{1,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{1,N} \end{pmatrix}$ and $\Lambda_2 = \begin{pmatrix} \lambda_{2,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2,N} \end{pmatrix}$.

. We are not going to prove this Lemma here. Look e.g. J. Math. Phys., 21 (1980), 411–421.

. Applying this Lemma, we get the following expression for the joint distribution of eigenvalues:

$$(10)\quad \begin{cases} P(\{\lambda_{k,i}\}_{k,i}) = \frac{1}{\tilde{Z}_{N,m}} \left(\prod_{i=1}^{N} e^{-\alpha_0 \cdot \lambda_{i,0}^2}\right) \cdot \left(\prod_{k=1}^{m-1} \prod_{i=1}^{N} e^{-\gamma_k \cdot \lambda_{k,i}^2}\right) \cdot \left(\prod_{i=1}^{N} e^{-\alpha_{m-1} \cdot \lambda_{m-1,i}^2}\right) \cdot \\ \cdot \Delta_N(\lambda_1) \cdot \left(\prod_{k=0}^{m-1} \det\left(e^{\beta_k \cdot \lambda_{k,i} \cdot \lambda_{k+1,j}}\right)_{1 \le i,j \le N}\right) \cdot \Delta_N(\lambda_m)\, d\lambda_0 \dots d\lambda_m \end{cases}$$

since the Vandermonde determinants coming from (9) are all canceled by the ones coming from the $dH_k = \Delta_N^2(\lambda_k)\, d\lambda\, dU_k$ except for one at the beginning and at the end.

. The measure (10) is of the form needed to apply theorem at page ④ of chapter 7.1). We do not do the computations here (analogue to the ones of the N non-intersecting Brownian Bridges). The final result is that the eigenvalues forms a space-time extended determinantal point process with Kernel

$$(11)\quad \begin{cases} K_N(t_1,x_1;t_2,x_2) = \begin{cases} \sum_{k=0}^{N-1} e^{-\frac{k(t_2-t_1)}{2N}} \cdot P_k(x) P_k(y) \cdot e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1 \le t_2, \\ -\sum_{k=N}^{\infty} e^{-\frac{k(t_2-t_1)}{2N}} \cdot P_k(x) P_k(y) \cdot e^{-\frac{x^2+y^2}{4N}}, & \text{for } t_1 > t_2, \end{cases} \\ \text{where } P_k(x) = \frac{1}{\sqrt[4]{2\pi N}\sqrt{2^k k!}} \cdot H_k\left(\frac{x}{\sqrt{2N}}\right), \text{ with } H_k \text{ the standard Hermite polynomials.} \end{cases}$$

. The kernel (11) is called <u>extended Hermite kernel</u>.

. Edge scaling: The edge scaling is the following. Let

$\lambda_i(t)$ be the i-th largest eigenvalue of the stationary solution of (4) for $\beta = 2$. Then, define

$$(12) \qquad \lambda_i^{\text{resc}}(s) \doteq \frac{\lambda_i(2s \cdot N^{2/3}) - 2N}{N^{1/3}} .$$

. Then one can prove that: $\lim_{N\to\infty} \lambda_1^{\text{resc}}(s) = \mathcal{A}_2(s)$, the Airy process.

7.5) The LGV Theorem ("discrete version" of Karlin-Mc Gregor).

. In a lot of applications, it is useful the following result on non-intersecting paths on directed graphs.

. Consider a graph (V, E) of vertices V and edges E:

. The edges are directed.

. A path π is a sequence of consecutive vertices joined by directed edges.

. Let $\mathcal{P}(u,v)$ denote the set of all paths starting from $u \in V$ and ending at $v \in V$.

. Two paths π and π' intersects if they have a common vertex.

. To every edge assign a weight $w(e)$, $e \in E$. Then, the weight of a path π is given by

$$w(\pi) \doteq \prod_{e \in \pi \cap E} w(e), \quad \text{and}$$

define the total weight of paths from u to v is defined by

$$h(u,v) \doteq \sum_{\pi \in \mathcal{P}(u,v)} w(\pi) .$$

. The final important condition on the graph is the following:
Given initial points $(u_1, \dots, u_m)$ and final points $(v_1, \dots, v_m)$, there exists (at most) a unique permutation $\sigma \in S_m$ s.t. we can connect u_i to $v_{\sigma(i)}$, $i = 1, \dots, m$, by a set of non-intersecting paths.

. We say that $(u_1, \dots, u_m)$ and $(v_1, \dots, v_m)$ are compatible if there exist a way of connecting them. Then we choose the numbering of the v_i's s.t. the above permutation is the identity.

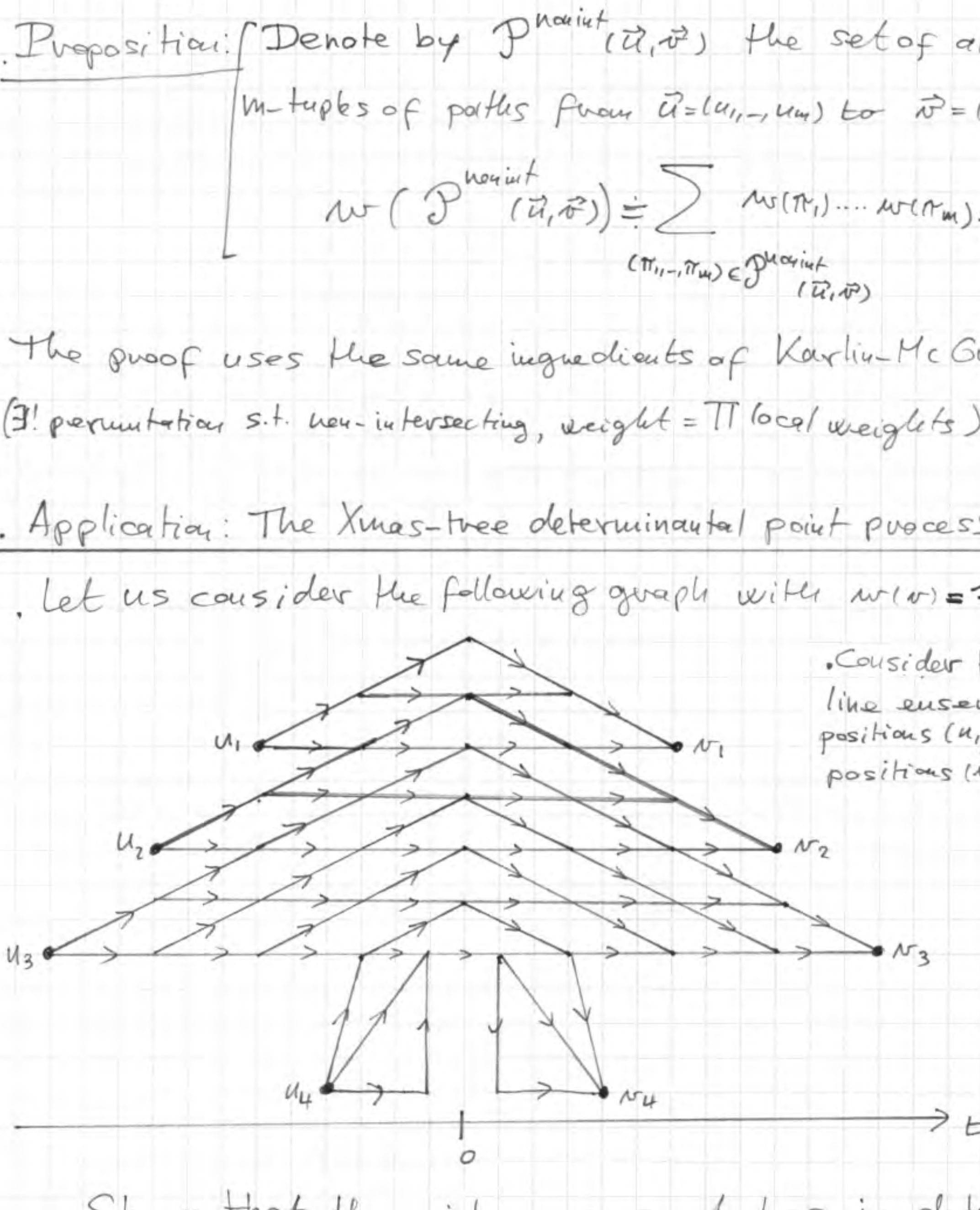
. Proposition: Denote by $\mathcal{P}^{\text{nonint}}(\vec{u}, \vec{v})$ the set of all non-intersecting m-tuples of paths from $\vec{u} = (u_1, \dots, u_m)$ to $\vec{v} = (v_1, \dots, v_m)$. Then,

$$w\left(\mathcal{P}^{\text{nonint}}(\vec{u}, \vec{v})\right) \doteq \sum_{(\pi_1, \dots, \pi_m) \in \mathcal{P}^{\text{nonint}}(\vec{u}, \vec{v})} w(\pi_1) \cdots w(\pi_m) = \det\left(h(u_i, v_j)\right)_{1 \le i,j \le m}.$$

. The proof uses the same ingredients of Karlin-McGregor theorem ($\exists!$ permutation s.t. non-intersecting, weight $= \prod$ local weights).

. Application: The Xmas-tree determinantal point process.

. Let us consider the following graph with $w(v) = 1$, $\forall$ vertex.



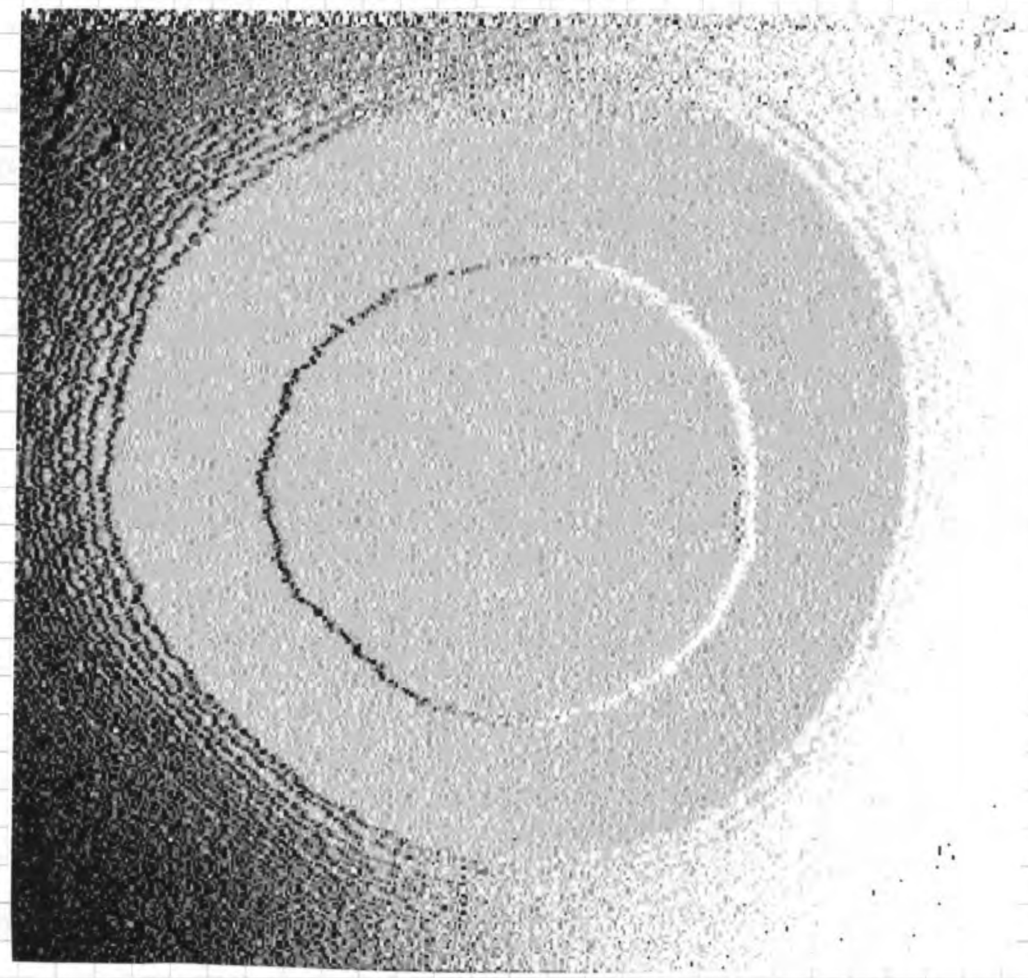
. Consider the non-intersecting line ensembles with initial positions (u_1, u_2, u_3, u_4) and final positions (v_1, v_2, v_3, v_4).

. Show that the point process at $t=0$ is determinantal and compute the correlation kernel. Have a nice Xmas!

8) Application to the "3D Ising corner at T=0".

. The model considered is a simplified model which describes a crystal at low temperature (with short-range interactions).

. A real image of a crystal at low (i.e., much below the melting and the roughening temperatures) is taken by using an electronic microscope:



. Observations: One sees that there is a facet (with an extra island inside) and then a rounded part.

. Goal: Describe the interface between the facet and the rounded part.

. We consider the following simplified model to answer to the question.

8.1) The model.

. At each $x \in \mathbb{Z}^d$, one has an occupation variable $n_x \in \{0, 1\}$.

. Let N_a be the total number of atoms forming the crystal.

. The interaction is nearest-neighbor with strength $-J$, for some $J > 0$, namely the Hamiltonian is:

$$H(\{n\}) = -J \cdot \sum_{|x-y|=1} n_x \cdot n_y .$$

. This is a ferromagnetic Ising interaction if we replace $n_x = \frac{1+\sigma_x}{2}$, σ_x the spin at site x.

. We want to consider the equilibrium (thermal) situation, therefore the appropriate measure is the Gibbs measure with temperature T:

$$\mu(\{n\}) = e^{-H(\{n\})/k_B T} .$$

. Of course, as stated above, μ is an infinite measure (by translation-invariance). Thus what one really have to consider are configurations up to translation, i.e., consider the equivalence classes of configurations.

. To model what happens at low temperature, we look the limiting case $T = 0$. How to extend the results in a mathematically rigorous way to $T > 0$ is an open problem.

. At $T=0$, only the configurations which minimize the energy are allowed. Here the number of atoms is fixed, thus the minimization of the energy $\equiv$ minimization of the surface area.

8.1.1) d=2 case.

. Although our goal is $d=3$, we start with $d=2$ to understand what happens.

. If $N_a = N^2$ for some $N \in \{1,2,3,\dots\}$, then there is a unique equilibrium configuration (class of).

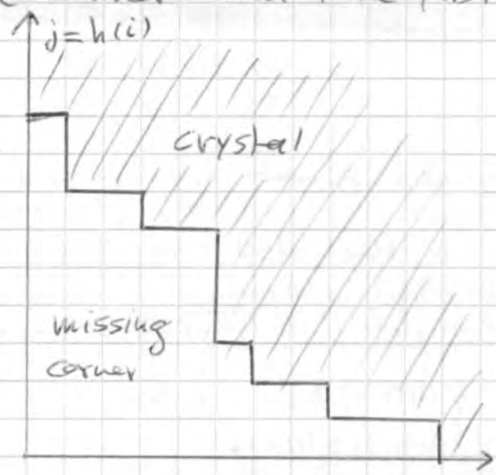
. If, however, $N_a = N^2 - V$, with $0 < V < N-1$, then the minimal energy configurations are non-unique: we need to take away V atoms from a perfect cube by keeping the surface area unchanged.

$\Downarrow$

. No overhangs!

. The further simplification can be made if $V \ll N^{1/2}$, so that we can consider the four corners independently.

Therefore, now we consider V to be the number of atoms missing from a single corner and take first $N \to \infty$ (i.e, $N_a \to \infty$).



. No overhangs

$\Downarrow$

. Crystal border described by a height function h:

$h(i) \geq 0$, $i \geq 0$ and decreasing: $h(i) \geq h(i+1)$ s.t.

$$\sum_{i \geq 0} h(i) = V.$$

Question: What is the limit form and the fluctuations in the limit $V \to \infty$?

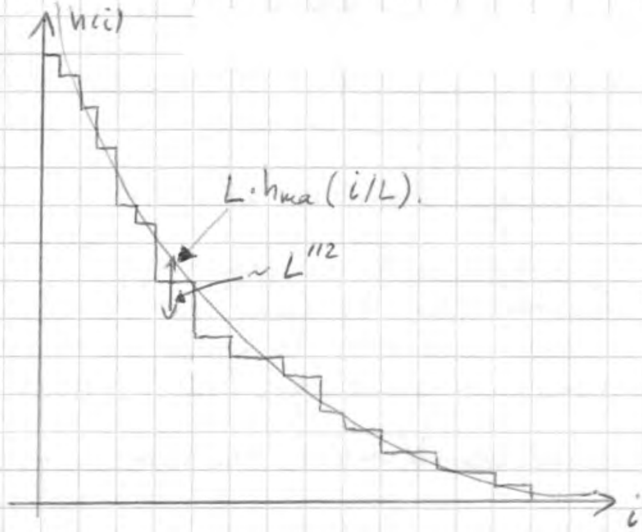
. Answer: Take $V = [L^2]$, so that L describes the linear scale in atomic units.

ⓐ Limit form: Macroscopically we have a deterministic shape: $\lim_{L\to\infty} \frac{1}{L} h([L\cdot u]) = h_{ma}(u)$, $u>0$

almost surely ($\exists$ large deviations estimates); with

$$e^{-h_{ma}(u)} + e^{-u} = 1.$$

ⓑ Fluctuations: Gaussian fluctuations on $L^{1/2}$ scale.

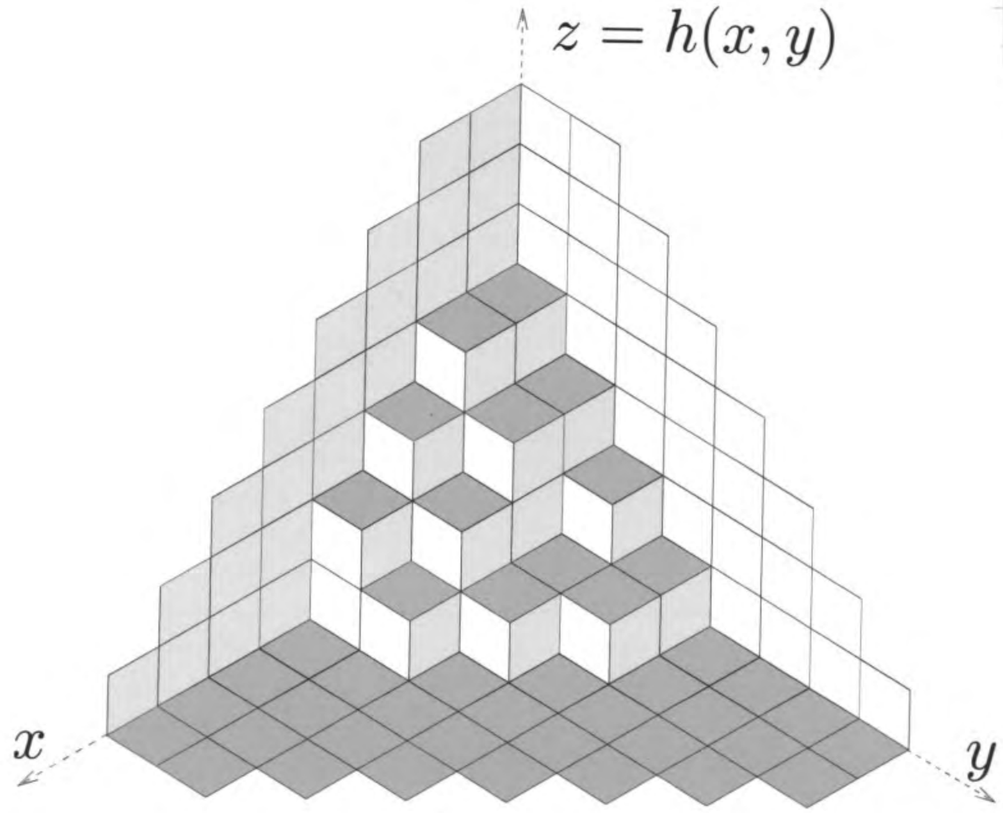


. 8.1.2) $d=3$ case.

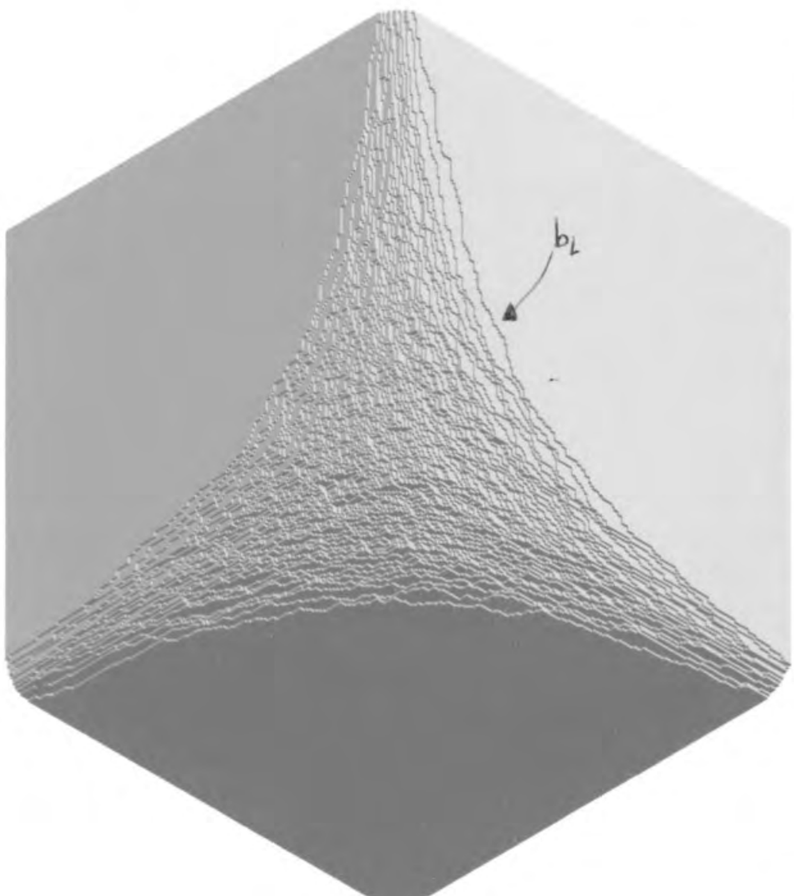
. The same arguments used in the $d=2$ case carry over in $d=3$. This time we have a height function, decreasing in both variables:

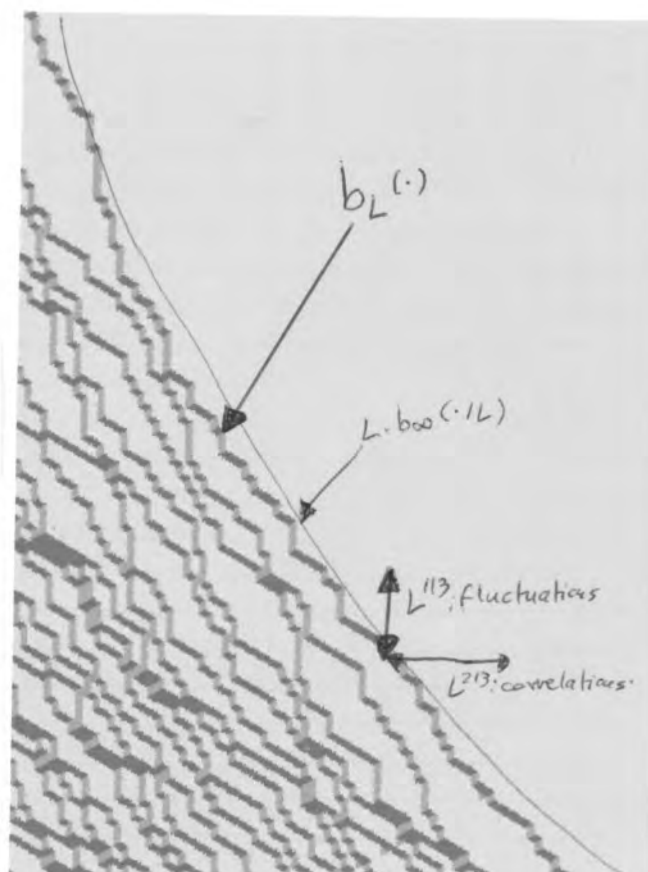
$$(x,y) \longmapsto z = h(x,y) \quad \text{with} \quad \begin{cases} h(x,y) \geq 0 & , x,y \geq 0 \\ h(x,y) \geq h(x,y+1) \\ h(x,y) \geq h(x+1,y) \end{cases}$$

with $V = \sum_{x,y\geq 0} h(x,y)$ fixed.



. Observable: Facet border $b_L(y) \equiv h(0,y)$.





(a) Limit shape: Let $V = [L^3]$. Then Cerf and Kenyon [2001] prove that

$\exists$ deterministic (non-random) limit shape h_∞:

$$\lim_{L\to\infty} \frac{1}{L} \cdot h([Lu],[Lv]) = h_\infty(u,v) \quad , \text{ a.s.}$$

In particular, the limit shape of the border is:

$$b_\infty(\tau) \doteq \lim_{L\to\infty} \frac{1}{L} b_L([\tau L]) = -2 \cdot (1 - e^{-\tau/2}).$$

$\Rightarrow$ For the fluctuations we focus, for $\tau > 0$ fixed,

around $\begin{cases} x = 0 , \\ y = \tau L , \\ z = b_\infty(\tau) \cdot L . \end{cases}$

(b) Fluctuations: results [Ferrari, Spohn 2003]

(1) . Fluctuations live on a $L^{1/3}$ scale.

(2) . Spatial correlations live on a $L^{2/3}$ scale.

(3) . The limit process is the Airy process, $\mathcal{A}$.

. Rescaling of the edge: $$b_L^{edge}(s) = \frac{b_L([\tau L + s L^{2/3}]) - L \cdot b_\infty(\tau + s \cdot L^{-1/3})}{L^{1/3}}.$$

. Then, $\lim_{L\to\infty} b_L^{edge}(s) = \mathcal{A}(s \cdot \frac{\varkappa}{2}) \cdot \varkappa^{-1}$, $\varkappa = b_\infty''(\tau)$, in the sense of finite-dimensional distributions, i.e.,

for any $m \in \mathbb{N}$, $s_i, a_i \in \mathbb{R}$,

$$\lim_{L\to\infty} \mathbb{P}\left(\bigcap_{i=1}^{m} \{ b_L^{edge}(s_i) \le a_i \}\right) = \mathbb{P}\left(\bigcap_{i=1}^{m} \{ \mathcal{A}(s_i \tfrac{\varkappa}{2}) \le \tfrac{a_i}{\varkappa} \}\right).$$

8.2) How to get the d=3 results.

8.2.1) Canonical → Gran canonical ensemble.

- The fix missing volume V condition (canonical ensemble) is as usual difficult to implement since it is a global condition.

- Since we are interested in the $V \to \infty$ limit, it is more convenient to use the gran-canonical ensemble, where V instead of being fixed, is a random variable geometrically distributed with mean $\bar{V} \propto L^3$:

$$\underline{V(h) = [L^3]} \ \underset{\text{by}}{\overset{\text{replaced}}{\Longrightarrow}} \ \underline{\exp\left(-\frac{V(h)}{L}\right) \doteq q^{V(h)}} \text{ with } q = e^{-\frac{\lambda}{L}}.$$

with λ chosen s.t. $\bar{V} \equiv \mathbb{E}(V(h)) = L^3$.

Q: Is this step justified?

A: It depends on the observable one is interested in. For example, if one is interested in the fluctuations of the volume, it is obviously not true that the two descriptions agree in the $L \to \infty$ (thermodynamic) limit.

- We, however, consider an observable (the facet border) which fluctuates as $L^{1/3}$ in the canonical ensemble and in the gran-canonical one will need to average it over volumes around $\bar{V} \sim L^3$ and fluctuation on the scale $\bar{V}^{1/2} \sim L^{3/2}$.

- So, let X_L our observable for fixed L, s.t. $\frac{X_L - \alpha \cdot L}{L^{1/3}} \xrightarrow{L \to \infty} \xi$.

- Then, in the gran-canonical ensemble, we have $X_L^{g.c.} = \mathbb{E}_L^{g.c.}(X_L)$.

So, Since $X_L \cong \alpha \cdot L + L^{1/3} \cdot \xi = \alpha \cdot \bar{V}^{1/3} + \xi \cdot \bar{V}^{1/9}$,

$$X_L^{g.c.} \cong \alpha \cdot \bar{V}^{1/3} \cdot (1 + \bar{V}^{-1/2})^{1/3} + \xi \cdot \bar{V}^{1/9} (1 + \bar{V}^{-1/2})^{1/9}$$

$$\cong \alpha \cdot \bar{V}^{1/3} + \frac{1}{3} \cdot \alpha \cdot \bar{V}^{-1/6} + \xi \cdot \bar{V}^{1/9} = \alpha L + \xi L^{1/3} + \mathcal{O}(L^{-1/2}).$$

Therefore, we will also have $\frac{X_L^{g.c.} - \alpha L}{L^{1/3}} \xrightarrow{L \to \infty} \xi$.

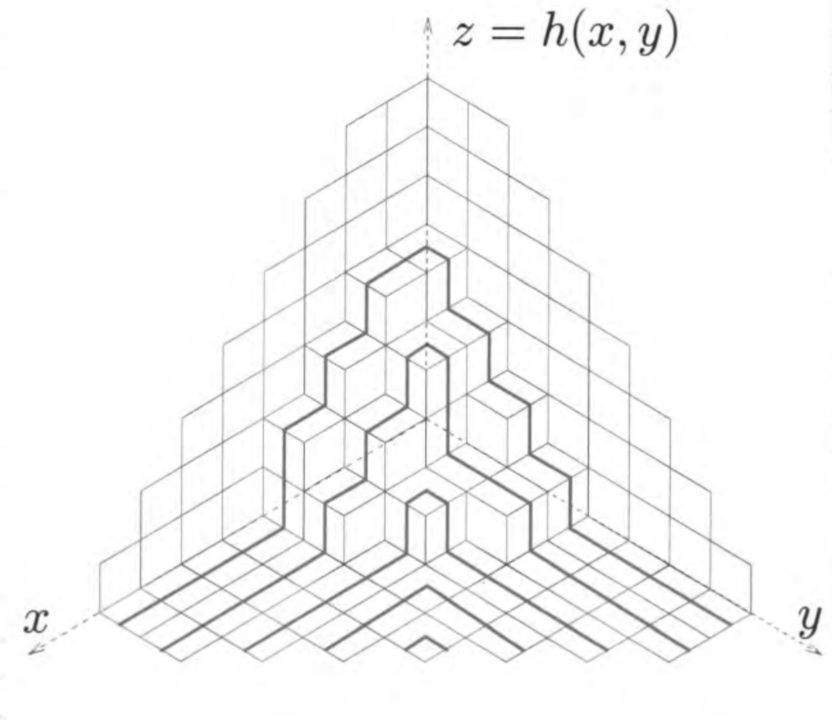
- Therefore, the fluctuations of the volume do not have any effect in the $L \to \infty$ limit to the fluctuations of our observable, and the replacement of the canonical ensemble by the gran-canonical one is fine.

Rem.: If we would have an observable scaling as $Y = \bar{V} + \zeta \cdot \bar{V}^{1/3}$, then the two ensembles would not be aequivalent, since in the gran-canonical ensemble one would have $Y^{g.c.} = \bar{V} + \zeta \cdot \bar{V}^{1/3} + \underbrace{O(\bar{V}^{1/2})}_{\text{Volume fluctuations } \gg \bar{V}^{1/3}}$.

8.2.2) Mapping to non-intersecting line ensembles.

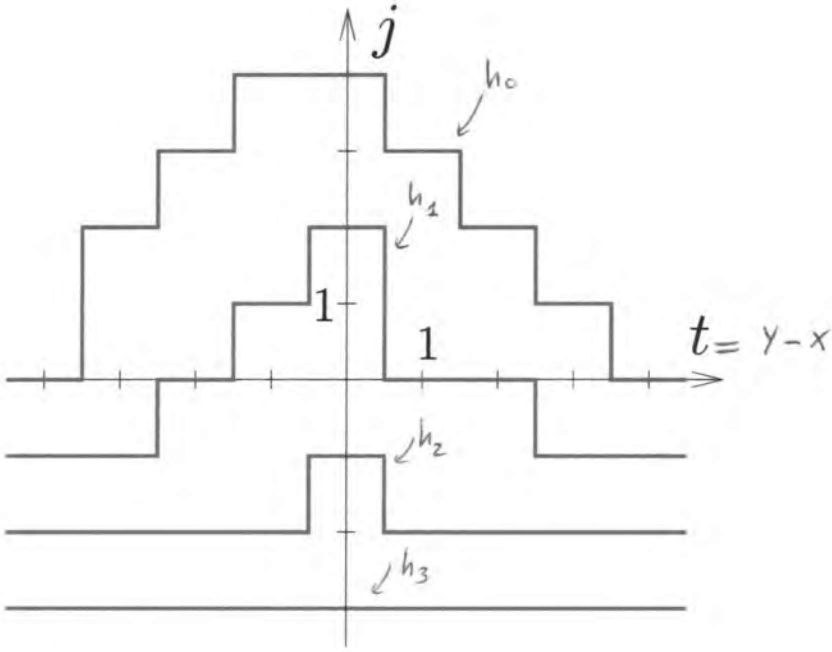
- Consider now the gran-canonical ensemble, i.e., give an extra weight $q^{V(h)}$ to the zero-temperature configurations.

- We first consider the gradient lines as in figure below:



- The next step is just a geometric transformation changing the directions of the x- and y-axis to $(-1,0)$ and $(1,0)$ respectively:

. We get, in the case of the above example:



. The <u>top line</u> is the border of the facet.

. We introduce names of the lines: $h_e(t)$, $\underline{t\in\mathbb{Z}}$, $e\geq 0$. They satisfy the non-intersecting conditions:

ⓐ $\begin{cases} h_e \text{ is increasing for } t\leq 0, \text{ decreasing for } t\geq 0; \\ h_e(t)\leq h_e(t+1),\quad t<0,\quad h_e(t)\geq h_e(t+1),\quad t\geq 0, \end{cases}$

have limits: ⓑ $\lim_{t\to\pm\infty} h_e(t) = -e,$

and are non-∩: ⓒ $h_{e+1}(t) < h_e(t-1),\ t\leq 0\ ;\ h_{e+1}(t) < h_e(t+1),\ t\geq 0.$

<u>. Measure on lines: Local!</u>

. In the above picture, we already extended the height functions $h_e(t)$ from $t\in\mathbb{Z}$ to $t\in\mathbb{R}$, with jumps at $\mathbb{Z}+1/2$.

. For a given line h_e, let $t_{e,1}<\dots<t_{e,k(e)}<0$ be the left jump times of height $s_{e,1},\dots,s_{e,k(e)}$, and $0<t_{e,k(e)+1}<\dots<t_{e,k(e)+n(e)}$ be the right jump times of height $-s_{e,k(e)+1},\dots,-s_{e,k(e)+n(e)}$. Then,

$$\underline{V(h) = \sum_{e\geq 0}\sum_{j=1}^{k(e)+n(e)} s_{e,j}\cdot|t_{e,j}|}$$

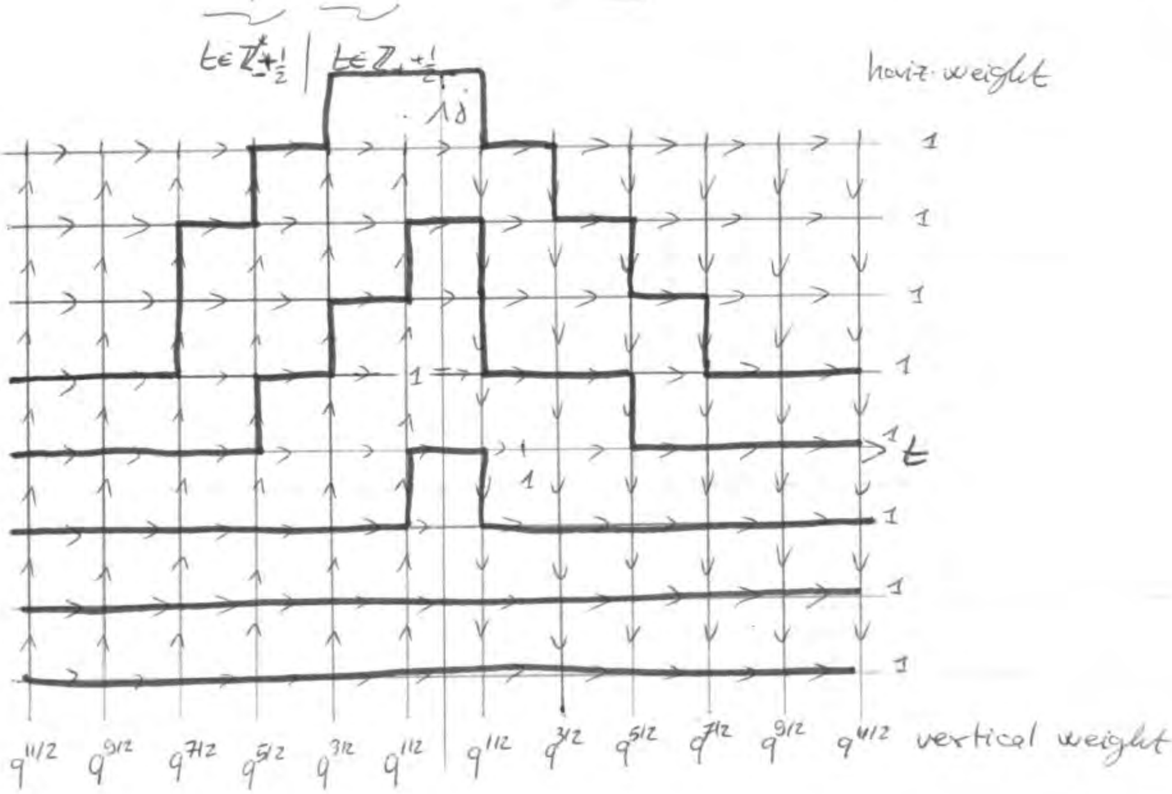
. This means that the gran-canonical weight $q^{V(h)}$ can be encoded in the jumps of the line ensembles: in this description it is a local weight!

. 8.2.3) Non-intersecting line ensemble and LGV graph.

. In our original work ("Step fluctuations for a faceted crystal" in J.S.P.) at this point we used the fermionic approach with notations usual for physicists but not well known by mathematicians. Here I present it in another way, by associating an LGV graph and then applying the theorem we learned.

. LGV graph: $\rightarrow$: weight 1

$\uparrow$, $\downarrow$: weight $q_t = q^{|t|}$, with $|t|$ = distance to the $t=0$ axis.



. This is the LGV graph with the weight indicated at the boundaries. It is inhomogeneous in the t-variable.

. We plotted the lines of the above example too.

. To apply the LGV theorem, we need to start with some fixed initial and final positions and with a finite number of particles, N.

. Starting points : $\{Y_e = (-M, -e)\}_{e\geq 0}^{N-1}$,

. Final points : $\{X_e = (M, -e)\}_{e\geq 0}^{N-1}$.

. Let us now consider m values of t in $\mathbb{Z}$, say $-M < t_1 < t_2 < \dots < t_m < M$.

. The associate point process (extended)

$$\eta(t, j) = \begin{cases} 1, & \text{a line crosses } (t, j), \\ 0, & \text{no lines cross } (t, j), \end{cases}$$

is, by the LGV theorem, determinantal [see Lectures, sect 7.5].

. In fact, the measure on $\mathbb{Z} \times (t_1, \dots, t_m)$ writes

$$(*) \quad \det\left(T((-M, -i) \to (t_1, h_j(t_1)))\right)_{0\leq i,j\leq N-1} \cdot \prod_{k=1}^{m-1} \det\left(T((t_k, h_i(t_k)) \to (t_{k+1}, h_j(t_{k+1})))\right)_{0\leq i,j\leq N-1} \cdot$$

$$\cdot \det\left(T((t_m, h_i(t_m)) \to (M, -j))\right)_{0\leq i,j\leq N-1}.$$

where T is the transition probability of a single line on the LGV graph.

. Then, the measure $(*)$ is of the form so that Theorem in sect. 7.1, page 4, can be applied.

. <u>Remark</u> : . Below we will write explicitly the transition probability. Since $q<1$, the $M\to\infty$ limit is straightforward. A little bit more care will be needed to control the $N\to\infty$ limit. We will explain what happens without entering in the technical details of the $N\to\infty$ limit (which can be found in the references we will give).

. Integral representation for the transition probability.

. It is convenient to use the Fourier representation or its contour integral analogue since on the vertical axis we have translation-invariance.

Case $t<0$: $T((t,x_1)\to(t+1,x_2)) = q_{t+\frac{1}{2}}^{x_2-x_1}\cdot \mathbb{1}_{[x_2\geq x_1]}$) can be rewritten as

$$= \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{x_2-x_1+1}}\cdot\frac{1}{(1-q_{t+\frac{1}{2}}\cdot w)},$$

where Γ_0 is any simple loop encircling $w=0$ and no other poles of the integrand and anticlockwise oriented.

. To verify it, just use: $\displaystyle \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x+1}}\cdot\frac{1}{1-qw} \underset{|w|<\frac{1}{q}}{=} \sum_{n\geq 0}\underbrace{\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x+1}}\cdot q^n w^n}_{=\delta_{n,x}\cdot q^x}.$

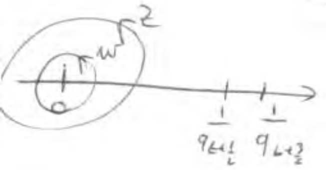
. Two-steps for $t<-1$:

$$T((t,x_1)\to(t+2,x_2)) = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_2-x_1+1}}\cdot\frac{1}{(1-q_{t+\frac{1}{2}}w)(1-q_{t+\frac{3}{2}}w)}.$$

. In fact, $\displaystyle T((t,x_1)\to(t+1,x_2)) = \sum_{y\geq x_1} T((t,x_1)\to(t+1,y))\cdot T((t+1,y)\to(t+2,x_2)) =$

$$= \sum_{y\geq x_1}\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}dz\oint_{\Gamma_0}dw\,\frac{1}{w^{x_2-y+1}}\cdot\frac{1}{(1-q_{t+\frac{3}{2}}w)}\cdot\frac{1}{z^{y-x_1+1}}\cdot\frac{1}{(1-q_{t+\frac{1}{2}}z)}$$

$$\overset{|z|>|w|}{=} \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0}dz\oint_{\Gamma_0}}_{|w|<|z|}dw\,\frac{1}{(1-q_{t+\frac{3}{2}}w)}\,\frac{1}{(1-q_{t+\frac{1}{2}}z)}\cdot\frac{1}{w^{x_2-x_1+1}}\cdot\underbrace{\sum_{y\geq x_1}\left(\frac{w}{z}\right)^{y-x_1}\cdot\frac{1}{z}}_{=\frac{1}{z-w}}$$



$$= \frac{1}{2\pi i}\oint_{\Gamma_0}dw\,\frac{1}{(1-q_{t+\frac{3}{2}}w)}\cdot\frac{1}{(1-q_{t+\frac{1}{2}}w)}\cdot\frac{1}{w^{x_2-x_1+1}}. \quad \#$$

pole at $z=0$ vanished, but now simple pole at $z=w \Rightarrow$ residue

. General case is analogue, as well the case $t \geq 0$ by symmetry.

. For $-\infty \leq t_1 < t_2 \leq 0$:

$$T\big((t_1,x_1)\to(t_2,x_2)\big) = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{x_2-x_1+1}} \prod_{t=t_1}^{t_2-1} \frac{1}{(1-q_{t+\frac{1}{2}} w)}$$

. For $0 \leq t_1 < t_2 \leq \infty$:

$$T\big((t_1,x_1)\to(t_2,x_2)\big) = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{x_1-x_2+1}} \prod_{t=t_1}^{t_2-1} \frac{1}{(1-q_{t+\frac{1}{2}} w)}$$

. From these formulas is easy to see that $M\to\infty$ limit is immediate.

. In the case of N particles, define $A_N = [A_N]_{0\leq i,j\leq N-1}$ given by

$$[A_N]_{i,j} = T\big((-\infty, i)\to(\infty,j)\big) = \sum_{z\in\mathbb{Z}} T\big((-\infty, i)\to(0,z)\big)\cdot T\big((0,z)\to(\infty, j)\big).$$

Then, by the theorem of Sect. 7.1, we get that the kernel of our determinantal point process is given by:

$$K_N\big((t_1,x_1);(t_2,x_2)\big) = -T\big((t_1,x_1)\to(t_2,x_2)\big)\cdot \mathbb{1}_{[t_1<t_2]} + \sum_{i,j=-N+1}^{0} T\big((t_2,x_2)\to(\infty, i)\big)\cdot [A_N^{-1}]_{i,j}\cdot T\big((-\infty, j)\to(t_1,x_1)\big)$$

. The $N\to\infty$ limit: A_N is a $N\times N$ matrix, a minor of a Toeplitz matrix, i.e., $[A_N]_{i,j}$ depends only on $j-i$. In our case is even symmetric. The inverse of infinite-dimensional(*) Toeplitz matrices are "easy" to compute. Moreover, the transitions T in the kernel K_N depending on i or j goes to zero exponentially fast as $i,j\to -\infty$. This is the

(*) $i,j\leq 0$ instead of $i,j\in[-N+1,0]$.

consequence that the probability that the line l_e is not straight goes to ~~zero~~ exponentially fast (in e) as $e \to \infty$.

. In our case, one can also get that the entries of A_N go rapidly (exponentially) to zero, in the distance to the diagonal: $|[A_N]_{i,j}| \le C_1 \cdot e^{-C_2 \cdot |j-i|}$ for some $C_1, C_2 > 0$.

- The matrix A_N is then "almost" like a band matrix and the $N \to \infty$ limit is unproblematic. The details of the justification are like in Section 2 of "arXiv math/0006097" by Eric Rains ("Correlation functions for symmetrized increasing subsequences").

. Denote by A_∞ the half-infinite matrix, limit of A_N as $N \to \infty$.

Inverse of the matrix for $N = \infty$.

. Denote by $(T_+)_{x,y} = T((-\infty, x) \to (0, y))$ and $(T_-)_{x,y} = T((0,x) \to (0,y))$

Then, let A be the infinite matrix, $A = [A_{xy}]_{x,y \in \mathbb{Z}}$, which is the product of T_+ and T_-: $\quad A = T_+ \cdot T_-$.

- First remark that $T_+ \cdot T_- = T_- \cdot T_+$ since $A = A^t$ is a Toeplitz matrix.

. Moreover, $[A_\infty]_{x,y} = [A]_{x,y}$, $x, y \le 0$.

. The matrices $T_\pm$ are triangular, $T_+ = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$, $T_- = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$.

. Denote by $P_+ =$ projection on $\{1, 2, \dots\}$,
$P_- =$ " " $\{\dots, -1, 0\}$ and decompose $\ell^2 = P_- \ell^2 \oplus P_+ \ell^2$.

Then, the bloc representations of $T_\pm$ are: $T_+ = \begin{pmatrix} a' & c' \\ 0 & b' \end{pmatrix}$, $T_- = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$,

so that $A = \begin{pmatrix} aa' & * \\ * & * \end{pmatrix}$. <u>We need to compute $(aa')^{-1} = (a')^{-1} \cdot (a)^{-1}$.</u>

. <u>a) Inverse of a:</u> $\quad a_{x,y} = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{w^{x-y+1} g(w)}$, $\quad g(w) = \prod_{t=0}^{\infty} (1 - q_{t+\frac{1}{2}} w)$

Then, $\quad a^{-1}_{x,y} = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dz}{z^{x-y+1}} \cdot g(z)$.

. In fact: $\sum_{y\le 0} a_{x,y}\, a^{-1}_{y,x'} = \sum_{y\le x} a_{x,y}\, a^{-1}_{y,x'}$

$$= \sum_{y\le x} \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} \frac{dw}{w^{x-y+1}} \oint_{\Gamma_0} \frac{dz}{z^{y-x'+1}} \cdot \frac{g(z)}{g(w)}$$

$$= \frac{1}{(2\pi i)^2} \underbrace{\oint_{\Gamma_0} \frac{dw}{g(w)\, w} \oint_{\Gamma_0} \frac{dz\, g(z)}{z^{x-x'+1}}}_{|z|<|w|} \cdot \underbrace{\sum_{y\le x} \frac{z^{x-y}}{w^{x-y}}}_{=\frac{w}{w-z}}$$

$$= \underbrace{\underbrace{\oint dz \oint dw}_{|z|>|w|}}_{=0,\ \text{since no poles at } w=0} + \int (\text{Residue at } w=z)\, dz = \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{x-x'+1}} = \delta_{x,x'}.$$

. Similarly, one verifies $\sum_{y\le 0} a^{-1}_{x,y}\cdot a_{y,x'} = \delta_{x,x'}$.

b) Inverse of a': $\quad a'_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{y-x+1}} \cdot \frac{1}{\tilde g(w)}$, with $\tilde g(w) = \prod_{t=-\infty}^{-1} (1-q_{t+\frac12} w)$

and $(a')^{-1}_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dz}{z^{y-x+1}} \cdot \tilde g(z)$.

. Finite size Kernel.

. Therefore, the Kernel of our determinantal point process is

$$K((t_1,x_1);(t_2,x_2)) = -T((t_1,x_1)\to(t_2,x_2))\, \mathbb{1}_{[t_1<t_2]}$$
$$+ \sum_{x,y\le 0} T((t_2,x_2)\to(\infty,x)) \sum_{\ell\le 0} (a')^{-1}_{x,\ell}\, (a^{-1})_{\ell,y} \cdot T((-\infty,y)\to(t_1,x_1)).$$

↳ replaced by $u\in\mathbb{Z}$, since $x\le 0$ and $(a')^{-1}_{x,u}=0$ if $u>x$.

. To our purpose we need only $t_1,t_2\ge 0$. In this case,

$$T((t_2,x_2)\to(\infty,x)) = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{x_2-x+1}} \cdot \prod_{t=t_2}^{\infty} \frac{1}{(1-q_{t+\frac12} w)}$$

and $$T((-\infty,y)\to(t_1,x_1)) = \sum_{v\in\mathbb{Z}} (a')_{y,v} \left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{v-x_1+1}}\cdot\prod_{t=0}^{t_1-1}\frac{1}{(1-q_{t+\frac12}z)}\right)$$

$v \doteq y+x_1-\tilde v$

$$= \sum_{\tilde v\in\mathbb{Z}} \left(\frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{x_1-\tilde v+1}}\,\frac{1}{\tilde g(w)}\right)\cdot\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{y-\tilde v+1}}\cdot\prod_{t=0}^{t_1-1}\frac{1}{(1-q_{t+\frac12}z)}\right)$$

($\tilde v \le y$ is enough)

· Putting all together we get:

$$K((t_1,x_1),(t_2,x_2)) = -T((t_1,x_1)\to(t_2,x_2))\,\mathbb{1}_{[t_1< t_2]} + \bar{K}((t_1,x_1),(t_2,x_2)),$$

where

$$\bar{K}((t_1,x_1),(t_2,x_2)) = \sum_{\ell\le 0}\left.\begin{array}{l}\left(\sum_{x\le 0}\frac{1}{2\pi i}\oint_{\Gamma_c}\frac{dw}{w^{x_2-x+1}}\cdot\prod_{t=t_2}^{\infty}\frac{1}{(1-q_{t+\frac12}w)}\right.\\ \left.\cdot\frac{1}{2\pi i}\oint_{\Gamma_c}\frac{dz}{z^{\ell-x+1}}\cdot\prod_{t=-\infty}^{-1}(1-q_{t+\frac12}z)\right).\end{array}\right\}\ (\alpha)$$

$$\left.\begin{array}{l}\cdot\left(\sum_{y\le 0}\sum_{v(\le y)}\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_1-v+1}}\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\right.\\ \cdot\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{y-v+1}}\prod_{t=0}^{t_1-1}\frac{1}{(1-q_{t+\frac12}z)}\\ \left.\cdot\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{d\tilde z}{\tilde z^{\ell-y+1}}\prod_{t=0}^{\infty}(1-q_{t+\frac12}\tilde z)\right)\end{array}\right\}\ (\beta)$$

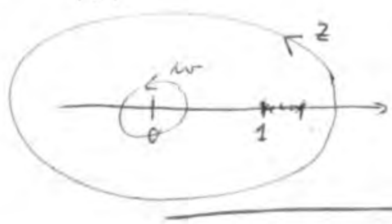
· First we do a simplification of (β): Since $\ell\le 0$, we can replace the double sum by $\sum_{v\le 0}\sum_{y\ge v}$ (the restriction $y\le 0$ is not needed, the last term being exactly zero). Then we perform the sum over $y\ge v$ of the last two terms and obtain finally:

$$\boxed{(\beta)} = \sum_{v\le 0}\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w^{x_1-v+1}}\cdot\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\right)\left(\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z^{\ell-v+1}}\cdot\prod_{t=t_1}^{\infty}(1-q_{t+\frac12}z)\right).$$

$$= \sum_{v\le \ell}\left(\quad '' \quad\right)\cdot\left(\quad '' \quad\right)$$

$u \doteq \ell - v$

$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_c}dw\oint_{\Gamma_0}dz}_{|z|>\frac{1}{|w|}}\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\cdot\prod_{t=t_1}^{\infty}(1-q_{t+\frac12}z)\cdot\underbrace{\sum_{u\ge 0}\frac{(wz)^{\ell}}{(wz)^u\cdot w^{x_1+1}z^{\ell+1}}}_{=\frac{1}{w^{x_1-\ell+1}(z-\frac1w)}}$$



$$= \underbrace{\oint_{\Gamma_c}dw\oint_{\Gamma_0}dz}_{|z|<\frac{1}{|w|}} + \boxed{\frac{1}{2\pi i}\oint_{\Gamma_c}dw\prod_{t=-\infty}^{-1}\frac{1}{(1-q_{t+\frac12}w)}\prod_{t=t_1}^{\infty}\left(1-q_{t+\frac12}\cdot\frac1w\right)\cdot\frac{1}{w^{x_1-\ell+1}}}$$

$\to 0$, since no pole at $z=0$ anymore

Residue at $z=\frac1w$

. Similarly,

$$(\alpha) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \prod_{t=t_2}^{\infty} \frac{1}{(1-q_{t+\frac{1}{2}} w)} \cdot \prod_{t=-\infty}^{-1} (1 - q_{t+\frac{1}{2}} \tfrac{1}{w}) \cdot \frac{1}{w^{x_2-e+1}}$$

. It is also possible to check that $\sum_{e\in\mathbb{Z}} (\alpha)(\beta) = T((t_1,x_1)\to(t_2,x_2))$.

. Therefore, the final formula for the Kernel writes:

$$K((t_1,x_1),(t_2,x_2)) = \begin{cases} \sum_{e\leq 0} \left[\frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{\prod_{t=-\infty}^{-1}(1-q_{t+\frac12}\frac{1}{w})}{\prod_{t=t_2}^{\infty}(1-q_{t+\frac12}\cdot w)} \cdot \frac{1}{w^{x_2-e+1}}\right] \\ \quad \cdot \left[\frac{1}{2\pi i}\oint_{\Gamma_0} dz \frac{\prod_{t=t_1}^{\infty}(1-q_{t+\frac12}\frac{1}{z})}{\prod_{t=-\infty}^{-1}(1-q_{t+\frac12} z)} \cdot \frac{1}{z^{x_1-e+1}}\right], & \text{for } t_1 \geq t_2, \\ -\sum_{e>0} [\;''\;]\cdot[\;''\;], & \text{for } t_1 < t_2. \end{cases}$$

. Scaling limit.

What remains to be done is the asymptotic analysis in the scaling limit:

$$\begin{cases} t_i = [\tau L + s_i L^{2/3}] \\ x_i = [b_\infty(\tau)\cdot L + b_\infty'(\tau)\cdot s_i L^{2/3} + \frac12 b_\infty''(\tau)\cdot s_i^2 L^{1/3} + r_i L^{1/3}], \end{cases}$$

i.e., the rescaled point process is

$$\eta_L^{edge}(r,s) \doteq L^{1/3}\cdot \eta_L(x,t).$$

Similarly, the rescaled kernel is:

$$K_L^{edge}((r_1,s_1),(r_2,s_2)) \underset{\substack{\uparrow \\ \text{up to} \\ \text{a conjugation}}}{\cong} L^{1/3}\cdot K((t_1,x_1),(t_2,x_2)).$$

. In "Step fluctuations for a faceted crystal", J. Stat. phys. 113.(2003), 1-46, we prove that $\lim_{L\to\infty} K_L^{edge} = K_{Airy}$. The convergence is such that, it follows the finite-dimensional distribution of the rescaled process to the Airy process.

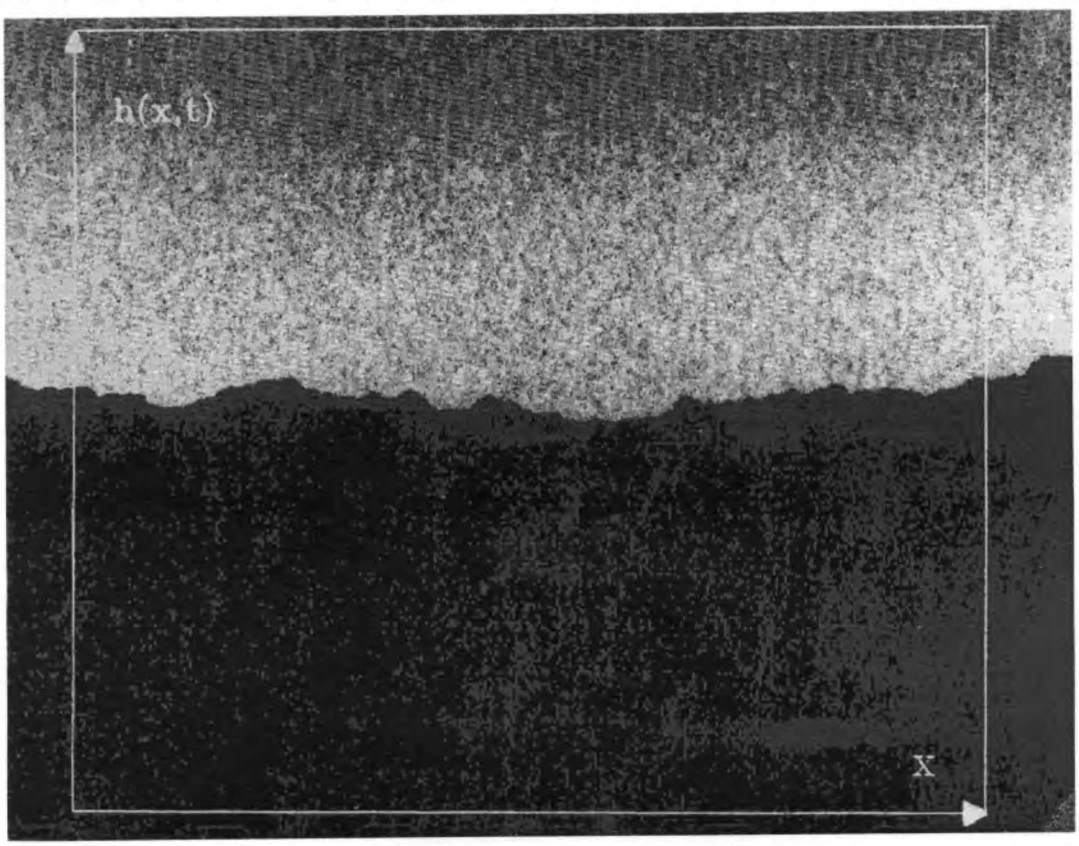
9) Application to the "PNG droplet".

. With "PNG droplet" we mean a corner growth model, which we will define later. First a few words on the class of model to which it belongs.

9.1) Generalities, KPZ class.

. There are a lot of different kind of growth processes. For example, a crystal can grow due to atomic deposition, or when a porous medium is put in contact to some liquid, then the growing quantity is the wetted region.

. As illustration, below there is a piece of paper burned from below (black region). The interface is clearly visible.



[From Barabasi-Stanley book "Fractal Concepts in Surface Growth"]

. On a macroscopic scale the interface is roughly flat, but we would like to say something about its roughness.

. Universality picture: The statistical properties of the interface, for large growth time t, should depend only on a few global properties of the dynamics like:

. substrate dimension,

. locality of growth,

. symmetries,

. conservation laws.

. The model we consider belongs to the Kardar-Parisi-Zhang (KPZ) universality class in 1+1 dimension. Kardar, Parisi, and Zhang wrote down a macroscopic equation which describes a stochastically growing interface $x \mapsto h(x,t)$:

$$\frac{\partial h(x,t)}{\partial t} = \nu \cdot \Delta h(x,t) + \frac{1}{2}\lambda \cdot (\nabla h(x,t))^2 + \eta(x,t),$$

where : . $\nu \Delta h$ (with $\nu > 0$) is smoothening (surface tension)

. $\lambda > 0$: lateral growth

. η : space-time uncorrelated white noise.

. This equation is the simplest continuous equation for an irreversible, local, non-linear random growth.

. The smoothening makes the surface "macroscopically deterministic", i.e.,

$$\lim_{t\to\infty} \frac{h(\xi t, t)}{t} = h_{ma}(\xi) \text{ is non-random.}$$

. Thus, we focus at the fluctuations:

$$H(x,t) = h(x,t) - t \cdot h_{ma}(x/t).$$

. Question: On which scale we have to focus to see non-trivial fluctuations / correlations ?

. Fluctuation exponent: α s.t. $H(x,t) \simeq t^{\alpha}$

. Correlation exponent: β s.t. in order to have $|H(x,t) - H(x',t)| \simeq t^{\alpha}$ we need to move apart of $|x - x'| \simeq t^{\beta}$.

. KPZ exponent in 1+1 dimension: $\alpha = 1/3$, $\beta = 2/3$

. Therefore, one will have to rescale the height as follows:

$$h_t^{resc}(u) \doteq \frac{h(a \cdot t + u \cdot t^{2/3}, t) - t \cdot h_{ma}(a + u \cdot t^{-1/3})}{t^{1/3}}$$

. Question: Can we determine the limit process describing the height fluctuations, i.e., $\lim_{t\to\infty} h_t^{resc}$? Is it depending on initial conditions or not?

. To try to answer to these questions, we consider a simplified model, the "polynuclear growth (PNG) model".

. As far we know now, starting with an interface without fluctuations, say $h(x,0) = 0$ for example, then the limit process should be:

. Airy_2 process if h_{ma} is curved, [$\text{Airy}_2 \equiv$ Airy process].

. Airy_1 process if h_{ma} is flat.

[See math.PR/0105240 and 0707.4207 on the www.arXiv.org].

. Now I'll define the PNG model and analyze the particular geometry "corner growth" which leads to a curved limit shape $\Rightarrow$ Airy_2 process.

9.2) The polynuclear growth model; droplet case.

. The model: . Configurations: The configurations are given by integer-valued functions $x \mapsto h(x,t) \in \mathbb{Z}$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$.

$\Rightarrow$ We can describe the configurations by telling the positions of the up- and down-jumps [by convention, h is upper semi-continuous].

. If at the same position we have an up- and a down-jump, we call it nucleation.

. Dynamics: It consists in a deterministic part and a stochastic part:

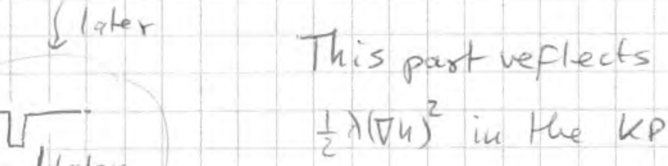
ⓐ Deterministic part: $\begin{cases} \text{. up-steps move to left, unit speed} \\ \text{. down-steps move to right, " "} \\ \text{. when they meet, they merge} \end{cases}$

This part reflects both the term $\nu \Delta h$ and $\frac{1}{2}\lambda(\nabla h)^2$ in the KPZ equation.

ⓑ Stochastic part: . nucleations are added as a space-time Poisson process with some intensity $\varrho(x,t)$.

This part reflects the noise term η in the KPZ equation.

. PNG droplet geometry: . The intensity of nucleations for the PNG droplet is taken to be: $\varrho(x,t) = \begin{cases} 2, & |x| \le t, \\ 0, & \text{otherwise.} \end{cases}$



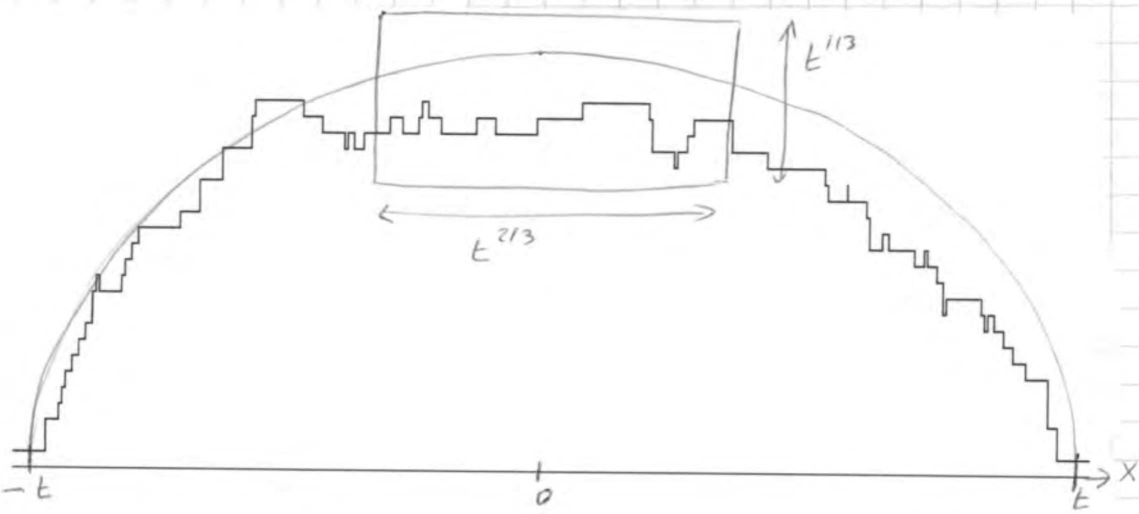
. Equivalently, the PNG droplet is obtained by nucleating with fixed intensity <u>above</u> a first island starting at the origin $(x,t)=(0,0)$.

. <u>Theorem [Prähofer, Spohn '02]</u> . Let $h(x,0)=0$ and $g(x,t)=\begin{cases} 2, & |x|\le t, \\ 0, & \text{otherwise}. \end{cases}$

In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty} \frac{h(u\cdot t^{2/3},t)-t\, h_{ma}(u t^{-1/3})}{t^{1/3}} = \mathcal{A}_2(u),$$

with $h_{ma}(\xi) = 2\sqrt{[1-\xi^2]_+}$, and $\mathcal{A}_2$ is the Airy process.



. Snapshot of the PNG droplet
[double jumps at a position are just due to the pixel-based animation, in continuous space they do not occur].

. How was this result obtained?

The answer is first the model is extended to a multilayer version and <u>for this particular geometry</u> (i.e., for this choice of poisson points density) it turns out that it is equivalent to the non-intersecting condition.
So, one can then use the methods we learned so far.

9.3) Multilayer PNG droplet and RSK construction.

. The extension to multilayer PNG is as follows.
Instead of a single height function $h(x,t)$, we have a collection of lines $\{h_e(x,t)\}_{e\leq 0}$, where $h_0(x,t) \equiv h(x,t)$ and $h_e(x,t)$ evolves as follows:

. Initial conditions: . $h_e(x,0) = e$, $e = 0,-1,-2,\dots$

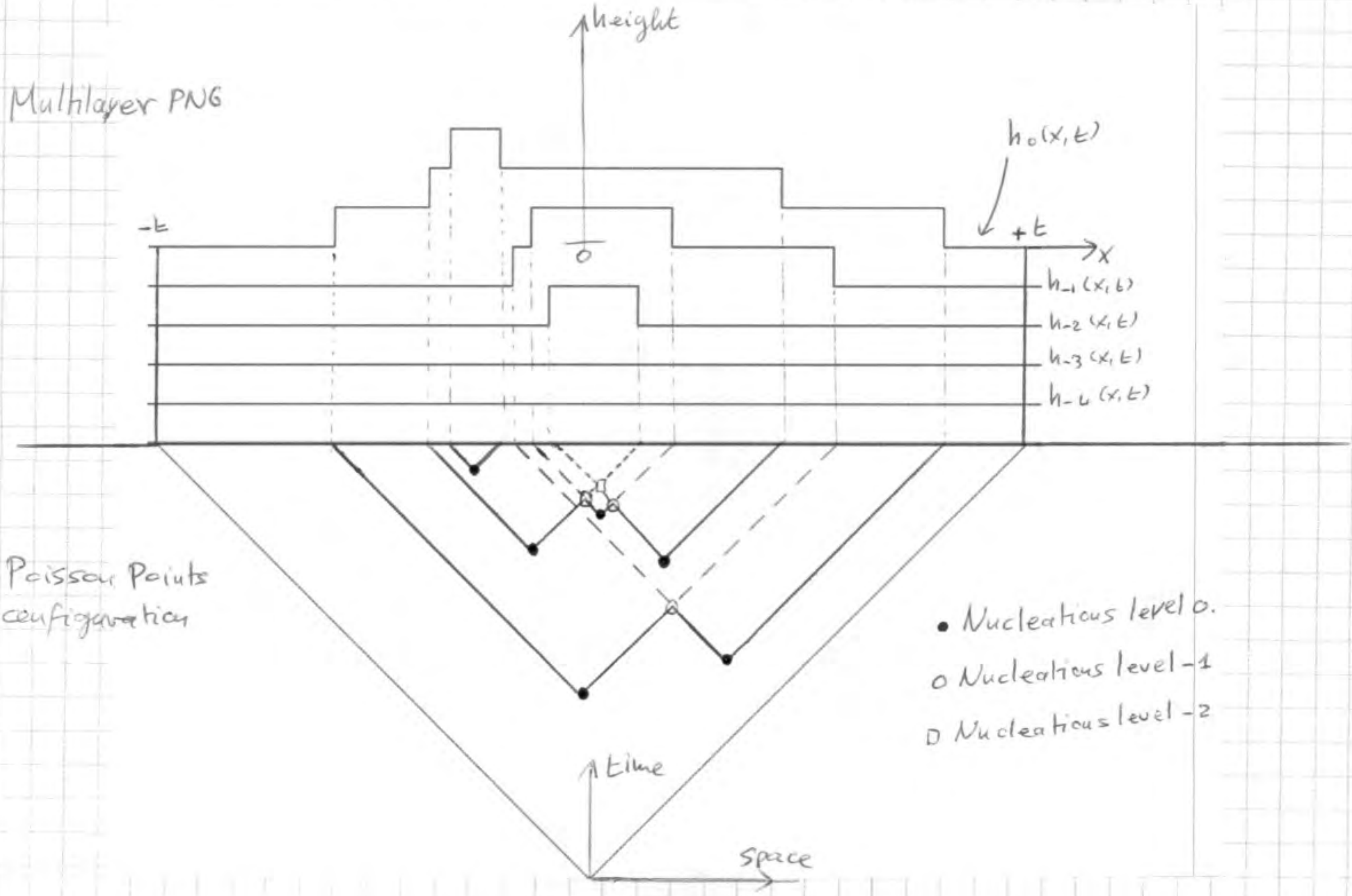
. Evolution: . $h_0(x,t)$ evolves as PNG,

. $h_e(x,t)$ has the same deterministic PNG evolution, but the nucleations of level e occurs in space-time when there is a merging at level $e+1$.

. The important property is that the set of lines, by construction: . are non-intersecting: $h_e < h_{e+1}$.

. contains the same information as the Poisson points (the space-time locations of the nucleations).

Multilayer PNG



Poisson Points configuration

- To construct the multilayer from the nucleation points one can do the following geometric construction (which is equivalent to the RSK algorithm on permutation, therefore it was given the name RSK construction; RSK = Robinson, Schensted, Knuth).

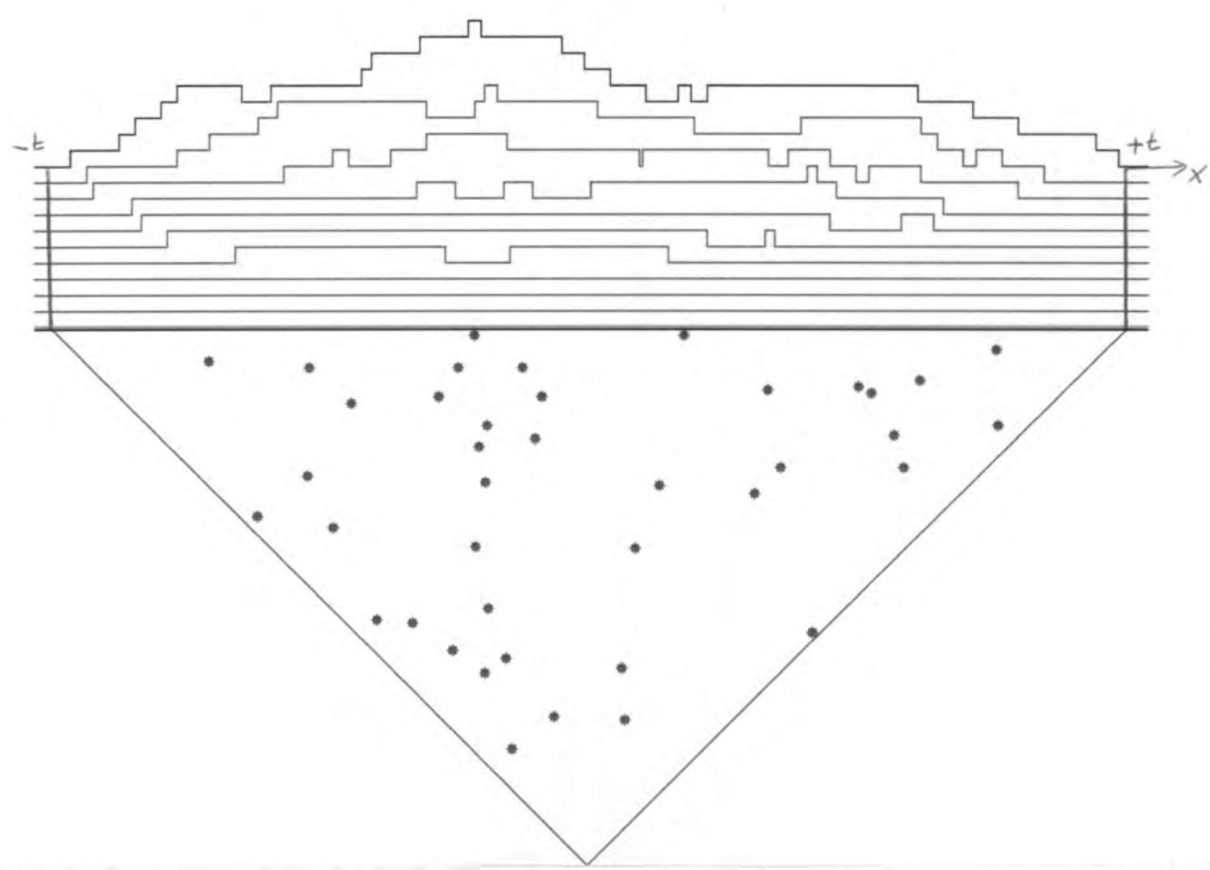
- One draws the forward light cones of the nucleations until they intersect. These are the space-time positions of the up and down steps of h_0.

- Then, iteratively, the intersection points for level $\ell+1$ becomes nucleations for level ℓ.

- The multilayer PNG at some given time t, is then easily constructed by looking at the positions of the light-cones at time t. An example is shown in the previous picture (page 6).

- <u>Remark</u>: From the multilayer, it is easy to recover the positions of the nucleations, by doing backwards in time the construction starting from the lowest "excited" line.

- <u>A larger snapshot of PNG multilayer.</u>



. In their paper "Scale invariance of the PNG droplet and the Airy process", Prähofer and Spohn prove the following.

. Denote by $-t < Y^+_{1,e} < Y^+_{2,e} < \dots < Y^+_{n_e,e} < t$ the positions of the up-jumps of h_e and by $-t < Y^-_{1,e} < Y^-_{2,e} < \dots < Y^-_{n_e,e} < t$ the positions of the down-jumps.

. Set $\vec{n} = (n_0, n_{-1}, n_{-2}, \dots)$ and $|\vec{n}| = \sum_{e \le 0} n_e$.

. By the RSK construction, $|\vec{n}| = \#$ Poisson points. Since we use density two, $\mathbb{P}(|\vec{n}| = k) = e^{-a_t} \frac{a_t^k}{k!}$, with $a_t = 2 \cdot t^2$, so $|\vec{n}|$ is a.s. finite.

. Denote by $\Gamma_t(\vec{n})$ the set of all step configurations $((Y^+_{j,e}, Y^-_{j,e})_{1 \le j \le n_e})_{e \le 0}$ resulting from an admissible line configuration $(h_e(x,t))_{e \le 0} \in \Lambda_t$ ($\Lambda_t \equiv$ set of line configurations). $\Gamma_t(0) \cong \phi$ and $\Gamma_t(\vec{n})$ in naturally embedded in $[-t,t]^{2|\vec{n}|}$. Then, $\Gamma_t = \bigcup_{|\vec{n}| < \infty} \Gamma_t(\vec{n})$.

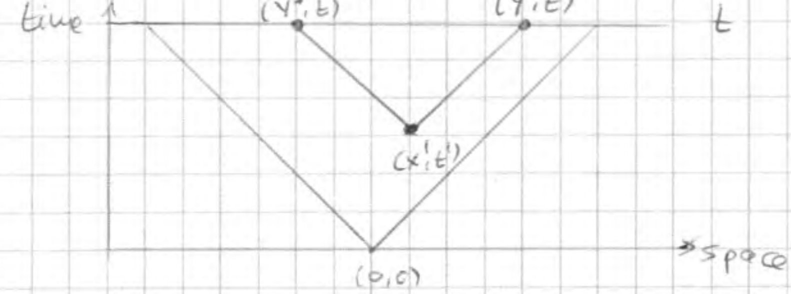
. By the RSK construction, we have a <u>bijective map</u> $S : \Lambda_t \to \Gamma_t$

<u>Theorem:</u> Let w_t be the uniform measure on Γ_t, i.e., $w_t(\Gamma_t(0)) = 1$, and $w_t|_{\Gamma_t(\vec{n})}$ is the $2|\vec{n}|$-dimensional Lebesgue measure on $\Gamma_t(\vec{n})$.

Then, $\int w_t \equiv Z(t) = \exp(2 \cdot t^2)$ and $\mu_t = \frac{w_t}{Z(t)}$ is a proba. measure on Γ_t. If the height functions $\{h_e\}_{e \le 0}$ evolves by the RSK dynamics, then μ_t is the joint distribution of $\{h_e(x,t), x \in \mathbb{R}, e \le 0\}$ under the map S.

. To see that this theorem holds, we have just to see that the measure (on the step positions) inherited by the Poisson points measure on $\{(x',t'), 0 \le t' \le t, |x'| \le t'\}$ is μ_t. For this is enough to see that we have the right measure for every N, the number of Poisson points

- To be precise, to a Poisson point at (x', t'), it corresponds an up-step at $Y^+ = x' - (t-t')$ and a down-step at $Y^- = x' + (t-t')$.



- To a configuration of N Poisson points it corresponds a set of up- and down-jumps $\{(Y_i^+, Y_i^-)_{i=1,\dots,N}\}$ such that

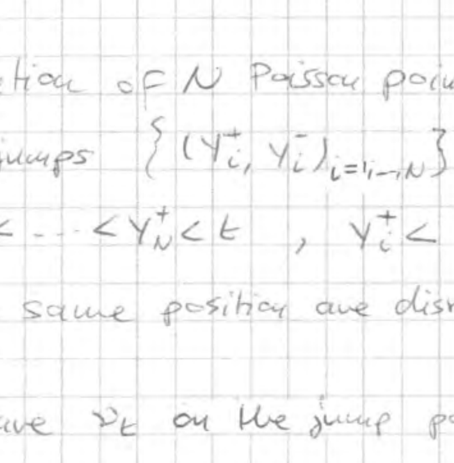
$$-t < Y_1^+ < \dots < Y_N^+ < t \quad , \quad Y_i^+ < Y_i^-, \; i = 1, \dots, N$$

(two jumps at the same position are disregarded, since they occur with probability 0).

- Then, the measure ν_t on the jump positions induced by the Poisson process with intensity $\rho = 2$ is:

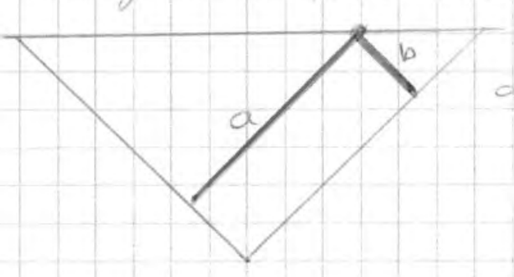
$$\cdot \; \nu_t(\text{no jumps}) = e^{-2t^2}$$

$$\cdot \; \nu_t\big|_{2N \text{ jumps}} = e^{-2t^2} \cdot dy_1^+ \dots dy_N^+ \, dy_1^- \dots dy_N^-$$



$\Rightarrow$ On the line time $= t$, by the geometric factor $\sqrt{2}$ (dilatation) we have a Lebesgue measure time $\sqrt{\frac{\rho}{2}} = 1$ with $\rho = 2$.

- The simple measure on the step positions is due to the particular geometry, since at every point the length of the backwards light cone intersection with the forward light cone from $(0,0)$ is constant $= t\sqrt{2}$, so is the step intensity!



$a + b = t\sqrt{2}$.

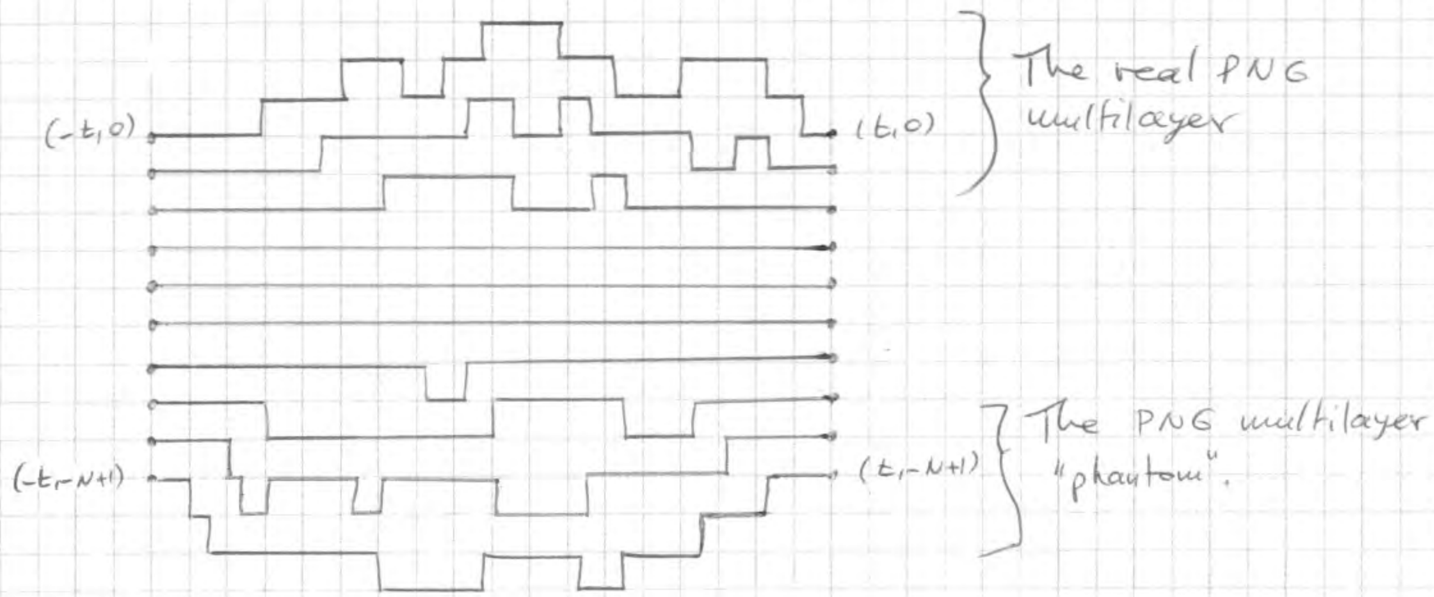
9.4) Non-intersecting lines and extended determinantal point process

. The multilayer PNG is a set of non-intersecting lines with fixed initial and final positions. The measure on the jumps is just Lebesgue measure, i.e., jumps occurs up and down with intensity one (provided the lines do not intersect).

. Moreover, by the RSK construction backwards in time, to any non-intersecting lines it corresponds a set of Poisson points.

. To apply Karlin-McGregor theorem, we start with N non-intersecting lines and take first $N \to \infty$ and later $t \to \infty$ under appropriate edge scaling.

. This time, an effect that was not present in the 3D-Ising corner, arises. For finite, large N, we will get essentially two independent PNG multilayers instead of only one. But since the lower one is at positions $\approx -N$, it does not influence the statistics of the upper one as $N \to \infty$ (with exponentially decreasing influence).



. So, we have to study the system of N non-intersecting lines starting from $(x_i(-t)=-i)_{i=0,1,\dots,N-1}$ and ending at $(x_i(t)=-i)_{i=0,1,\dots,N-1}$, under the non-intersecting constraint but otherwise doing continuous time random walks with jump rates one ($\equiv$ Lebesgue measure on jumps).

. By Karlin-McGregor, we need to determine first the transition probability of a single free path. To go from x to y during a time interval τ, it will be given by $\langle y, e^{-\tau H} x\rangle$, where $H\psi(u) = -[\psi(u+1)+\psi(u-1)]$ for $\psi \in \ell^2(\mathbb{Z})$. In other words,

<u>for $\tau \geq 0$:</u> $P_\tau(x;y) = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dz}{z^{y-x+1}} \cdot e^{\tau\left(z+\frac{1}{z}\right)}$, where Γ_0 is any anticlockwise simple loop around $z=0$.

[To check it, just look $\tau \to 0$: $P_\tau(x;y) = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dz}{z^{y-x+1}} \left(1+\tau\left(z+\frac{1}{z}\right)+O(\tau^2)\right)$
$$= \delta_{x,y} + \tau\cdot(\delta_{x,y+1}+\delta_{x,y-1}) + O(\tau^2)$$
and use the fact that we have a semi-group].

. Therefore, the point process associated with the line ensemble has kernel given by:

$$K_N(t_1,x_1;t_2,x_2) = -P_{t_1-t_2}(x_1;x_2)\,\mathbb{1}_{[t_2<t_1]}$$
$$+\sum_{x,y\leq 0} P_{t-t_2}(x_2;x)\cdot[A_N^{-1}]_{x,y}\cdot P_{t+t_1}(y;x_1)$$

where $[A_N]_{x,y} = P_{2t}(x;y)$.

. The $N \to \infty$ limit is as for the 3D-Ising corner model. So, we need to determine the inverse of $A = [P_{2t}(x,y)]_{x,y\leq 0}$.

. We consider A as a matrix on $\ell^2(\mathbb{Z})$ and the inverse to be computed is on $\ell^2(\{\ldots,-2,-1,0\})$ only.

. We use the same decomposition of the 3D-Ising corner, page 14;

let $P_+ =$ projector on $\{1,2,\ldots\}$ and $P_- =$ projector on $\{\ldots,-1,0\}$.

. Denote by $a_+(\tau)$ the matrix with upjumps only, i.e.,

$$[a_+(\tau)]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw\, e^{\tau w}}{w^{y-x+1}}, \qquad \Rightarrow \text{upper-triangular,}$$

and by $a_-(\tau)$ the matrix with downjumps only, i.e.,

$$[a_-(\tau)]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw\, e^{\tau/w}}{w^{y-x+1}}. \qquad \Rightarrow \text{lower-triangular.}$$

. The block decompositions are: $a_+(\tau) = \begin{pmatrix} m_1(\tau) & m_2(\tau) \\ 0 & m_3(\tau) \end{pmatrix}$, $a_-(\tau) = \begin{pmatrix} \tilde{m}_1(\tau) & 0 \\ \tilde{m}_2(\tau) & \tilde{m}_3(\tau) \end{pmatrix}$

$$\Rightarrow A = a_+(2t)\, a_-(2t) = a_-(2t)\, a_+(2t) = \begin{pmatrix} \tilde{m}_1(2t)\, m_1(2t) & * \\ * & * \end{pmatrix}$$

. We need to compute: $\boxed{P_-\left(P_+ + P_-\, a_-(2t) a_+(2t) P_-\right)^{-1} P_- = a_+(2t)^{-1} P_-\, a_-(2t)^{-1}.}$

In fact, $a_+(2t)^{-1} = \begin{pmatrix} m_1^{-1} & -m_1^{-1}\cdot m_2 \cdot m_3^{-1} \\ 0 & m_3^{-1} \end{pmatrix}$ and $a_-(2t)^{-1} = \begin{pmatrix} \tilde{m}_1^{-1} & 0 \\ -\tilde{m}_3^{-1}\cdot \tilde{m}_2 \cdot \tilde{m}_1^{-1} & \tilde{m}_3^{-1} \end{pmatrix}$

Then,
$$a_+(2t)^{-1} P_-\, a_-(2t)^{-1} = \underbrace{\begin{pmatrix} m_1^{-1} & -m_1^{-1} m_2\, m_3^{-1} \\ 0 & m_3^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} m_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} \tilde{m}_1^{-1} & 0 \\ -\tilde{m}_3^{-1}\tilde{m}_2 \cdot \tilde{m}_1^{-1} & \tilde{m}_3^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} m_1^{-1}\cdot \tilde{m}_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and
$$P_-(P_+ + P_- a_- a_+ P_-)^{-1} P_- = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{m}_1 m_1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} m_1^{-1}\tilde{m}_1^{-1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} m_1^{-1}\cdot \tilde{m}_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

· Inverse of a_+ and a_- on $\ell^2(\mathbb{Z})$:

$$\cdot\left[a_+^{-1}(2t)\right]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} dw\, \frac{e^{-2tw}}{w^{y-x+1}} \quad \text{and}$$

$$\cdot\left[a_-^{-1}(2t)\right]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw\, e^{-2t/w}}{w^{y-x+1}}.$$

$$\Rightarrow K(t_1,x_1;t_2,x_2) = -\left[a_-(t_1-t_2)a_+(t_1-t_2)\right]_{x_1,x_2}\cdot \mathbb{1}_{[t_2<t_1]}$$

$$+\sum_{x,y\le 0}\left[a_-(t-t_2)a_+(t-t_2)\right]_{x_2,x}\cdot\left[a_+(2t)^{-1}P_-\, a_-(2t)^{-1}\right]_{x,y}\cdot$$

$$\cdot\left[a_-(t+t_1)a_+(t+t_1)\right]_{y,x_1}$$

Now, we can extend the sum over all $x,y\in\mathbb{Z}$, since the middle term is zero for x or $y>0$.

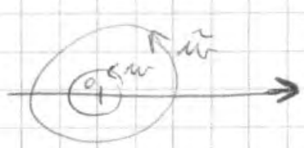
Therefore,

$$K(t_1,x_1;t_2,x_2) = -\left[a_-(t_1-t_2)\,a_+(t_1-t_2)\right]_{x_1,x_2}\cdot \mathbb{1}_{[t_2<t_1]}$$

$$+\sum_{\ell\le 0}\left[a_-(t-t_2)\cdot a_+^{-1}(t+t_2)\right]_{x_2,\ell}\cdot\left[a_-^{-1}(t-t_1)\cdot a_+(t+t_1)\right]_{\ell,x_1}$$

$$\cdot\quad \left[a_-(t-t_2)\,a_+^{-1}(t+t_2)\right]_{x,y} = \sum_{z\in\mathbb{Z}}\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\frac{dw\, e^{(t-t_2)/w}}{w^{z-x+1}}\cdot\oint_{\Gamma_0}\frac{d\tilde w\, e^{-(t+t_2)\tilde w}}{\tilde w^{y-z+1}}$$

$$= \sum_{z\le y}\left(\quad '' \quad\right)$$



$$= \sum_{\ell = y-z\ge 0}\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}dw\oint_{\Gamma_0}d\tilde w\, \frac{e^{(t-t_2)/w}\cdot e^{-(t+t_2)\tilde w}}{w^{y-x+1}\cdot\tilde w}\cdot\left(\frac{w}{\tilde w}\right)^{\ell}$$

$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0}dw\oint_{\Gamma_0}d\tilde w}_{|w|<|\tilde w|}\,\frac{e^{(t-t_2)/w}\, e^{-(t+t_2)\tilde w}}{w^{y-x+1}}\cdot\underbrace{\sum_{\ell\ge 0}\left(\frac{w}{\tilde w}\right)^{\ell}\frac{1}{\tilde w}}_{=\frac{1}{\tilde w - w}}$$

$$\overset{\text{Residue at } \tilde w = w}{=} \frac{1}{2\pi i}\oint_{\Gamma_0}dw\, \frac{e^{\frac{t-t_2}{w}}\cdot e^{-(t+t_2)w}}{w^{y-x+1}}.$$

and similarly,

$$\left[a_-^{-1}(t-t_1)\,a_+(t+t_1)\right]_{x,y} = \frac{1}{2\pi i}\oint_{\Gamma_0} dw\, \frac{e^{-(t-t_1)/w}\cdot e^{(t+t_1)w}}{w^{y-x+1}}.$$

<u>Parenthesis on Bessel functions:</u> For $b \geq a$ and $n \in \mathbb{Z}$,

(a) $\bullet$

$$\frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dz}{z}\cdot\frac{e^{b(z-\frac{1}{z})}\cdot e^{a(z+\frac{1}{z})}}{z^n} \underset{z=\frac{1}{w}}{=} \frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w}\cdot\frac{e^{b(\frac{1}{w}-w)}\cdot e^{a(w+\frac{1}{w})}}{w^{-n}}$$

$$= \left(\frac{b+a}{b-a}\right)^{n/2}\cdot J_n\left(2\sqrt{b^2-a^2}\right),$$

where J_n are the standard Bessel functions.

$$\Rightarrow \sum_{e\leq 0}\left[a_-(t-t_2)\,a_+^{-1}(t+t_2)\right]_{x_2,e}\cdot\left[a_-^{-1}(t-t_1)\,a_+(t+t_1)\right]_{e,x_1} =$$

$$= \sum_{e\leq 0}\left(\frac{t-t_2}{t+t_2}\right)^{\frac{x_2-e}{2}}\cdot\left(\frac{t+t_1}{t-t_1}\right)^{\frac{x_1-e}{2}}\cdot J_{x_2-e}\left(2\sqrt{t^2-t_2^2}\right)\cdot J_{x_1-e}\left(2\sqrt{t^2-t_1^2}\right)$$

(b) $\bullet$

$$\frac{1}{2\pi i}\oint_{\Gamma_0}\frac{dz}{z}\cdot\frac{e^{t(z+\frac{1}{z})}}{z^n} = I_{|n|}(2t)$$

, where I_n is the modified Bessel function.

(c). One can also compute:

$$\sum_{e\in\mathbb{Z}}\left[a_-(t-t_2)\,a_+^{-1}(t+t_2)\right]_{x_2,e}\cdot\left[a_-^{-1}(t-t_1)\,a_+(t+t_1)\right]_{e,x_1} = \frac{1}{2\pi i}\oint_{\Gamma_0} dw\,\frac{e^{(t_1-t_2)(w+\frac{1}{w})}}{w^{x_1-x_2+1}}$$

Therefore the kernel can be written as follows:

$$K(t_1,x_1;t_2,x_2) = \begin{cases} \sum_{e\leq 0}\left(\frac{t-t_2}{t+t_2}\right)^{\frac{x_2-e}{2}}\cdot\left(\frac{t+t_1}{t-t_1}\right)^{\frac{x_1-e}{2}}\cdot J_{x_2-e}\left(2\sqrt{t^2-t_2^2}\right) J_{x_1-e}\left(2\sqrt{t^2-t_1^2}\right), & \text{for } t_2\leq t_1 \\ \sum_{e>0} \quad \text{''} & \text{, for } t_2 > t_1. \end{cases}$$

This is the form of the extended (Bessel) kernel obtained in the original paper by Prähofer and Spohn 02.

Back to the main track.

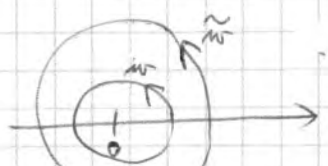
The main part of the kernel writes then:

$$\sum_{\ell\le 0} (\dots)_{x_2,\ell}\,(\dots)_{\ell,x_1} = \sum_{\ell\le 0}\frac{1}{(2\pi i)^2}\cdot\oint_{\Gamma_0}dw\oint_{\Gamma_0}d\tilde{w}\; e^{t(\frac{1}{w}-w)}\cdot e^{-t_2(w+\frac{1}{w})}$$

$$\cdot\, e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}\cdot\frac{w^{x_2-1}}{\tilde{w}^{x_1+1}}\cdot\left(\frac{\tilde{w}}{w}\right)^{\ell}$$

$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0}dw\oint_{\Gamma_0}d\tilde{w}}_{|\tilde{w}|\ge|w|}\cdot e^{t(\frac{1}{w}-w)}\, e^{-t_2(w+\frac{1}{w})}\cdot e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}$$

$$\cdot\,\frac{w^{x_2-1}}{\tilde{w}^{x_1+1}}\underbrace{\sum_{\ell\le 0}\left(\frac{\tilde{w}}{w}\right)^{\ell}}_{=\frac{\tilde{w}}{\tilde{w}-w}}$$



$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0}dw\oint_{\Gamma_0}d\tilde{w}}_{|\tilde{w}|>|w|}\cdot\frac{e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}}{e^{t(w-\frac{1}{w})}\cdot e^{t_2(w+\frac{1}{w})}}\cdot\frac{w^{x_2-1}}{\tilde{w}^{x_1}}\cdot\frac{1}{\tilde{w}-w}$$

which can be reexpressed also as

$$= \frac{1}{(2\pi i)^2}\underbrace{\oint_{\Gamma_0}dw\oint_{\Gamma_0}d\tilde{w}}_{|w|>|\tilde{w}|}\;\frac{e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}}{e^{t(w-\frac{1}{w})}\cdot e^{t_2(w+\frac{1}{w})}}\cdot\frac{w^{x_2-1}}{\tilde{w}^{x_1}}\,\frac{1}{\tilde{w}-w}$$

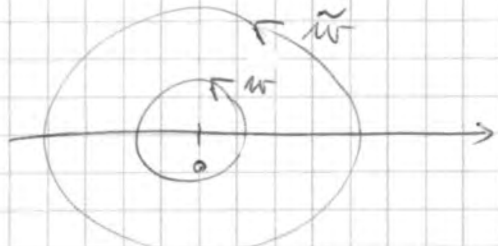
$$+\frac{1}{2\pi i}\oint_{\Gamma_0}dw\;\frac{e^{(t_1-t_2)(w+\frac{1}{w})}}{w^{x_1-x_2+1}}.$$

Remark: the last term is, for $t_1>t_2$, exactly the extra term in the complete kernel. This happens all the time and it is not particular of the PNG model.

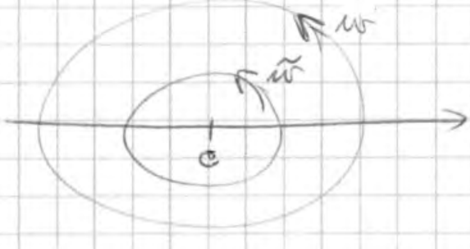
. Therefore, the final formula for the Kernel is:

$$\boxed{K(t_1,x_1;t_2,x_2) = \frac{1}{(2\pi i)^2}\oint_{\Gamma_0} dw \oint_{\Gamma_0} d\tilde{w}\, \frac{e^{t(\tilde{w}-\frac{1}{\tilde{w}})}\cdot e^{t_1(\tilde{w}+\frac{1}{\tilde{w}})}}{e^{t(w-\frac{1}{w})}\cdot e^{t_2(w+\frac{1}{w})}}\, \frac{w^{x_2-1}}{\tilde{w}^{x_1}}\cdot\frac{1}{\tilde{w}-w}}$$

where: $\begin{cases} \underline{\text{for } t_1 \le t_2}, \text{ the integrand satisfy: } w \text{ inside path } \tilde{w}, \\ \underline{\text{for } t_1 > t_2}, \text{ the integrand satisfy: } \tilde{w} \text{ inside path } w. \end{cases}$



Case: $t_1 \le t_2$



Case: $t_1 > t_2$

9.5) Edge scaling and convergence to the Airy process.

. Let $\eta_t(x,j) = \begin{cases} 1, & \text{a line crosses } (x,j), \\ 0, & \text{otherwise}. \end{cases}$

Then, since we want to analyze $\dfrac{h(ut^{2/3},t) - 2t + u^2t^{1/3}}{t^{1/3}} \equiv h_t^{resc}(u)$, our point process η_t rescales as:

$$\eta_t^{edge}(u,s) = t^{1/3}\cdot\eta_t\left(ut^{2/3}, [2t - u^2t^{1/3} + st^{1/3}]\right),$$

and the associated Kernel as:

$$K^{edge}(u_1,s_1;u_2,s_2) = t^{1/3}\cdot K\left(u_1t^{2/3}, [2t-u_1^2t^{1/3}+s_1t^{1/3}]; u_2t^{2/3}, [2t-u_2^2t^{1/3}+s_2t^{1/3}]\right).$$

Let $\begin{cases} f_0(z) \doteq z - \frac{1}{z} - 2\cdot\ln z \\ f_1(z;u) \doteq \left(z+\frac{1}{z}\right)u \\ f_2(z,u,s) \doteq -(s-u^2)\ln z \end{cases}$

. Consider the case $t_1 \le t_2$, i.e, $\underline{u_1 \le u_2}$. The case $u_1 > u_2$ follows similarly.

Then, $$K_t^{edge}(u_1,s_1;u_2,s_2) = \frac{t^{1/3}}{(2\pi i)^2} \underbrace{\oint_{\Gamma_0} dw \oint_{\Gamma_0} dz}_{\substack{w \text{ inside} \\ z\text{-path}}} \cdot \frac{e^{t f_0(z) + t^{2/3} f_1(z,u_1) + t^{1/3} f_2(z,u_1,s_1)}}{e^{t f_0(w) + t^{2/3} f_1(w,u_2) + t^{1/3} f_2(w,u_2,s_2)}} \cdot \frac{1}{w(z-w)}.$$

. <u>Asymptotic analysis:</u> u_1 and u_2 remain fixed.

(a) Convergence on bounded set (uniformly): do the asymptotic analysis of K^{edge} for $s_1, s_2 \in [-L, L]$, for $L \gg 1$ fixed.

Needed to control the cvg. of the Fredholm determinants.

(b) Obtain a bound for $(s_1,s_2) \in [-L, \varepsilon t^{2/3}]^2 \setminus [-L,L]$ for L large enough and ε small enough.
Usually one can get: $|K^{edge}| \underset{\substack{\uparrow \\ \text{(up to ev. a} \\ \text{conjugation)}}}{\leq} C \cdot e^{-(s_1+s_2)}$.

(c) Obtain a bound for $(s_1,s_2) \in [-L,\infty)^2 \setminus [-L, \varepsilon t^{2/3}]^2$.

. For part (a), see the details below. The leading contribution comes from the neighborhood of the double critical point of $f_0(z)$.

. To get the part (b), one can modify the path used in (a) locally around the old critical points, since now one will have two real critical points of the term proportional to t. One can estimate by taylor series for $\varepsilon \ll 1$.

. To get part (c), one usually can keep the path of (b) for $s_i = \frac{\varepsilon}{2} t^{2/3}$ and see that the difference is exponentially small.

. Now we explain how to do the asymptotic analysis for s_1, s_2 in a bounded set.

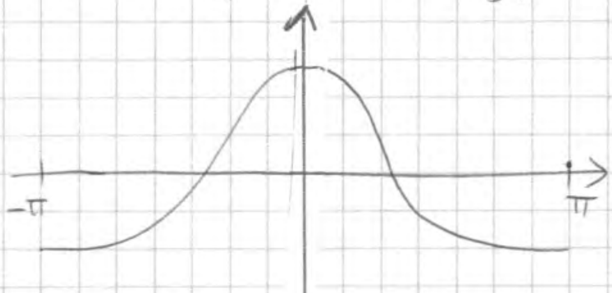
. <u>Step 1</u>: Find integration paths s.t. $\mathrm{Re}[f_0(z)]$ decreases by going away from the critical point and $\mathrm{Re}(-f_0(w))$ too.

. <u>Critical point</u>: $\frac{df_0(z)}{dz} = 1 + \frac{1}{z^2} - \frac{2}{z} = \frac{1+z^2-2z}{z^2} = \frac{(z-1)^2}{z^2}$
$= 0 \Rightarrow$ <u>$z=1$: double critical point</u>

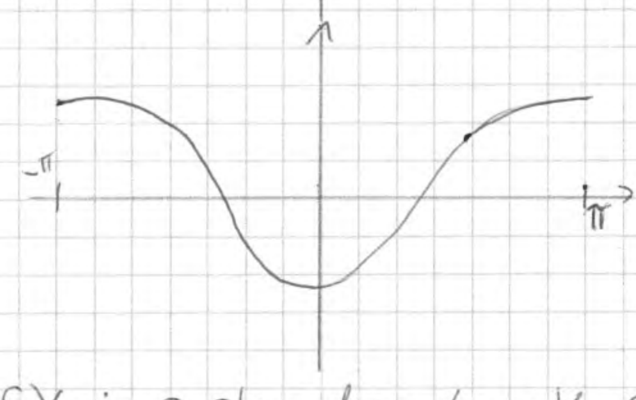
. Consider the path: $\gamma_\rho = \{z = \rho e^{i\varphi},\ \varphi \in [-\pi, \pi)\}$.

Then, $\mathrm{Re}(f_0(z)) = (\rho - \frac{1}{\rho})\cos\varphi - 2\ln\rho$

$\boxed{\rho>1}$



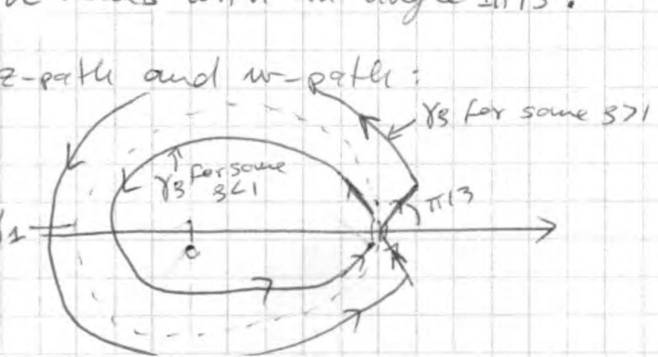
$\boxed{\rho<1}$



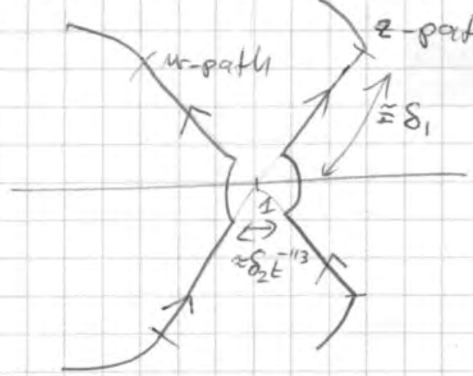
$\Rightarrow \begin{cases} \cdot\ \gamma_\rho \text{ is a steep descent path for } z \text{ if } \rho>1 \\ \cdot\ \gamma_\rho \text{ is a } \text{" " " " } w \text{ if } \rho<1 \end{cases}$

. Moreover, close to $z_c = 1$, $f_0(z) \simeq f_0(z_c) + \frac{1}{3}(z-z_c)^3$, which means that a path leaving $z=z_c$ is <u>steepest descent</u> $\Rightarrow$ it leaves with an angle $\pm\pi/3$.

. $\Rightarrow$ Choice of z-path and w-path:



. Zoom close to the critical point:



- We need to keep a small distance $\mathcal{O}(t^{-1/3})$ to the critical point because the two paths can't intersecate due to the $\frac{1}{z-w}$ term.

- We can fix a small δ_1 s.t. the $\mathrm{Re}(f_0(t))$ decreases along $e^{i\pi/3}\cdot x$, $0 \le x \le \delta_1$, and we already have that along γ_δ, $\delta > 1$, we are fine too.

- Once we get the steep descent path, we can replace the integrals in K^{resc} by the ones restricted to a δ-neighborhood of the critical point. The error can be estimated $\mathcal{O}(e^{-\mu(\delta)t})$ where $0<\mu(\delta)\sim\delta^3$. Denote by $\Gamma_z^\delta, \Gamma_w^\delta$ these paths.

. Step 2: Taylor series around the critical point.

We have:
$$\begin{cases} f_0(z) = f_0(z_c) + \frac{1}{3}(z-z_c)^3 + \mathcal{O}((z-z_c)^4) \\ f_1(z,u) = f_1(z_c,u) + u\cdot(z-z_c)^2 + \mathcal{O}(|z-z_c|^3) \\ f_2(z,u,s) = f_2(z_c,u,s) + (u^2-s)(z-z_c) + \mathcal{O}((z-z_c)^2) \end{cases}$$

$$\Rightarrow K_t^{edge} \cong \frac{t^{1/3}}{(2\pi i)^2} \cdot \int_{\Gamma_w^\delta} dw \int_{\Gamma_z^\delta} dz \, \frac{e^{\frac{1}{3}t(z-z_c)^3 + t^{2/3}u_1(z-z_c)^2 + t^{1/3}(u_1^2-s_1)(z-z_c)}}{e^{\frac{1}{3}t(w-z_c)^3 + t^{2/3}u_2(w-z_c)^2 + t^{1/3}(u_2^2-s_2)(w-z_c)}} \cdot \frac{1}{(z-w)z_c}$$

$$\cdot\, e^{\mathcal{O}(t(z-z_c)^4,\, t^{2/3}(z-z_c)^3,\, t^{1/3}(z-z_c)^2;\ \text{same with } w)} \, (1+\mathcal{O}(z-z_c))$$

conjugated by
$$\left(\frac{e^{t^{2/3}f_1(z_c,u_1)+t^{1/3}f_2(z_c,u_1,s_1)}}{e^{t^{2/3}f_1(z_c,u_2)+t^{1/3}f_2(z_c,u_2,s_2)}}\right)^{-1}$$

. At this point we use: $|e^{x}-1| \leq |x| \cdot e^{|x|}$ applied

for $x = \mathcal{O}(\cdots) \Rightarrow$ If we replace by zero the error term, we do an error given by the same main integrand times an extra $\mathcal{O}(\cdots)$.

. By the change of variable: $\begin{cases} (z-z_c)t^{1/3} = Z, \\ (w-z_c)t^{1/3} = W, \end{cases}$

one then sees that this error is bounded by $\mathcal{O}(t^{-1/3})$

. So, we obtained:

$$K_t^{edge} \underset{\text{conj.}}{=} \mathcal{O}(t^{-1/3}, e^{-\mu(\delta)t}) + \frac{t^{1/3}}{(2\pi i)^2} \cdot \int_{\Gamma_w^\delta} dw \int_{\Gamma_z^\delta} dz \, \frac{e^{\frac{1}{3}t(z-z_c)^3 + t^{2/3}u_1(z-z_c)^2 + t^{1/3}(u_1^2-s_1)(z-z_c)}}{e^{\frac{1}{3}t(w-z_c)^3 + t^{2/3}u_2(w-z_c)^2 + t^{1/3}(u_2^2-s_2)(w-z_c)}} \cdot \frac{1}{z_c \cdot (z-w)}$$

. By the change of variable indicated above, this last integral becomes $(z_c = 1)$

$$\frac{1}{(2\pi i)^2} \int dW \int dZ \, \frac{e^{\frac{Z^3}{3} + u_1 \cdot Z^2 + (u_1^2 - s_1) Z}}{e^{\frac{W^3}{3} + u_2 W^2 + (u_2^2 - s_2) W}} \cdot \frac{1}{Z - W}$$

where the integration paths can be continued to:

. for Z: $e^{\pm i\pi/3} \cdot \infty$,

. for W: $e^{\pm 2\pi i/3} \cdot \infty$.

The error made will be once more only $\mathcal{O}(e^{-\mu(\delta)t})$.

. Therefore, we showed that for s_1, s_2 in a bounded set,

$$\lim_{t\to\infty} K_t^{resc}(u_1,s_1;u_2,s_2) \underset{\substack{\uparrow \\ \text{conjugation} \\ \text{too}}}{\equiv} \frac{1}{(2\pi i)^2} \int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} dW \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} dZ \, \frac{e^{\frac{Z^3}{3}+u_1 Z^2+(u_1^2-s_1)Z}}{e^{\frac{W^3}{3}+u_2 W^2+(u_2^2-s_2)W}} \cdot \frac{1}{Z-W}.$$

. Claim: This is the Airy Kernel (up to conjugation)

In fact, since $\mathrm{Re}(z-w)>0$, $\displaystyle\frac{1}{z-w} = \int_0^\infty e^{-\lambda(z-w)}\,d\lambda$

$$\Rightarrow \int_0^\infty d\lambda \left(\frac{1}{2\pi i}\int dw\, e^{-\left(\frac{w^3}{3}+u_2w^2+(u_2^2-s_2-\lambda)w\right)}\right) \cdot \left(\frac{1}{2\pi i}\int dz\, e^{\frac{z^3}{3}+u_1z^2+(u_1^2-s_1-\lambda)z}\right)$$

$$\underset{\substack{w=\tilde w-u_2\\ z=\tilde z-u_1}}{=} \int_0^\infty d\lambda \left(\frac{1}{2\pi i}\int d\tilde w\, e^{-\left(\frac{\tilde w^3}{3}-(s_2+\lambda)\tilde w\right)} \cdot e^{-\lambda u_2} \cdot e^{\frac{u_2^3}{3}-u_2 s_2}\right) \left(\frac{1}{2\pi i}\int d\tilde z\, e^{\frac{\tilde z^3}{3}-(s_1+\lambda)\tilde z} \cdot e^{\lambda u_1} \cdot e^{-\frac{u_1^3}{3}+u_1 s_1}\right)$$

↳ in the conjugation

$$= \int_0^\infty d\lambda\, e^{-\lambda(u_2-u_1)} \cdot Ai(s_2+\lambda)\, Ai(s_1+\lambda)$$

where we used:
$$\begin{cases} \displaystyle\frac{1}{2\pi i}\int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} dw\, e^{-\frac{w^3}{3}+\alpha w} = Ai(\alpha) \\ \text{and} \quad \displaystyle\frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} dz\, e^{\frac{z^3}{3}-\beta z} = Ai(\beta) \end{cases}.$$

Finished.