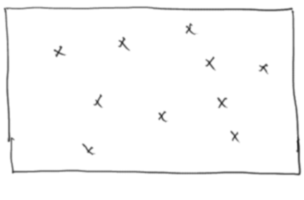
# Point Processes Lecture

## Motivation

Energy levels of large atoms [Electron levels]


Modelled using large matrices with random entries and study eigenvalues – Random Matrix Theory

Ant Mounds



Randomly located in space

Correlation between points: Ant mounds randomly located in space and do not want to be too close.

Last lecture: Poisson point process – no correlations between points

## Definitions of PP

Think: Point processes are measureable mappings from $(\Omega, \mathcal{F}, \mathbb{P})$ to a space of point measures. – i.e point processes are random point measures.

First define $(\Omega, \mathcal{F}, \mathbb{P})$: Let $\Lambda$ be the particle space – complete separable metric space [i.e $\mathbb{R}^d, \mathbb{Z}^d, \mathbb{R} \times \{1, 2, \dots n\}$]

Let $\Omega$ be the space of all localy finite particle configurations i.e. $\eta = (x_i)$ $x_i \in \Lambda$ $i \in \mathbb{N}$ and for all bounded sets $B \subset \Lambda$, $\eta(B) = (\# x_i \in B) < \infty$

Let $C_n^B = \{\eta \in \Omega : \eta(B) = n\}$ $\forall B \subset \Lambda$ bounded and for any $k \geq 0$. These define cylinder sets.

$\mathcal{F}$ is the $\sigma$-algebra generated by $C_n^B$. Denote $\mathbb{P}$ to be the probability measure on $(\Omega, \mathcal{F})$.

Second: define space of point measures:

**Definition:** Let $\mathcal{B}(\Lambda)$ be the Borel $\sigma$-algebra of $\Lambda$. A point measure on $\Lambda$ is a positive measure $\nu$ on $(\Lambda, \mathcal{B}(\Lambda))$ which is a locally finite sum of Dirac measures, i.e

$$\nu = \sum_{i \in I} \delta_{x_i} \quad \text{with } x_i \in \Lambda,\ I \subset \mathbb{N} \text{ and for any bounded } B \subset \Lambda,\ x_i \in B \text{ for only a finite number of } i \in I.$$

$M_p(\Lambda)$ = space of point measures on $\Lambda$

$\mathcal{M}_p(\Lambda)$ = $\sigma$-algebra generated by $\nu \mapsto \nu(f)$ of $M_p(\Lambda)$ to $\mathbb{N} \cup \{\infty\}$ where $f$ spans $\mathcal{B}(\Lambda)$.

**Definition** A point process $\eta$ on $\Lambda$ is a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(M_p(\Lambda), \mathcal{M}_p(\Lambda))$. The probability law of this point process is the image of $\mathbb{P}$ by $\eta$.

**Remark:** We can have (at the moment) $x_i = x_j$ for $i \neq j$ (multiple points)

From definition: if $\eta(A) < \infty$ and $A$ bounded, $\eta|_A = \sum_{i=1}^{\eta(A)} \delta_{x_i}$ for some $x_1, \dots, x_{\eta(A)} \in \Lambda$.

[and $n(A)$ is counting the number of points in $A$ [a random number]]

Definition: A simple point process is a p.p. s.t. $\mathbb{P}(n(\{x\}) \leq 1, \forall x \in \Lambda) = 1$

Remarks Not all p.p are simple. A simple p.p can be identified with the support of the random measure.

Correlation functions: Useful way to study p.p.

Construct $H_n = \sum_{x_{i_1} \neq \ldots \neq x_{i_n}} \delta_{(x_{i_1}, \ldots, x_{i_n})}$ n-tuple of points in the original point process including permutations of the points

Note: • multiplicity $k$ - $k$ distinct points occupying the same position.
• No ordering of the points.

$H_n$ is a new point process.

Define $M_n(A) = \mathbb{E}[H_n(A)]$ for all $A \in \mathcal{B}(\Lambda^n)$.

$M_n$ is a measure and is the expected number of n-tuples of distinct points falling into $A$.

[Assume $M_n, n \geq 1$ well-defined & $M_n(A) < \infty$ $\forall A$ bounded]

Set $\phi(x) = \sum_{j=1}^m a_j \chi_{A_j}(x)$, where $A_1 \ldots A_m$ disjoint measurables subsets of a bounded set $B \subseteq \Lambda$.

Then

(1) $$\prod_{i=1}^{n(B)} (1 + \phi(x_i)) = \sum_{n=0}^\infty \frac{1}{n!} \sum_{x_{i_1} \neq \ldots \neq x_{i_n}} \phi(x_{i_1}) \ldots \phi(x_{i_n}) = \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j)\, H(d^n x).$$

Assume that $\sum_{n=0}^\infty \frac{\|\phi\|_\infty^n}{n!} M_n(B^n) < \infty$ for $B \subseteq \Lambda$ bounded then

$$\sum_{n=0}^\infty \frac{1}{n!} \mathbb{E}\left[\left|\int_{\Lambda^n} \prod_{j=1}^n \phi(x_j)\, H(d^n x)\right|\right] \leq \sum_{n=0}^\infty \frac{\|\phi\|_\infty^n}{n!} M_n(B^n)$$

$[\|\phi\|_\infty = \sup_{x \in \Lambda} |\phi(x)|]$

Apply Fubini in (1):

(2) $$\mathbb{E}\left[\prod_{i=1}^{n(B)} (1 + \phi(x_i))\right] = \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j)\, M_n(d^n x)$$

Since $A_1, \ldots, A_m$ are disjoint and subsets of $B \subseteq \Lambda$ bounded:

$$1 + t\,\phi(x) = \prod_{j=1}^m (1 + t a_j)^{\chi_{A_j}(x)} \quad \text{for all } |t| \leq 1$$

so

$$\prod_{i=1}^{n(B)} (1 + t\phi(x_i)) = \prod_{j=1}^m (1 + t a_j)^{n(A_j)}$$

$$= \prod_{j=1}^m \sum_{n_j=0}^{n(A_j)} \frac{n(A_j)!}{n_j!\,(n(A_j) - n_j)!} (t a_j)^{n_j} \quad \text{-Binomial}$$

$$= \sum_{n_1 \ldots n_m = 0}^\infty \prod_{j=1}^m \frac{(t a_j)^{n_j}}{n_j!} \prod_{j=1}^m \frac{n(A_j)!}{(n(A_j) - n_j)!} \quad \text{where } \frac{1}{n_j!} = 0 \text{ if } n_j < 0$$

$$= \sum_{n=0}^\infty \frac{t^n}{n!} \sum_{n_1 + \ldots + n_m = n} \binom{n}{n_1, \ldots, n_m} \prod_{j=1}^m a_j^{n_j} \prod_{j=1}^m \frac{n(A_j)!}{(n(A_j) - n_j)!} \quad \text{(ressumming)}$$

Take expectations:

③ $\mathbb{E}\left[\prod_{i=1}^{h(B)}(1+t\phi(x_i))\right] = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{n_1+\dots+n_m=n}\binom{n}{n_1\dots n_m}\prod_{j=1}^{m}a_j^{n_j}\prod_{j=1}\mathbb{E}\left[\frac{h(A_j)!}{(h(A_j)-n_j)!}\right]$ (by Fubini)

But from (2)

④ $\mathbb{E}\left[\prod_{i=1}^{h(B)}(1+t\phi(x_i))\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\prod_{k=1}^{n}\left(t\sum_{j=1}^{m}a_j\chi_{A_j}(x_k)\right)M_n(d^nx)$ (Expanding out)

$$= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{n_1+\dots+n_m=n}\binom{n}{n_1\dots n_m}\prod_{j=1}^{m}a_j^{n_j}\,M_n(A_1^{n_1}\times\dots\times A_m^{n_m})$$

Definition: The factorial moment measure: for any disjoint bounded Borel sets $A_1\dots A_m$ in $\Lambda$ and $n_i$, $1\le i\le m$ s.t $1\le n_i\le n$ and $n_1+\dots+n_m=n$

$$M_n(A_1^{n_1}\times\dots\times A_m^{n_m}) = \mathbb{E}\left[\prod_{j=1}^{m}\frac{h(A_j)!}{(h(A_j)-n_j)!}\right]$$

Remark $\mathbb{E}[X^k]$ – $k^{th}$ moment

$\mathbb{E}[X(X-1)\dots(X-k+1)]$ – $k^{th}$ factorial moment

Definition If $M_n$ is absolutely continuous w.r.t to $\lambda$, the natural reference measure on $\Lambda$, i.e

$$M_n(A_1\dots A_n) = \int_{A_1\times\dots\times A_m}\rho^{(n)}(x_1\dots x_n)\,d\lambda(x_1)\dots d\lambda(x_n)$$

for all $A_i\in\mathcal{B}(\Lambda)$, we call $\rho^{(n)}(x_1\dots x_n)$ the $n^{th}$ correlation function.

Remark 1) In many cases these uniquely determine the point process

2) For a simple p.p on $\mathbb{R}$: $[x_i, x_i+\Delta x_i]$ be infinitesimally small sets, then

$$\rho^{(n)}(x_1\dots x_n) = \lim_{\substack{\Delta x_i\to 0\\ i=1,\dots n}}\frac{P[\text{a particle in each of } [x_i,x_i+\Delta x_i]\ \forall i\in\{1,\dots n\}]}{\Delta x_1\dots\Delta x_n}$$

– joint particle densities. $\rho^{(1)}$ – density of particles.

3) For a simple p.p on $\mathbb{Z}$: $\rho^{(n)}$ is the probability of particles at $x_1\dots x_n$.

Wrap up computations with two propositions:

Proposition 1: Consider a point process $h$ whose correlation functions exist. Let $\phi:\Lambda\to\mathbb{C}$ be bounded measurable function with supp $\phi\subset B$, $B$ bounded, $B\subseteq\mathcal{B}(\Lambda)$ and

$$\sum_{n=0}^{\infty}\frac{\|\phi\|_\infty^n}{n!}\int_{B^n}\rho^{(n)}(x_1\dots x_n)\,d^n\lambda(x)<\infty.$$

Then $$\mathbb{E}\left[\prod_{i=1}^{h(B)}(1+\phi(x_i))\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\prod_{j=1}^{n}\phi(x_j)\,\rho^{(n)}(x_1\dots x_n)\,d^n\lambda(x)$$

Proposition 2: Let $(u_n)_{n\ge1}$ be a sequence of measurable function $u_n:\Lambda^n\to\mathbb{R}$. Assume that for any simple measurable function $\phi$ with bounded support, the p.p $h$ satisfies

$$\mathbb{E}\left[\prod_{i=1}^{h(B)}(1+\phi(x_i))\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\prod_{j=1}^{n}\phi(x_j)\,u^{(n)}(x_1\dots x_n)\,d^n\lambda(x)\quad\forall B\subseteq\mathcal{B}(\Lambda)\text{ bounded.}$$

Then all correlation functions $\rho_n, n\ge1$ exist and $\rho_n=u_n$.